LATTICES, CIRCLE AND SPHERE ARRANGEMENTS

Thesis of the PhD dissertation

Végh Attila

Consultant: Dr. G. Horváth Ákos
BME, Institute of Mathematics
Department of Geometry

Budapest University of Technology and Economics
Department of Applied Mathematics

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1 Introduction

My dissertation is concerned with applications of root-lattices in different fields of geometry. I do not discuss the historical preliminaries at length. Results of past research are included in the single chapters. Chapter 1 is a short summary of my results and it also shows the relevance of my findings to root-lattices. Chapter 2 deals with the role of lattice E_6 when examining the minimal vectors of 6-dimensional lattices. Chapter 3 investigates lattices Z_n, A_n and D_n for the orthogonal projections of the DV cells of lattices and also examines the DV cell of lattice E_8, which is in connection with the Voronoi-conjecture. Finally, in chapter 4 lattices A_n, D_n , E_7 and E_8 give the maximal thickness of certain $\langle p, q\rangle$ point-systems. In the following we define these special root-lattices. The n-dimensional cube-lattice is

$$Z_n = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{Z}\},$$

where $\mathbb{Z}$ is the set of integer numbers. We can get lattice A_n from the $(n+1)$-dimensional cube-lattice:

$$A_n = \{(x_0, x_1, \dots, x_n) \in Z_{n+1} : x_0 + x_1 + \cdots + x_n = 0\}.$$

Lattice D_n, which is also referred to as checkerboard lattice is defined as

$$D_n = \{(x_1, x_2, \dots, x_n) \in Z_n : x_1 + x_2 + \cdots + x_n \text{ even}\}.$$

And finally, see the definition of lattice E_8, which will have a prominent role in the forthcoming discussion. Lattices E_7 and E_6, based on the definition of E_8, are also given below.

$$E_8 = \{(x_1, x_2, \dots, x_8) : x_i \in \mathbb{Z} \text{ or } x_i \in \mathbb{Z} + \frac{1}{2} \text{ for every } x_i,\ \sum x_i \equiv 0 \pmod 2),$$

$$E_7 = \{(x_1, x_2, \dots, x_8) \in E_8 : x_1 + x_2 + \cdots + x_8 = 0\},$$

$$E_6 = \{(x_1, x_2, \dots, x_8) \in E_8 : x_1 + x_8 = x_2 + \cdots + x_7 = 0\}.$$

2 The minimum vectors of lattices

Let $\mathbb{E}^n\left(\mathbf{0}, \mathrm{V}^n\left(\mathbb{R}, \langle\ ,\ \rangle\right)\right)$ be the Euclidean n-space with a distinguished origin $\mathbf{0}$, with the n-vector space V^n, over the set of real numbers $\mathbb{R}$ and with a positive definite symmetric scalar product $\langle\ ,\ \rangle : \mathrm{V}^n \times \mathrm{V}^n \to \mathbb{R}, (\mathbf{x}, \mathbf{y}) \to \langle \mathbf{x}, \mathbf{y}\rangle$. Let $\mathrm{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} = \{\mathbf{a}_i\}$ be a basis of V^n with the Gramian $G := (a_{ij}) := (\langle \mathbf{a}_i, \mathbf{a}_j\rangle)$. A $\mathbb{Z}$-lattice to the basis A is defined as

$$\Lambda(\mathrm{A}, \mathbb{Z}) = [\mathbf{a}_1, \dots, \mathbf{a}_n] = \left\{\sum_{i=1}^{n} x_i \mathbf{a}_i : x_i \in \mathbb{Z}\right\}.$$

The minimum $m(\Lambda)$ of the lattice Λ is defined by

$$m(\Lambda) \in \mathbb{R}^+ : m(\Lambda) = |\mathbf{m}| \leq |\mathbf{v}| \text{ for an } \mathbf{m} \in \Lambda \setminus \{\mathbf{0}\} =: \dot{\Lambda} \text{ and for any } \mathbf{v} \in \dot{\Lambda}.$$

We may assume (by similarity in $\mathbb{E}^n$) that $m(\Lambda) = 1$. The set of minimum vectors is called the minima of Λ and is denoted by $\mathrm{M}(\Lambda)$. That is

$$\mathrm{M}(\Lambda) := \{\mathbf{m} \in \Lambda : |\mathbf{m}| = m(\Lambda) = 1\}.$$

The maximal A-coordinate of the minima of Λ is defined by

$$L(\mathrm{A}) := \max\left\{x_i \in \mathbb{Z} : \sum_{i=1}^{n} x_i \mathbf{a}_i = \mathbf{m}, \mathbf{m} \in \mathrm{M}(\Lambda)\right\} \in \mathbb{N}.$$

Consider the minimum of these maximal A-coordinates of the minima of Λ when changing the basis A in Λ, i.e. define

$$L(\Lambda) := \min\{L(\mathrm{A}) \in \mathbb{N} : \mathrm{A} \text{ is any basis of } \Lambda\}.$$

Finally, vary the lattices Λ in $\mathbb{E}^n$. Then

$$L_n := L(\mathbb{E}^n) := \max\left\{L(\Lambda) \in \mathbb{N} : \Lambda \text{ is any lattice of } \Lambda \in \mathbb{E}^n\right\}.$$

In general, the problem is the determination of L_n. In other words in any lattice of $\mathbb{E}^n$ find a basis, in which the maximal coordinate of the minima of the lattice is the possible smallest. Á. G. Horváth in [30] proved that $L_n = 1$ for $n \leq 5$ and in [31] that for root-lattices $L(Z_n) = L(A_n) = L(D_n) = L(E_6) = L(E_7) = 1$ and $L(E_8) = 2$ hold.

Lattice Λ is the *extension* of lattice $\bar{\Lambda}$ if $\bar{\Lambda} \subset \Lambda$. This extension is *admissible* if $m(\Lambda) = m(\bar{\Lambda})$, i.e. the minimum does not decrease under the extension. The *index* of the admissible extension is defined by the number $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right) = v\left(\bar{\Lambda}\right)/v\left(\Lambda\right)$, where $v\left(\Lambda\right)$ is the volume of a basic parallelepiped in lattice Λ (see [30], [31], [29]). S.S. Ryshkov [50] and N.V. Zaharova-N.V. Novikova [70] determined all admissible extensions up to $n \leq 8$ in $\mathbb{E}^n$.

By Ryshkov's observation [50] we may assume that any lattice $\Lambda \subset \mathbb{E}^n$ considered has n linearly independent minima of Λ. These minima $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ span a sublattice $\bar{\Lambda} \subset \Lambda$ such that $m(\Lambda) = m(\bar{\Lambda})$, i.e. Λ is an admissible extension of $\bar{\Lambda}$.

2.1 The 6-dimensional lattices

By applying the above result to $n = 6$ we get that the lattice $\Lambda \subset \mathbb{E}^6$ contains 6 independent minimal vectors. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_6\}$ be a basis of the sublattice $\bar{\Lambda} \subset \Lambda$, where $\mathbf{a}_i$ are minimum vectors of length 1. Lattice Λ is an admissible extension of lattice $\bar{\Lambda}$. Consider lattice $\bar{\Lambda}$ for which $\operatorname{ind}\left(\Lambda/\bar{\Lambda}\right)$ is maximal. The Gramian of lattice $\bar{\Lambda}$ is denoted by G.

It follows from the foregoing and [50] that there are three admissible extensions with index 2, one with index 3 and one with index 4 in 6-dimension. The proof of the theorem is divided into three statements according to these indices. In Statement 2.6 we shall investigate the case in which the index of the admissible extension is four. Up to similarity only one lattice has an admissible extension with index four. This follows easily from [50], but we shall give a new proof by estimating the sum of the elements of the Gram matrix. Such estimations will be useful in other cases, too. In this lattice the characteristic matrix, i.e the system of all different minima of the lattice, can be written easily. This matrix will be denoted by $[\mathbf{m}_1, \ldots, \mathbf{m}_\sigma]$ (See [50], [32], [33]) where $\pm\mathbf{m}_1, \ldots, \pm\mathbf{m}_\sigma$ are all different minima of the lattice. Finally, changing the basis of the lattice all elements of the characteristic matrix will be $0, -1, +1$, respectively.

In Statement 2.8 we study lattice extensions with index three and, by estimating the sum of the elements of the Gramian, we get very interesting conditions. In accordance with these conditions, we classify the lattices into two types. In the first case, writing all possible minima of each lattice of this class, we can change the basis of these lattices such that the coordinates of all possible minima of these lattices be $0, -1, +1$. In the second case, we prove that for any lattice Λ of this class $\mathrm{M}(\Lambda) \subseteq \mathrm{M}(E_6)$ holds, where $\mathrm{M}(\Lambda)$ denotes all the minima of Λ. Using theorem 2.3 and the properties of lattice E_6 (see [8],[31]), this case will also be solved. We also remark that in the proof we give a new construction for E_6, namely, as a special extension of a lattice $\bar{\Lambda}$.

In Statement 2.13 we examine admissible extensions with index two. The proof is divided into three parts, according to the three admissible extensions of index two. The basic idea of the proof is similar to that of [33]. In all three cases we write all possible minima of every lattice in E^6 and we prove that, with respect to a suitable basis of Λ, the coordinates of the minima of are $\pm 1, 0$. Summing up, we prove the following theorem:

Theorem 2.5. ([64]) *L_6 is equal to one, i.e., speaking in the above sense, for every Euclidean 6-lattice Λ there is a basis in which the maximal coordinate of all the minimum vectors of Λ is equal to at most 1.*

2.2 The 7-dimensional lattices

Consider a basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ of lattice Λ in an arbitrary orthonormal basis. Compose a matrix A from the coordinates of the basis vectors. Using matrix A we can define positive definite quadratic forms of lattice Λ:

$$Q(\mathbf{x}) = \langle A\mathbf{x}, A\mathbf{x} \rangle = \mathbf{x}^T A^T A \mathbf{x} = \mathbf{x}^T G \mathbf{x}, \mathbf{x} \in \mathbb{Z}^n,$$

where we use the Gramian $G = A^T A$ like above. Since results needed for the proof follow this terminology, we change to positive definite quadratic forms. Out of results concerning quadratic forms I only discuss those which are essential for my theorem. There are further definitions and results in [29], [8], [45]. According to the minima of lattice Λ we can define the homogeneous minimum of the positive definite quadratic form Q:

$$m(Q) := min\{Q(\mathbf{z}) : \text{where} \quad \mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}\}.$$

Let $\mathrm{M}(Q)$ the set of the points $\mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ where $Q(\mathbf{z})$ is minimal. A positive definite quadratic form Q is called *perfect* if it is determined uniquely by minimal vectors. That is, the solutions of the equation $Q(\mathbf{z}) = m(Q)$, $\mathbf{z} \in \mathbb{Z}^n$ are exactly minimal vectors $\mathbf{z}_i \in \mathrm{M}(Q)$.

Perfect forms are known for $n \leq 7$ dimensions. In the euclidean plane J.L. Lagrange [43] dealt with this question and he found one perfect form. In the space there is also one perfect form found by C.F. Gauss [26]. A. Korkine and G. Zolotareff [41], [42] discovered 2 and 3 perfect forms in 4 and 5 dimensions, respectively. E.S. Barnes [3] found 7 different perfect forms in 6 dimension. K.C. Stacey [54] investigated 7-dimensional perfect forms, but he omitted one from 33, which J.H. Conway- N.J.A Sloane [9] completed, finally D.O. Jaquet-Chiffelle [40] proved completeness of these. There is a list of the Gramian of these perfect forms in the appendix A. J. Martinet found more than 10916 perfect forms in 8 dimension. You can find the perfect forms for example in J.H. Conway- N.J.A Sloane's paper [9], and in C. Batut and J. Martinet's works [45], [4].

G.F. Voronoi's theorem in [68] has a fundamental role in determining L_7. According to G.F. Voronoi's statement, for every positive definite quadratic form $Q(\mathbf{x})$ there is a perfect form $Q^*(\mathbf{x})$ such that

$$\mathrm{M}(Q) \subseteq \mathrm{M}(Q^*).$$

Thus it is sufficient to investigate the Gramian of perfect quadratic forms listed in appendix A. The relationship between the Gram matrix and the coordinates of the minimal vectors is the following: if m is the length of the minimal vector $\mathbf{m}$, its coordinates for some basis of the lattice are $x_1, x_2, \ldots, x_n$, $D = det(G)$ and D_i is sub-determinant of the element a_{ii} of the Gramian in the basis of the lattice, then $|x_i| \leq m\sqrt{\frac{D_i}{D}}$. Further, it is well-known from linear algebra that the following transformation: add the c-fold of the ith row of the Gramian G to the jth row and in the new matrix add the c-fold of the ith column to the jth column is a basis transformation of lattice Λ. Using suitable basis transformations, I proved that the coordinates of minimal vectors of the lattice are smaller than 2 in every case. Thus the following theorem holds:

Theorem 2.18. *L_7 is equal to one, i.e. to every Euclidean 7-lattice Λ there is a basis in which the maximal coordinate of all the minimum vectors of Λ is equal to 1 at most.*

3 Dirichlet-Voronoi cells of lattices

The concept of the Dirichlet-Voronoi cell was introduced by Dirichlet [15] and Voronoi [68]. Voronoi polytope and Voronoi cells are also used instead of Dirichlet-Voronoi cell in higher dimensions. We use shortly DV cell in the following. Let us give a discrete point set L in the n-dimensional Euclidean space $\mathbb{E}^n$. The DV cell of a point P_i of the set L is the set of points which are at least as close to point P_i as to any other point P_j of the set L, i.e.

$$\mathrm{DV}(P_i) = \{x \in \mathbb{E}^n : \mathrm{dist}(x, P_i) \leq \mathrm{dist}(x, P_j) \text{ for every j}\}.$$

The DV cell is a special kind of a parallelotope where the parallelotope $\mathcal{P}$ is a convex polytope which fills the space face to face by its translation copies without intersecting by inner points. The centers of the parallelotopes form an n-dimensional lattice. Classical problems of DV cells can be seen for example in [8], [29], [32], [33]. B.A. Venkov [66] and later P. McMullen [47] proved the following important theorems for parallelotopes. A polytope $\mathcal{P}$ is a parallelotope if and only if
(i) $\mathcal{P}$ is centrally symmetric
(ii) each facet of $\mathcal{P}$ is centrally symmetric
(iii) 2-dimensional orthogonal projection along any $(n-2)$-face of $\mathcal{P}$ is either a parallelogram or a centrally symmetric hexagon.
A.D. Aleksandrov in [1] simplified B.A. Venkov's proof. The edges of the parallelogram and the centrally symmetric hexagon of the above property (iii) are the projections of facets of $\mathcal{P}$. These facets form a 2- and 3-belt, respectively. A zone of $(n-2)$-faces is a set of all mutually parallel $(n-2)$-faces of parallelotope $\mathcal{P}$. A zone of $(n-2)$-faces is a closed zone if each $(n-1)$-face of $\mathcal{P}$ has either two or none $(n-2)$-faces of this zone.

B.A. Venkov introduced the concept of a parallelotope of non-zero width in direction of a k-subspace X^k. A parallelotope $\mathcal{P}$ has non-zero width along X^k if the intersection $\mathcal{P} \cap (X^k + \mathbf{a})$ is either k-dimensional or empty for every translation vector $\mathbf{a}$. For $k = 1$ the k-subspace X^k is a line. The direction of this line is given by a vector $\mathbf{z}$ so the z width of a parallelotope $\mathcal{P}$ along $\mathbf{z}$ is the minimal length of the intersections with $\mathcal{P}$ of lines parallel to $\mathbf{z}$. If this minimal length is equal to zero then a parallelotope $\mathcal{P}$ is of zero width in direction $\mathbf{z}$. Denote by $S(z)$ the segment of the direction $\mathbf{z}$ and of the length z. V. Grishukhin in [28] proved that $\mathcal{P}$ has non-zero

width in direction $\mathbf{z}$ if and only if $\mathcal{P}$ has a closed edge zone parallel to $\mathbf{z}$ and $\mathcal{P}$ is the Minkowski sum of a segment $S(z)$ and a parallelotope $\mathcal{P}'$ of zero width in direction $\mathbf{z}$, i.e. $\mathcal{P} = \mathcal{P}' \oplus S(z)$. A d-dimensional *zonotope* in $\mathbb{E}^d$ is the Minkowski sum of n line segments, in other words it is the image of a regular n-cube under some orthogonal projection to the d-dimensional subspace. Thus for any parallelotope exactly one of the following statements holds:
(i) it is a zonotope or
(ii) it is a parallelotope of zero width in any direction or
(iii) it is the Minkowski sum of a zonotope with a parallelotope of zero width in any direction.

The Minkowski sum $\mathcal{P} \oplus S(z)$ is not necessarily a parallelotope. V. Grishukhin [27] gives necessary and sufficient conditions for this sum to be a parallelotope. The following assertions are equivalent for a parallelotope $\mathcal{P}$:
(i) the Minkowski sum $\mathcal{P} \oplus S(z)$ is a parallelotope
(ii) vector $\mathbf{z}$ is orthogonal to at least one facet vector of each 3-belt of $\mathcal{P}$.

3.1 The orthogonal projections of DV cells of lattices

Consider the n-dimensional lattice Λ^n. Intersect lattice Λ^n with a $(n-1)$-dimensional hyperplane H which contains the point P. Denote the resulting set of intersection points $\Lambda^n \cap H$ by Λ^{n-1}, if it is a $(n-1)$-dimensional lattice.

Definition 3.4. If an $(n-1)$-dimensional sublattice Λ^{n-1} of the lattice Λ^n of the filling parallelotope $\mathcal{P}$ has the following property: $\mathbb{R}^n \setminus \bigcup\{\mathcal{P} + \lambda_i \quad \lambda_i \in \Lambda^{n-1}\}$ has two path connected components, then the set $\bigcup\{\mathcal{P} + \lambda_i \quad \lambda_i \in \Lambda^{n-1}\} := [\Lambda^{n-1}(\mathcal{P})]$ is called a parallelotope *lamina*. The lattice Λ^{n-1} is called a *laminar lattice*.

Definition 3.7. If $\mathcal{P}$ is a DV cell, we can associate to all facet F a lattice vector showing to the center of a DV cell $\mathcal{P}_i$, where $F = \mathcal{P} \cap \mathcal{P}_i$. Thus this vector is a relevant vector of the facet F. Further, this vector is orthogonal to the facet F speaking about DV cells, thus this vector $\overrightarrow{PP_i}$ is the normal vector of the facet F. We shall call this special facet vector the *generalized facet vector* of F. Let now G be a $(n-2)$-dimensional face and assume that $G = \mathcal{P} \cap \mathcal{P}_i$ for another DV cell $\mathcal{P}_i$, but $\mathcal{P}$ and $\mathcal{P}_i$ do not have common facet. Such a face determines a 2-belt on the boundary of $\mathcal{P}$. We can also associate to it a lattice vector of Λ^n, which is the sum of the relevant (speaking about DV cells generalized facet) vectors of the facet containing this face (i.e. $\overrightarrow{PP_i} = \overrightarrow{PP_j} + \overrightarrow{PP_k}$). We call this vector the *generalized facet vector* of G.

In the following theorem we give a necessary and sufficient condition that the orthogonal projection of an n-dimensional DV cell of a lattice is an $(n-1)$-dimensional DV cell of the lattice Λ^{n-1}, i.e.

Theorem 3.5. ([65]) *The following statements are equivalent for DV cell* $\mathrm{DV}^n(P)$ *and vector* $\mathbf{z}$*:*

- **(i)** *The orthogonal projection of the cell* $\mathrm{DV}^n(P)$ *to the* $(n-1)$*-dimensional hyperplane* H *along* $\mathbf{z}$ *is an* $(n-1)$*-dimensional DV cell* $\mathrm{DV}^{n-1}(P)$ *of lattice* Λ^{n-1}*, where* $\Lambda^{n-1} = \Lambda^n \cap H$*.*
- **(ii)** *The vector* $\mathbf{z}$ *is orthogonal to at least one generalized facet vector of each 2 and 3-belt.*
- **(iii)** *The* $\mathbb{R}^n \setminus [\Lambda^{n-1}(\mathrm{DV}^n(P))]$ *has two path connected components. (By the definition of the parallelotope lamina* $[\Lambda^{n-1}(\mathrm{DV}^n(P))] = \bigcup\{\mathrm{DV}^n(P) + \lambda_i \quad \lambda_i \in \Lambda^{n-1}\}$*)*

In the dissertation we apply the above theorem for root-lattices Z_n and D_n, thus:

Theorem 3.6. ([65]) *The orthogonal projection of a DV cell of the n-dimensional cube-lattice Z_n in direction $(\pm\mathbf{e}_1 \pm \cdots \pm \mathbf{e}_n)$ is the DV cell of the root-lattice A_{n-1}.*

Theorem 3.8. ([65]) *The orthogonal projection of the DV cell $\mathrm{DV}^n(D_n)$ in direction $\pm\mathbf{e}_i$ is the DV cell $\mathrm{DV}^{n-1}(D_{n-1})$ and in direction $(\pm\mathbf{e}_1, \pm\cdots, \pm\mathbf{e}_n)$ the DV cell $\mathrm{DV}^{n-1}(A_{n-1})$.*

3.2 The DV cell of the lattice E_8

Consider the parallelotopes P and Q and further the relation $<$. $P < Q$ holds if and only if there exists a direction $\mathbf{v}$ for which $P \oplus \lambda\mathbf{v} = Q$, where $\oplus$ denotes Minkowski sum. In this case P is called the contraction of Q and Q is the extraction of P. It is easily seen that the set $\{\text{parallelotope}, <\}$ is partially ordered with maximal and minimal elements.

If the parallelotope $\mathcal{P}$ is of non-zero width in some direction $\mathbf{z}$ then the shadow boundary contains only $(n-1)$-facets which are called facets parallel to $\mathbf{z}$ by B.A. Venkov. As a generalization of this consider the following definitions. The generalized relevant vector can be defined similarly to the generalized facet vector.

Definition 3.8. The $(n-2)$-faces of the shadow boundary which do not belong to any $(n-1)$-face of the shadow boundary can determine a 2- or a 3-belt. If such an $(n-2)$-face belongs to a 2-belt then it is centrally symmetrical and the centre of it is the middle of a lattice vector. This lattice vector with starting point O in Λ^n is called a *generalized relevant vector*. This is the sum or the difference of two relevant vectors of the 2-belt. There is no generalized relevant vector of the $(n-2)$-face in 3-belt.

Definition 3.9. The lattice generated by the real and generalized relevant vectors of the shadow boundary in direction $\mathbf{z}$ is called *Venkov-lattice* of direction $\mathbf{z}$ and is denoted by Λ_z.

We remark that Á. G. Horváth in [36] proved a theorem similar to the one above concerning the orthogonal projection of DV cells for the orthogonal projection of the parallelotope: According to Á. G. Horváth the following statements are equivalent for a parallelotope $\mathcal{P}$:
(i) $\mathcal{P} \oplus S(z)$ is a parallelotope
(ii) the Venkov-lattice Λ_z of direction $\mathbf{z}$ is an $(n-1)$-dimensional lattice and the projection $\mathcal{P}|_{\mathbf{z}}$ is a parallelotope of the hyperplane $[\Lambda_z]$ with lattice Λ_z.

In the following we investigate the connection between the extraction of parallelotopes and the coordinates of relevant vectors and prove that if a parallelotope $\mathcal{P}$ can be extracted in a direction $\mathbf{z}$, then there exist a basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ of the lattice such that $[\Lambda_z] = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{n-1}]$ holds and the nth coordinates of the relevant vectors which do not belong to the Venkov-lattice Λ_z, are ± 1. Namely, if a parallelotope $\mathcal{P}$ can be extracted in a direction $\mathbf{z}$ then the Venkov-lattice is a primitive $(n-1)$-dimensional sublattice of Λ and all other relevant vectors have ± 1 nth coordinates in case of a suitable vector $\mathbf{e}_n$ completing a basis. Further, we prove if there exists a basis in which the coordinates of the relevant vectors of a parallelotope $\mathcal{P}$ are $0, \pm 1$ and the parallelotope $\mathcal{P}$ is an affine image of a DV cell, then there exists a direction $\mathbf{z}$ such that $\mathcal{P} \oplus S(z)$ is a parallelotope, too. Remark that by the theorem 3.13 and in case the above conditions hold the projection of the parallelotope $\mathcal{P}$ in the direction $\mathbf{z}$ is a parallelotope of the Venkov-lattice $\Lambda_{\mathbf{z}}$. By investigating DV cells instead of parallelotopes it can be shown that the projection is a DV cell, i.e. if there exists a basis in which the relevant vectors of the DV cell $\mathcal{D}$ have coordinates $0, \pm 1$, then there exists a direction $\mathbf{z}$ such that the projection of the cell $\mathcal{D}$ in direction $\mathbf{z}$ is the DV cell of the lattice $\Lambda_{\mathbf{z}}$.

In what follows we investigate the DV cells of root-lattices. According to J.H.Conway, N.J.A.Sloane's theorem[8] for any root-lattice Λ the DV cell around the origin is the union of the images of the fundamental simplex under finite reflection group or Weyl group $W(\Lambda)$. The facets of the DV cell are the images of the facets not belonging to the origin of the fundamental simplex under Weyl group $W(\Lambda)$. Consequently, the relevant vectors in a root-lattice are precisely the minimal vectors [8]. So we proved the following theorem:

Theorem 3.15. *For the DV cells $\mathcal{D}$ (and for their affine images $\mathcal{P}$) of root-lattices except lattice E_8 there exists a direction $\mathbf{z}$ such that $\mathcal{D} \oplus S(z)$ ($\mathcal{D} \oplus S(z')$) is a parallelotope.*

We remark that the significance of the theorem is that for the DV cell of lattice E_8 there is no direction in which it can be extracted. An analogue to the above theorem can be easily verify for projection:

Theorem 3.16. *For the DV cells $\mathcal{D}$ of root-lattices except lattice E_8 there exists a direction $\mathbf{z}$ such that the projection of a cell $\mathcal{D}$ in the direction of $\mathbf{z}$ is a DV cell of lattice $\Lambda_{\mathbf{z}}$.*

In following we discuss briefly how my foregoing results are related to the classification of parallelotopes. Two parallelotopes in the plane were well-known already in the antiquity: the centrally symmetric hexagon (primitive) and the parallelogram (not primitive). E.S. Fedorov in [20] described the 5 combinatorically different parallelotopes in 3-dimension among which the truncated octahedron is primitive and the others, namely the elongated octahedron, the rhombic dodecahedron, the hexagonal prism and the cube are not primitive. B.N. Delone [14] found 51 different types of the 4-dimensional parallelotopes. M.I. Shtogrin gave the missing 52nd in [53]. 17 of these are zonotope and the 35 others are the regular 24-cell and the Minkowski sum of this with some zonotope. Three out of these types are primitive. S.S. Ryshkov and E.P. Baranowskii [52] found 221 primitive 5-dimensional parallelotopes. One more was given by P. Engel and V. Grishukhin [18]. P. Engel in [16] and [17] gave 179372 combinatorically different types of 5-dimensional parallelotopes.

We can introduce a partial ordering on the combinatorical types of parallelotopes with the contractions of the closed edge-zones. An element is maximal if it cannot be extracted but trivially, i.e. there is no parallelotope from which it would be contracted. An element is minimal if it cannot be contracted. In 3-dimension the maximal element is the primitive truncated octahedron from which using contractions you can get all other parallelotopes. The minimal element is the cube (see [10] or [39]). In 4-dimension there are 4 maximal elements out of which only 3 are primitive. By contracting these we get 2 minimal elements, which, when extracted can result all other parallelotopes ([7], [60], [28]). It is clear that from the 3 primitive elements you can not get the all parallelotopes only with contractions.

P. Engel obtained the 5-dimensional parallelotopes from primitive parallelotopes first with contractions and then with the extractions of the obtained minimal elements. Unfortunately, in general not all parallelotopes can be obtained this way as according to theorem 3.15 the DV cell of lattice E_8 cannot be extracted in any direction. Further, as it is a parallelotope of zero width in each direction, it cannot be contracted either. *So there exists a non-primitive maximal element, which is also minimal and it cannot be obtained from primitive element.*

We remark that the above statement is closely connected to Voronoi's conjecture. G.F. Voronoi inquires whether each parallelotope is the affine image of a DV cell. He in [68] and [69] proved the conjecture for the case when the parallelotope is primitive. O.K. Zhitomirskii [71] extended the G.F. Voronoi's proof for $(n-2)$-primitive parallelotopes, i.e. when each belt of the parallelotope is a 3-belt. P. McMullen [46] proved the conjecture for zonotopes.

R.M. Erdahl gave another proof in [19].

We state the two theorems below without proof in the dissertation on the one hand because of lack of space on the other hand because the theme is not closed convincingly and other investigations are needed.

Theorem 3.17. *If the parallelotope $\mathcal{P} \oplus S(z)$ is the affine image of a DV cell $\mathcal{D} \oplus S(z')$ then the parallelotope $\mathcal{P}$ is the affine image of a DV cell $\overline{\mathcal{D}}$, too.*

I was able to prove only the reverse of the theorem under certain conditions, so the definition below is needed.

Definition 3.10. Two extractions of parallelotope $\mathcal{P}$ in directions $\mathbf{z}'$ and $\mathbf{z}''$ into parallelotopes $\mathcal{P} \oplus S(z')$, $\mathcal{P} \oplus S(z'')$ are equivalent if the shadow boundaries in directions $\mathbf{z}'$ and $\mathbf{z}''$ are equivalent so $\Lambda_{z'} = \Lambda_{z''}$. *The degree of liberty* $k_{\mathbf{z}}$ of the extraction in direction $\mathbf{z}$ is the dimension of the subspace determined by the directions of the extractions equivalent to $\mathbf{z}$. If the parallelotope cannot be extracted in any direction $\mathbf{z}$ then let $k_{\mathbf{z}} = 0$.

Theorem 3.18. *If an affine transformation L takes parallelotope $\mathcal{P}$ to a DV cell $\mathcal{D}$ and the parallelotope $\mathcal{P} \oplus S(z)$ exists where $k_{\mathbf{z}} = 1$ then the parallelotope $\mathcal{P} \oplus S(z)$ is the affine image of a DV cell $\overline{\mathcal{D}} \oplus S(\overline{z'})$.*

Summing up, parallelotopes, which can be obtained with contraction because of the theorem 3.17 from primitive and $(n-2)$-primitive parallelotopes (by O.K. Zhitomirskii's [71] result) then from these with extraction where $k_{\mathbf{z}} = 1$, because of the theorem 3.18, satisfy Voronoi's conjecture, i.e. they are affine images of DV cells. By the foregoing and V. Grishukhin's theorem 3.3 it is sufficient to prove Voronoi's conjecture only for parallelotopes which cannot be obtained from $(n-2)$-primitive parallelotopes and which are of zero width in each direction and cannot be obtained from these with the sequence of the extractions where $k_{\mathbf{z}} = 1$.

4 $\langle p, q \rangle$ point systems

J.H. Conway and N.J.A. Sloane [8] discuss in detail the role of root-lattices in case of packings and coverings in higher dimensions. It is well-known that in a lot of cases these and their dual lattices give the best results at present for the densest sphere packing and the thinnest sphere covering, respectively. Of course proof is given for the non lattice case only in the plane. We do not discuss the far-reaching results for packings and coverings. We investigate only the $\langle p, q \rangle$ point systems introduced in [37], which are analogous with the above problems. Let $p, q \geq 1$ be integers. The point set Σ is a $\langle p, q \rangle$ point system in a space of constant curvature, if $\exists\, r, R > 0$ such that each arbitrary open ball of radius r contains at most p points of Σ and at least q points of Σ belong to each arbitrary closed ball of radius R.

Definition 4.1. Let us denote by r_p the supremum of r and by R_q the infimum of R of the given $\langle p, q \rangle$ point system, respectively. Quotient $\frac{r_p}{R_q}$ is called the thickness of the $\langle p, q \rangle$ point system.

The problem is to find sup $\frac{r_p}{R_q}$ if the $\langle p, q \rangle$ point system is changed and determine the point set or sets where the maximal thickness is attained. Sup$\frac{r_p}{R_q}$ is denoted by $\kappa(n, p, q)$ where n is the dimension of the space. In other words, we consider point systems which have the maximal thickness and have the following properties: the open balls of radius r whose centres are points of the point system give a p-fold packing and the closed balls of radius R give a q-fold covering.

B.N. Delone [13] introduced the basic problem for (r, R) point systems which is the case of $p = 1,\ q = 1$ according to the above terminology. J. Horváth [37] investigated first this problem for $p, q > 1$. We discuss briefly the results in this field. The problem was solved for the case of $p = 1,\ q = 1$ in the plane by S.S. Ryshkov [51] and L. Fejes Tóth [23] and in the space by K. Böröczky [5]. The maximal thickness of the $\langle 1, 1\rangle$ point system is equal to $0,866...$ in the plane and $0,775...$ in the space. The lattice case was solved by J. Horváth [38] in 4 and 5 dimensions . The maximal thickness of the $\langle 2, 1\rangle$ point system in the plane in the non-necessarily lattice case and of the $\langle 2, 2\rangle$ point system in the plane in the lattice case is known [37]. Further, the problem is solved only for the lattice case and in the plane for all $p, q \leq 5$ and the $\langle 1, 6\rangle$, $\langle 3, 6\rangle$, $\langle 4, 6\rangle$ and $\langle 6, 6\rangle$ point systems [37], [56], [58], [72]. The lattice generated by the regular triangle, namely lattice A_2 gives the maximal thickness of the $\langle p, q\rangle$ point systems in many cases according to the results of J. Horváth [37], Á.H. Temesvári [56] and Á.H. Temesvári, A. Végh [58]. These results gave the starting point for further investigations. In the following at first we examine the problem for $p = 1$ and $q > 1$ in the non-necessarily lattice case and in higher dimensions. The results of these problems are connected to lattices investigated in previous chapters. Then we deal with the problem in the lattice case in the plane.

4.1 The lattice A_n

For a given integer $q > 1$ one would like to know the smallest radius R and find the optimal way to arrange n-dimensional unit spheres so that a) they form a packing and b) if each sphere is replaced with a sphere of radius R then they form a q-fold covering. In order to cover the centres of the original spheres at least twice, the radius R must be ≥ 2. Thus the following lemma holds:

Lemma 4.1. *If $q > 1$ and n is an arbitrary positive integer then $\kappa(n, 1, q) \leq \frac{1}{2}$.*

In the following we investigate lattices for which this simple bound is a sharp bound. Let $B(O, r)$ be the open ball and $\bar{B}(O, r)$ the closed one of radius r with centre O. The parallelepiped P^n is generated by the n-dimensional regular simplex. Let us suppose that the length of the edge of P^n is equal to 1. Consider an n-dimensional oblique-angled coordinate system, let $(0, \dots, 0)^T$ be a vertex of the n-dimensional regular simplex and let $(1, 0, \dots, 0)^T$, $(0, 1, \dots, 0)^T, \dots,$ $(0, 0, \dots, 1)^T$ be the other ones. So we gave a basis of lattice A_n. Then the Euclidean length of the vector $\mathbf{x}$ is $|\mathbf{x}|^2 = \mathbf{x}^T G\mathbf{x}$, where G is the Gram matrix of lattice A_n. The points $(x_1, x_2, \dots, x_n)^T$ $x_i \in \{0, 1\}$ are the vertices of P^n. The $(n-1)$-dimensional facets of P^n are $(n-1)$-dimensional parallelepipeds. Denote by $P^{n-1}_{k,l}$ those faces of P^n whose vertices satisfy the condition $x_k = l$, where $l \in \{0, 1\}$ and $k \in \{1, \dots, n\}$. H^{n-1}_i denotes the hyperplane with normal vector $(1, 1, \dots, 1)^T$ containing those vertices of P^n which have i non-zero coordinates i.e. $x_1+x_2+\dots+x_n = i$. Let $R^n_{i,i+1}$ be the convex hull: $\operatorname{conv}\left(\left(\operatorname{vert}P^n \cap H^{n-1}_i\right) \cup \left(\operatorname{vert}P^n \cap H^{n-1}_{i+1}\right)\right)$ and $R^{n-1}_{j,j+1} = R^n_{i,i+1} \cap \mathcal{P}^{n-1}_{k,l}$ where j depends on i, k, l. Solids $R^{n-1}_{j,j+1}$ are those faces of solid $R^n_{i,i+1}$ which do not lie in any of the hyperplanes H^{n-1}_i. So we proved that if $n \leq 7$ and the facets $R^{n-1}_{j,j+1}$ of the solid $R^n_{i,i+1}$ are q-fold covered by sphere $\bar{B}(X, 1)$, where $X \in \operatorname{vert}(R^{n-1}_{j,j+1})$, then solid $R^n_{i,i+1}$ is also q-fold covered by unit spheres around its vertices. So the following statement holds:

Theorem 4.3. ([63]) *$\kappa(n, 1, q) = \frac{1}{2}$, if $n = 2, 3, 4$ and $q \leq n + 1$ or $n = 5, 6, 7$ and $q \leq 5$ and in these cases lattice A_n gives a maximal thickness.*

4.2 The lattices D_n, E_7, E_8 and other lattices

According to H.S.M. Coxeter' notations [11], denote by α_n the n-dimensional regular simplex, by β_n the n-dimensional cross polytope, by $h\gamma_n$ the solid which is defined as follows: let $h\gamma_n = \operatorname{conv}(\{x \in \{0,1\}^n | \sum x_i \equiv 0\ (2)\})$, i.e. we omit and keep alternately the vertices of the n-dimensional cube and then we take the convex hull of the derived point set. This solid is called a half-cube by H.S.M. Coxeter. The L-partition of lattice D_n has two different types of solids by [8]: $h\gamma_n$ and β_n. We prove that first, in case of $n \geq 3$ the cross polytope β_n is $(n+1)$-fold covered by spheres of radius equal to the edges of the cross polytope around the vertices, second, the solid $h\gamma_5$ is 6-fold covered by spheres of radius equal to its edges around the vertices. So

Theorem 4.6. $\kappa(n,1,q) = \frac{1}{2}$, *if* $n = 3,4,5$ *and* $q \leq n+1$ *or* $n = 6,7,8$ *and* $q \leq 6$ *and in these cases lattice* D_n *gives a maximal thickness.*

We remark that this result surpasses the result which was given for lattice A_n, but it is not needless for two reasons. On the one hand, it is a new point system in general when the thickness is maximal. Of course, I cannot give each point system with maximal thickness (see also the case of the plane). On the other hand, the theorem below is a very important consequence of the application of lattices A_7 and D_8.

Theorem 4.7. $\kappa(7,1,q) = \frac{1}{2}$ *if* $q \leq 10$ *and* $\kappa(8,1,q) = \frac{1}{2}$ *if* $q \leq 12$ *and in these cases lattices* E_7 *and* E_8 *give a maximal thickness.*

I should like to mention that in the case of the above lattices it is not proved whether coverings more than the above manifold coverings are impossible. So it can happen that the investigated lattices give a maximal thickness for other $\langle p,q\rangle$ point systems, too. This holds for the result of the following theorem to an even greater extent.

According to J.H. Conway-N.J.A. Sloane [8] we introduce the concept of laminated lattice and we expand a few of its properties.

Definition 4.3. Let Λ_0 be the one-point lattice. For $n \geq 1$ we take all n-dimensional lattices with minimum 2 that have at least one sublattice Λ_{n-1}. Any such lattice is a *laminated lattice* Λ_n if it has a minimal determinant.

Let $\Lambda_{n-1}^{(i)} = \Lambda_{n-1} + i\mathbf{e}_n$ be the ith layer (or laminae) where $i \in \mathbb{Z}$. Then $\Lambda_n = \bigcup\{\Lambda_{n-1}^{(i)} \,|\, i \in \mathbb{Z}\}$. By [8] in $n \leq 48$ dimensions the laminated lattices are constructed from lattices $\Lambda_{n-1}^{(i)}$ as layers in a way that the orthogonal projection of a lattice point of layer $\Lambda_{n-1}^{(i+1)}$ is the farthest point from the lattice points of lattice $\Lambda_{n-1}^{(i)}$. On the basis of this we prove the following theorem:

Theorem 4.8. $\kappa(n,1,2) = \frac{1}{2}$ *if* $n \leq 48$ *and in these cases the laminated lattice* Λ_n *gives a maximal thickness.*

4.3 Point systems in the plane

In the following we discuss a few results in the plane on the one hand as this was the starting point of higher dimensional cases on the other hand, we intend to show that the determination of the maximal thickness of further lattices and general point systems with maximal thickness is a difficult problem in the plane, too. So we introduce a few new concepts.

A lattice Λ is reduced by Minkowski if basis vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy the following inequalities:

$$|\mathbf{a}| \leq |\mathbf{b}| \leq |\mathbf{b}-\mathbf{a}|,\ (AOB)\angle \leq \frac{\pi}{2}.$$

Introducing the notations $x = \frac{|\mathbf{a}|}{|\mathbf{b}|}$, $\alpha = (AOB)\angle$, $y = \cos\alpha$ the following equivalent inequalities hold:

$$0 < x \le 1,\ 0 \le y \le \frac{x}{2}.$$

We can choose the basis vectors of any lattice so that they satisfy the above conditions. So an ordered pair of the numbers (x, y) corresponds to all lattices reduced by Minkowski and a lattice reduced by Minkowski corresponds to all ordered pair of the numbers $(x, y) \ne (0, 0)$ (up to similarity). Thus there is a one-to-one correspondence up to similarity between lattices reduced by Minkowski and the points different from O of the triangle OPQ where $O(0, 0)$, $P(1, 0)$, $Q(1, \frac{1}{2})$.

Using the methods worked out for the determination of the densest p-fold lattice circlepacking (by Á. H.Temesvári, J. Horváth, N.N. Yakovlev [57]), and for the thinnest q-fold lattice circlecovering (by Á.H.Temesvári [55]) further the theorems on the possible latticecircles in [61], [62] we determined the thickness of the $\langle p, q\rangle$ point systems in the following cases [58]: for $p = 5$ and $q = 1, 2, 3, 4$ by Á. H.Temesvári, for $p = 1, 2, 3, 4$ and $q = 5$ by A. Végh. In the following, using the possibilities of the above general methods, we limit the set of the lattices where the thickness can be maximal for the $\langle 1, q\rangle$ and $\langle 3, q\rangle$ point systems where $q \in \mathbb{Z}^+$. We introduce the concept of the level-line. The lattice-line of the direction of the basis vector $\mathbf{a}$ containing the point O is called *O-level-line*. *j-level-line* is a lattice-line parallel to $\mathbf{a}$ containing the endpoint of the vector $j\mathbf{b}$. Let t_1 be a lattice transformation which does not move the O-level-line and move the point B orthogonally to the O-level-line. If the radius of the circumscribed circle of an arbitrary acute-angle lattice triangle Δ_i is R_i then, using the transformation t_1, $|\mathbf{a}|$ is constant and R_i decreases, so we proved the following theorem:

Theorem 4.12. ([58]) *Consider the point lattice $\langle p, q\rangle$ where $p = 1$ and $q \in \mathbb{Z}^+$. The supremum of the thicknesses of the point lattices may hold only for lattices for which $|\boldsymbol{a}| = |\boldsymbol{b}|$, i.e. $x = 1$.*

In what follows denote by C the endpoint of the vector $\overrightarrow{OA} + \overrightarrow{OB}$. Like above, let t_2 be a transformation for which line $f = OC$ is fixed and which moves point B orthogonally to line f. If the radius of the circumscribed circle of an arbitrary acute-angle lattice triangle Δ_i is R_i then, using the transformation t_2, $|\overline{OC}|$ is a constant and R_i decreases. Under transformation t_2 y increases while x decreases or $x = 1$ so:

Theorem 4.15. ([58]) *Consider the point lattice $\langle p, q\rangle$ where $p = 3$ and $q \in \mathbb{Z}^+$. The supremum of the thicknesses of the point lattices may hold only for lattices for which $y = \frac{x}{2}$, where $x \in \left[\sqrt{\frac{1}{7}}, 1\right]$ or $y = \frac{8x^2-1}{2x}$, where $x \in \left[\sqrt{\frac{1}{8}}, \sqrt{\frac{1}{7}}\right]$.*

We remark that by [37] the result of the point lattice $\langle 1, q\rangle$ cannot be generally improved as in case of the point lattice $\langle 1, 2\rangle$ the thickness is maximal if $x = 1$. In case of a non-necessarily lattice the thickness of other point systems is maximal, too. So it is a difficult problem to give all point systems $\langle 1, 2\rangle$ and $\langle 1, q\rangle$ even in the plane, to say nothing of higher dimensional cases. In these cases it seems to be impossible to give each point system with maximal thickness. In the dissertation we investigated this field only for special root-lattices. We added a short investigation in the plane to show where difficulties arise.

All **Theorem**s in this paper are my own results. The following list contains my own papers in the dissertation: [58], [59], [61], [62], [63], [64], [65]. * denotes the papers referred to the thesis of the dissertation, i.e. [58], [63], [64], [65].

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