Hydrodynamic behavior of hyperbolic two-component systems

PhD thesis

BENEDEK VALKÓ

Institute of Mathematics, Technical University Budapest

Supervisor:
Bálint Tóth

2004

Ezen értekezés bírálatai és a védésről készült jegyzőkönyv a későbbiekben a Budapesti Műszaki és Gazdaságtudományi Egyetem Természettudományi Karának Dékáni Hivatalában elérhető.

Alulírott Valkó Benedek kijelentem, hogy ezt a doktori értekezést magam készítettem, és abban csak a megadott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint, vagy azonos tartalomban, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelöltem.

. . . .
Valkó Benedek

Contents

Chapter 1

Introduction

The main problem of non-equilibrium statistical physics is the study of the dynamics of interacting particle systems, their behavior in space and time. We can think, for example about gas molecules in a room, or particles of a flowing fluid. Generally, the size of these systems is enormous (of order 10^{26}), thus the task of tracking every single particle is hopeless, even if we know everything about the microscopic dynamics. There is another, much more effective approach to this problem: we have to look at the 'big picture', i.e. the macroscopic evolution. This means that we characterize the state of our system with the local densities of certain physically relevant conserved quantities (particle number, momentum, energy) and the time-evolution of these functions, which are usually driven by coupled partial differential equations, gives us the needed description.

Hydrodynamic limit (hdl) is the device to get these pde-systems from the microscopic dynamics via some rescaling of space and time. In the physics literature there are a number of well-known phenomenological derivations of the hydrodynamic limit for several systems, starting with the classical work of Euler, Navier, Stokes, etc. See e.g. [31], [15]. It is a challenging and important program of mathematical physics to give mathematically rigorous versions of these derivations. For completely deterministic systems (e.g. which are governed by Newtonian mechanics) this is still an unsolved problem. However, if we add some stochastic elements to the evolution, the problem becomes more treatable (but far from trivial!). In the last couple of decades considerable advances have been made in the theory of hydrodynamic limits for stochastic systems (see the monographs [32, 13, 6]). Much effort has been made in the analysis of lattice gas models with conserved quantities (e.g. simple exclusion, zero range, Ginzburg-Landau models). These can be viewed as an approximation for the deterministic systems, but they also turn up as models for numerous phenomenona in biology, chemistry and physics (e.g. deposition and growth models, biological chemotaxis).

In this thesis we prove results for the hydrodynamic behavior of certain one dimensional lattice models motivated by a conjecture of B. Tóth and W. Werner [38]. In the upcoming sections we provide a brief overview of the thesis, starting with the description of the conjecture. Next we introduce the considered family of models, sketch our main results and give a short description of the method of the proofs.

1.1 Motivation

In [33] B. Tóth proved limit theorems about the so-called 'true self-avoiding walk' which is a discrete-time random process on $\mathbb{Z}$. The process is a negatively reinforced nearest-neighbor random walk: if the random walker is at a given lattice point then he chooses to go left or right in the next step with probabilities depending on the difference of the local times of the respective neighboring edges, giving more weight to the edge he has visited fewer times. Suppose the walker puts down a unit brick in each step on the edge he has just jumped through, building a wall during the course of its walk. Then the height of this wall at a given edge is equal to the respective value of the local time function and the movement of the walker is governed by the negative gradient of the wall (he rather goes 'downhill' than 'uphill', going through edges he has visited rarer).

In [37] the authors constructed a continuous process as a scaling limit of this discrete-valued process, we can view the process as the continuous-time movement of a particle on $\mathbb{R}$ which is building a wall (its local time) following similar rules as the discrete version. They also proved that the process obeys some dynamical driving mechanism corresponding to these rules. If we denote its position at time t by X_t and its local time (or the height of the wall) at a time t and position x by $h(t,x)$ then the movement of the particle is driven by the slope of the wall:

$$\text{'}dX_t = -\partial_x h(t, X_t)dt\text{'}, \tag{1.1}$$

and the wall is 'built up' by the presence of the particle:

$$\text{'}\partial_t h(t,x) = \delta(X_t - x)\text{'}. \tag{1.2}$$

Of course, these equations do not make sense in this form (hence the inverted commas), but they can be made rigorous. For the details of the construction and primary properties of the process we refer the reader to the original paper. We only remark one interesting and unusual feature: the process X_t has the 2/3 scaling: $(\alpha^{-2/3}X_{\alpha t}, t \geq 0)$ has the same law as $(X_t, t \geq 0)$. In fact $((\alpha^{-2/3}X_{\alpha t}, \alpha^{-1/3}h(\alpha^{2/3}x), \alpha t), t \geq 0, x \in \mathbb{R})$ has the same joint law as $((X_t, h(x,t)), t \geq 0, x \in \mathbb{R})$.

It is natural to consider the case when instead of one particle we have many building the same wall. Corresponding to the discrete case, in [38] a 1 dimensional particle system with 2 conserved quantities was introduced (the 'bricklayer' model): we have several particles (bricklayers) positioned on the lattice sites of $\mathbb{Z}$ who are building a wall from unit-bricks piled above the edges of the lattice. Each bricklayer jumps to a neighboring site with rates depending on the negative gradient of the wall at its position (with the 'downhill' jump getting bigger rate than the 'uphill' jump) and at each jump a unit brick is deposited to the column above the respective edge. This way holes in our walls are tend to be filled quickly by the bricklayers. The two conserved quantities are the particle number and discrete negative gradient of the wall. In the next section we will discuss a whole family of similar models in more detail.

In the continuous setting we would get the following picture: a continuously distributed cloud of particles is building a wall with their movement driven by the slope of the wall. If we denote the density of the particles at x and time t by $\rho(t,x)$ and $u(t,x) := -\partial_x h(t,x)$, then from (1.1), (1.2) and some formal computations we get the following partial differential equation system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t u + \partial_x \rho &= 0. \end{cases} \tag{1.3}$$

As noted in [38], this pde-system can also be viewed as a general phenomenological description of a deposition/domain growth – or, in biological term: chemotaxis – mechanism: $\rho(t,x)$ is the density of population performing the deposition $h(t,x)$ is the height of the deposition and $u(t,x) := -\partial_x h(t,x)$. The physics of the phenomenon is contained in the following two rules:

(a) The velocity field of the population is proportional to the *negative gradient of the height* of the deposition. That is, the population is pushed towards the local decrease of the deposition height. This rule, together with the conservation of total mass of the population leads to the continuity equation, the first equation in (1.3).

(b) The deposition rate is proportional to ρ:

$$\partial_t h \sim \rho, \tag{1.4}$$

i.e. the deposition is done additively by the population. This, after differentiating with respect to x, gives the second equation of (1.3).

In [38] the previously mentioned bricklayer model was derived using formal, non-rigorous hydrodynamic limit and low density perturbational analysis. It was conjectured there that the arguments can be made rigorous and should hold for a large class of models with two conserved quantities. In the paper there were made connections to the Kardar-Parisi-Zhang equation (KPZ equation) which is one of the most famous models in the physical literature for growing interfaces (c.f. [12]). It gives a general phenomenological description of a surface growing to normal direction to its boundary, with a 'tension' that tries to keep the surface together (fills the holes quickly). This resembles properties of our growing wall built by the bricklayers. The KPZ equation itself is (in mathematical sense) an ill-posed non-linear pde with a stochastic term which takes the following form in one dimension:

$$\partial_t h = \nabla h - (\partial_x h)^2 + W \tag{1.5}$$

where $W = W(t,x)$ is a space-time white noise.

Motivated by the KPZ equation we modify rule (b) in the phenomenological description of the deposition/domain growth mechanism of [38] by adding term proportional to $(\partial_x h)^2$ in (1.4):

$$\partial_t h \sim \rho + \gamma(\partial_x h)^2. \tag{1.6}$$

This means that the deposition is not only done solely by the population, but there is also some self-generated deposition (in the spirit of KPZ). Differentiating this with respect to x we get the following pde system instead of (1.3):

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t u + \partial_x(\rho + \gamma u^2) &= 0, \end{cases} \tag{1.7}$$

where γ is a real parameter. This is a system of hyperbolic conservation laws with behavior depending largely on the value of γ. Hyperbolicity means that the Jacobian of the pde has two different real eigenvalues. In sections 3.1 and 3.11 we collect some of the main features of this system. We only want to note one important property which is true for hyperbolic conservation laws in general: there is no global strong solution, arbitrary smooth initial conditions yield shocks in finite time (apart from some very specially prepared initial conditions).
Now we are ready to state the main objective of the thesis:
We want to derive (1.7) as a mathematically rigorous universal low-density hydrodynamic limit for a large class of one dimensional interacting particle systems with two conserved quantities.

In Section 1.3 we describe in detail the result which treats the previous statement and also the other results of the thesis. But first, we introduce the models we will work with.

1.2 Models

We consider one dimensional lattice models and in order to keep the state space finite we work with the discrete tori $\mathbb{T}^n := \mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$. We will denote the continuous torus $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$. We have a finite local spin state denoted by Ω which is the set of the possible observables at a given site. The state space of the interacting particle system is

$$\Omega^n := \Omega^{\mathbb{T}^n},$$

configurations will be denoted by

$$\underline{\omega} := (\omega_j)_{j \in \mathbb{T}^n} \in \Omega^n.$$

The dynamics of our process will be Markovian in continuous time and we only allow elementary jumps (changes of the configuration) which effect two neighboring sites. If at a given time the process is in the configuration $\underline{\omega}$, then at sites $j, j+1$ the spins ω_j, ω_{j+1} can change to ω', ω'' with some rate depending only on $\omega_j, \omega_{j+1}, \omega', \omega''$. Thus the dynamics is governed by translation invariant local rules. We will consider models with two (discrete) conserved quantity which are denoted by

$$\zeta : \Omega \to \mathbb{Z}, \qquad \eta : \Omega \to \mathbb{Z},$$

we also use the notations $\zeta_j = \zeta(\omega_j)$, $\eta_j = \eta(\omega_j)$. We only allow elementary jumps which conserve the sums $\sum_{j\in\mathbb{T}^n} \zeta_j$, $\sum_{j\in\mathbb{T}^n} \eta_j$. This means that if we have an elementary jump which changes (ω_j, ω_{j+1}) to $(\omega'_j, \omega'_{j+1})$ with a positive rate then

$$\begin{aligned} \zeta_j + \zeta_{j+1} &= \zeta'_j + \zeta'_{j+1}, \\ \eta_j + \eta_{j+1} &= \eta'_j + \eta'_{j+1}. \end{aligned}$$

The conserved quantities have to be different: we assume that ζ, η and the constant 1 function on Ω are linearly independent.

It is easy to see, that (possibly shifting η to be always nonnegative) we can always interpret our model locally as a growing or decaying wall made by unit bricks piled on the edges of the lattice and built by bricklayers positioned on lattice sites where η_j is the number of bricklayers at site j and ζ_j is the difference between the height of the columns on $(j-1, j)$ and $(j, j+1)$. See Figure 1.1. Then an elementary move of the process affecting the sites $(j-1, j)$ corresponds to one of the following changes:

– a couple of particles jump from $j-1$ to j, or vice versa,

– a couple of bricks are deposited on or removed of the top of the column standing on the edge $(j-1, j)$,

– a combination of the two previous things.

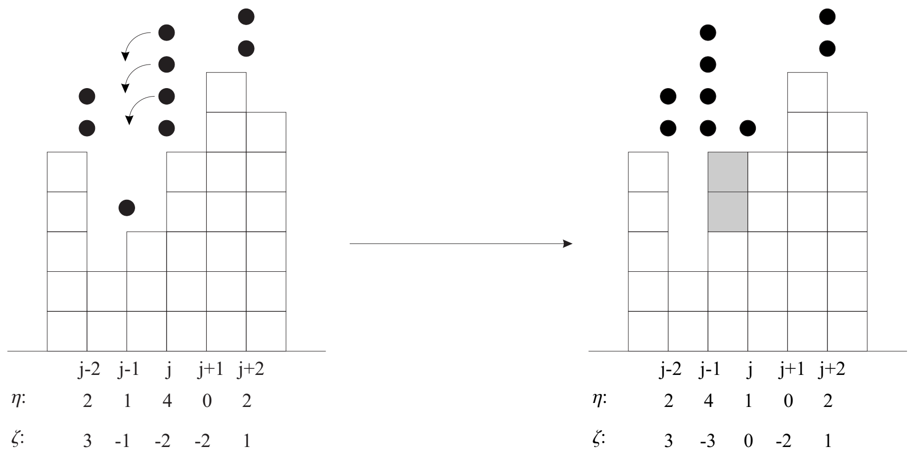


Figure 1.1: The picture shows a possible elementary move in the bricklayer setting. 3 particles jumped from j to $j-1$ and 2 bricks were deposited on the top of the column standing on $(j-1, j)$.

We will also impose some technical, mostly combinatorial conditions on the rate function, these are described in detail in the respective chapters of the thesis. One of the main consequence of these conditions is the existence of a probability measure π on S which puts positive weight on every element of S and for which the measure

$$\pi^n := \prod_{j\in\mathbb{T}^n} \pi$$

is stationary for the process on Ω^n. Actually, we will have a whole family of stationary measures with a similar product structure. Suppose, that $(u, v) \in \mathcal{D} := \mathrm{co}\{(\zeta(\omega), \eta(\omega)) : \omega \in \Omega\}$ where co stands for convex hull. $\mathcal{D}$ is the so-called physical domain. Then there exists a unique probability measure $\pi_{u,v}$ on Ω such that

$$\mathbf{E}_{\pi_{u,v}}\zeta(\omega) = u, \qquad \mathbf{E}_{\pi_{u,v}}\eta(\omega) = v$$

and the measure

$$\pi^n_{u,v} := \prod_{j\in\mathbb{T}^n} \pi_{u,v}$$

is stationary on Ω^n.

With the rate function one can easily define the Markov-generator L^n of the process and because of the nearest neighbor interactions we get that it acts on the conserved quantities as follows:

$$\begin{aligned} L^n\zeta_i &= -\phi(\omega_i, \omega_{i+1}) + \phi(\omega_{i-1}, \omega_i) =: -\phi_i + \phi_{i-1}, \\ L^n\eta_i &= -\psi(\omega_i, \omega_{i+1}) + \psi(\omega_{i-1}, \omega_i) =: -\psi_i + \psi_{i-1}. \end{aligned}$$

The explicitly computable functions ϕ, ψ are called *microscopic fluxes* and their expectations with respect to $\pi^2_{u,v}$ are called *macroscopic fluxes*:

$$\begin{aligned} \Phi(u, v) &:= \mathbf{E}_{\pi^2_{u,v}}\phi(\omega_1, \omega_2), \\ \Psi(u, v) &:= \mathbf{E}_{\pi^2_{u,v}}\psi(\omega_1, \omega_2). \end{aligned}$$

As we will see, the macroscopic flux functions $\Phi(u, v), \Psi(u, v)$ will govern the macroscopic evolution of density-profiles of the conserved quantities. These functions depend on the microscopic model.

There are several detailed concrete examples of microscopic models in Section 2.4.

1.3 Results

Hydrodynamic limit gives the macroscopic behavior of the density-profiles of the conserved quantities via some suitable scaling of space and time. The scaling of space will be the same in all the results of the thesis: we rescale it by n. This means that we represent $\mathbb{T}^n$ with n sites on the continuous torus $\mathbb{T}$, with distance $1/n$ between the neighboring sites.

There are several heuristical derivations which (formally) yield that under Eulerian scaling (which means the rescaling of time and space by n) the macroscopic density-profiles of the conserved quantities ζ, η evolve according to the equation

$$\begin{cases} \partial_t u + \partial_x \Phi(u,v) &= 0, \\ \partial_t v + \partial_x \Psi(u,v) &= 0, \end{cases} \tag{1.8}$$

which is usually a hyperbolic conservation law.

This means the following. Suppose that $u_0(\cdot), v_0(\cdot)$ are real functions on $\mathbb{T}$ with $(u_0(x), v_0(x)) \in \mathcal{D}$ for $x \in \mathbb{T}$. Fix a microscopic model and take its versions for every n on Ω^n with space rescaled by n. Assume that we have initial (random) configurations of our processes such that the densities of ζ, η approximate the functions $u_0(\cdot), v_0(\cdot)$. Then letting the systems run up to time nt the density-profiles of ζ, η will approximate the functions $u(t,\cdot), v(t,\cdot)$ which are the solutions of (1.8) with initial conditions $u(0,x) = u_0(x), v(0,x) = v_0(x)$. The approximation of a deterministic function by the density-profile can be defined in several ways. A natural definition is the following weak approximation: for any smooth test function $g : \mathbb{T} \to \mathbb{R}$

$$\begin{aligned} &\frac{1}{N} \sum_{j \in \mathbb{T}^n} g(j/N) \zeta_j(nt) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x) u(t,x)\, dx, \\ &\frac{1}{N} \sum_{j \in \mathbb{T}^n} g(j/N) \eta_j(nt) \xrightarrow{\mathbf{P}} \int_{\mathbb{T}} g(x) v(t,x)\, dx. \end{aligned} \tag{1.9}$$

Then the (heuristic) result may be summarized the following way: if the previous limits hold for $t = 0$ for any test function g then they will hold at any $t > 0$.

In Chapter 2 we give a rigorous version of this result. There is essentially one robust, model-independent method for proving hydrodynamic limits which works for hyperbolic interacting particle systems with two conserved quantities: H.T. Yau's relative entropy method, introduced in [41]. For attractive one component systems there exist stronger results (c.f. [26]), but these cannot be extended to our case. The relative entropy method does not depend much on the microscopic properties on the model, but this great generality has one drawback: the proof only works for smooth solutions of the limiting pde. (Smooth actually means some finite differentiability conditions.) However, as we already mentioned, hyperbolic conservation laws with generic initial conditions cannot have globally smooth solutions. This means that the relative entropy method can only apply up to a finite time, before the first appearance of shocks. Applying Yau's method we prove that in Eulerian scaling we indeed get the pde (1.8) as a hydrodynamic limit, at least in the regime of smooth solution (see Theorem 1 in Subsection 2.3.2). The proof follows the standard steps of the relative entropy method, but there is one novelty. In order to complete the proof, we need some sort of symmetry relation between the macroscopic fluxes Φ, Ψ, reminiscent of the Onsager reciprocity relations which can be proved using the existence of a stationary measure with product structure. This symmetry relation (see Lemma 1 in Subsection 2.5.2) is an important element of the proof, but also allows us to show some interesting (although not surprising) facts about the pde (1.8), for example that it is indeed (at least weakly) hyperbolic inside the domain $\mathcal{D}$. (Meaning, that the Jacobian can be diagonalized in a real sense.)

Chapter 3 deals with our main objective, the derivation of pde (1.7) as a universal hydrodynamic limit. Using the results of the Eulerian scaling we can give a formal, non-rigorous derivation the pde:
Suppose that $\min_{\omega\in S}\eta(\omega)=0$ (this is a natural assumption if we interpret η as the number of particles at a given site) and that our microscopic model has a left-right reflection-symmetry. This means, that changing the direction of the lattice, our wall evolves with the same dynamics (a more precise definition can be found in Subsection 3.2.1). We note that the pde (1.7) has this reflection-symmetry: if $(\rho(t,x),u(t,x))$ is a solution then so is $(\rho(t,-x),-u(t,-x))$. Under the previous assumptions (and using ρ instead of the variable v) we get the following asymptotics for the macroscopic fluxes:

$$\begin{aligned} \Phi(\rho,u) &= a\,(\rho+\gamma u^2)\left(1+\mathcal{O}(\rho+u^2)\right), \\ \Psi(\rho,u) &= b\,\rho u\left(1+\mathcal{O}(\rho+u^2)\right), \end{aligned} \tag{1.10}$$

if $\rho, |u| \ll 1$. Using these asymptotics the pde (1.7) may be derived by perturbing the constant $(0,0)$ solution of the Eulerian pde

$$\begin{cases} \partial_t u + \partial_x \Phi(\rho,u) = 0, \\ \partial_t \rho + \partial_x \Psi(\rho,u) = 0. \end{cases} \tag{1.11}$$

Let $\rho_0(x)$ and $u_0(x)$ be given profiles and assume that $\rho^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ is a solution of the Eulerian pde (1.11) with initial condition

$$\rho^\varepsilon(0,x)=\varepsilon^2\rho_0(x), \quad u^\varepsilon(0,x)=\varepsilon\, u_0(x).$$

Then, at least formally, if $\varepsilon\to 0$

$$\varepsilon^{-2}\rho^\varepsilon(\varepsilon^{-1}t,x)\to\rho(t,x), \quad \varepsilon^{-1}u^\varepsilon(\varepsilon^{-1}t,x)\to u(t,x),$$

where $\rho(t,x)$, $u(t,x)$ is solution of the pde (1.7) with initial conditions

$$\rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x).$$

Actually, the constants a, b from the asymptotics also appear in the equation, but they can be scaled out by simple linear transformations to get (1.7). It is important to note that the constant γ cannot be scaled out of the equations and that is the only trace left of the microscopic model in the pde-system.

From the perturbational analysis we can guess how we should derive (1.7) as a universal hydrodynamic limit. Fix a microscopic model with the previous assumptions ($\min\eta=0$ and reflection-symmetry), a small constant $\beta > 0$ and suppose that $\rho(t,x)$, $u(t,x)$ is solution of the pde (1.7) with initial conditions

$$\rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x).$$

If at time $t = 0$ the density-profiles of η, ζ approximate the functions $n^{-2\beta}\rho_0(\cdot), n^{-\beta}u_0(\cdot)$ then at microscopic time $n^{1+\beta}t$ they should approximate the functions $n^{-2\beta}\rho(t,\cdot), n^{-\beta}u(t,\cdot)$. Note that this is not Eulerian scaling! In Chapter 3 we give rigorous version of this statement. We impose some additional technical conditions on our microscopic model and since we apply the relative entropy method our result holds only in the regime of smooth solutions. There is another constraint: the proof only works if $\gamma > 1$. Surprisingly (or maybe predictably) the $\gamma = 1$ value is also a turning point in the behavior of the pde-system from the pde point of view (see Subsection 3.1.1). This is a good indication of the fact that the proof is not merely a simple application of the relative entropy method, but also uses nontrivial elements of pde theory. In order to control the fluctuations of some terms with Poissonian (rather than Gaussian) decay coming from the low density approximations we have to apply refined pde estimates, in particular Lax entropies of these pde systems play a key role in the main part of the proof.

The results of Chapter 3 may be interpreted as the description of the hydrodynamic behavior for the perturbation of a *singular* equilibrium point. Indeed, the point $(0,0)$ (around which we considered the perturbation) is not hyperbolic for the Eulerian pde (1.11), the Jacobian is a multiple of the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

However, most of the points in $\mathcal{D}$ are hyperbolic, thus it is a natural question to ask what the behavior of the perturbation will be if we perform it around a 'common' hyperbolic point. Chapter 4 deals with this question. Applying standard perturbation techniques it can be shown (at least formally) that small perturbations of the constant solutions of the Eulerian pde (1.8) at a hyperbolic point (u_0, v_0) will evolve according to two *decoupled* partial differential equations (usually Burgers' equations). We show, that the hydrodynamic picture follows the formal perturbation analysis. Take a microscopic model, and fix a small parameter $\beta > 0$. If the initial density profiles are perturbations of order $n^{-\beta}$ of the constant (u_0, v_0) profile then rescaling time by $n^{1+\beta}$ the evolution of the density profiles are governed by two decoupled equations in the hydrodynamic limit. See Subsection 4.3.2 for more details and a precise formulation. The proof is an application of the relative entropy method (thus it only applies in the regime of smooth solutions) and it also strongly relies on the Onsager-type relations of Lemma 1 of Subsection 2.5.2. The reason for the decoupling of the resulting pde system is the hyperbolicity, basically, the two different eigenvalues (sound speeds) cause the equations to separate. The result is an extension of the results of [28] and [34] were the analogue result was proved for one-component systems.

1.4 Structure of the thesis

The PhD thesis is based on the three papers [35], [36] and [40] which are contained in Chapters 2, 3 and 4 without significant changes. This means that the respective chapters are more or less self-contained, but also that there are some overlaps (mainly in the introductions and the descriptions of the models). The first two papers are joint works with my supervisor Bálint Tóth. [35] has been published, [36] and [40] are submitted for publication. We have another published joint paper connected to this field: in [34] we give the analogue to the main result of [40] for one component systems. Since in [40] we use similar techniques in a more complex setting, we chose not to include the paper [34] in the thesis.

Chapter 2

Onsager Relations and Eulerian Hydrodynamic Limit for Systems with Several Conservation Laws*

We present the derivation of the hydrodynamic limit under Eulerian scaling for a general class of one-dimensional interacting particle systems with two or more conservation laws. Following Yau's relative entropy method it turns out that in case of more than one conservation laws, in order that the system exhibit hydrodynamic behavior, some particular identities reminiscent of Onsager's reciprocity relations must hold. We check validity of these identities whenever a stationary measure with *product structure* exists. It also follows that, as a general rule, *the equilibrium thermodynamic entropy* (as function of the densities of the conserved variables) *is a globally convex Lax entropy* of the hyperbolic systems of conservation laws arising as hydrodynamic limit. The Onsager relations arising in this context and its consequences seem to be novel. As concrete examples we also present a number of models modelling deposition (or domain growth) phenomena. The fact that equilibrium thermodynamic entropy is Lax entropy for the arising Euler equations was noticed earlier in the context of Hamiltonian systems with weak noise, see [21].

2.1 Introduction

We investigate the hydrodynamic behavior of a very general class of one dimensional interacting particle systems with two or more conserved observables. The systems are not reversible and the hydrodynamic limit under Eulerian scaling is investigated. We apply Yau's relative entropy method and obtain validity of the hydrodynamic limit up to the occurrence of the first shock wave in the solution of the limiting pde. There is no novelty in the standard steps of the relative entropy proof, so we only sketch these. The real novelty appears when it turns out that, in case of more than one conserved quantity, in order to complete the relative entropy proof, a class

*This chapter contains the published paper [35].

of identities should hold, relating the macroscopic fluxes appearing in the hydrodynamic pdes. These identities are much reminiscent of Onsager's reciprocity relations. As far as we know these relations have not been pointed out in the context mathematically rigorous Eulerian hydrodynamics. We check the validity of these relations assuming only the existence of a stationary product measure. As a consequence of the Onsager relations it follows that the systems of partial differential equations (systems of conservation laws) arising as hydrodynamic limit are by force of hyperbolic type and the equilibrium thermodynamic entropy of the system (as function of the densities of the conserved quantities) is globally convex Lax entropy of the hydrodynamic equations. This fact may be not so surprising, as it is commonly accepted physical fact. So much so that hyperbolic systems of conservation laws possessing a globally convex Lax entropy are commonly called *of physical type*, see [29]. In the context of Hamiltonian systems perturbed by a weak noise a similar result was established in [21]. Nevertheless, as far as we know, this fact has not been emphasized in the context of mathematically rigorous derivation of hydrodynamic behavior. It is worth noting that given a hyperbolic system of conservation laws the existence of convex Lax entropies is far from trivial: in the case of two component systems the local existence of convex Lax entropies was established in the very technical work [17]. In case of more than two components in general the pdes defining Lax entropies are overdetermined, so in general Lax entropies do not exist at all. It turns out from our result that the hyperbolic systems of conservation laws arising as hydrodynamic limit are of very special type: they always possess a globally convex Lax entropy, namely the equilibrium thermodynamic entropy of the system.

Beside the general framework we also present a number of concrete examples of deposition models with two conserved quantities to which the general result applies, deriving in this way systems of pdes (hyperbolic systems of conservation laws) which describe macroscopically domain growth phenomena in 1+1 dimension.

Our general results are easily extended to more than one dimensions, however the formalism becomes more complicated. We were mostly motivated by the (one dimensional) deposition models presented as concrete examples.

The paper is organized as follows: In section 2.2 we present the general formalism and the conditions under which the hydrodynamic limit is derived. In section 2.3 we state the main results of the paper. In section 2.4 we present a number of concrete examples to which the general framework applies. We hope that the models introduced in this section could be of interest in the context of deposition/domain growth phenomena. In section 2.5 we sketch the proof of the main result formulated in section 2.3. The sketchy proof is broken up into several parts. We only hint at the standard steps of the relative entropy proof, referring the reader to the original work [41] or the monographs [13] or [6]. The essential parts of this section are subsections 2.5.2 and 2.5.3 where the Onsager relations and their consequences are derived. Finally, in section 2.6 we extend the results formulated in the previous sections from two to arbitrary number of conserved quantities.

2.2 Microscopic models

2.2.1 State space, conserved quantities

Throughout this paper we denote by $\mathbb{T}^n$ the discrete tori $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. We will denote the local spin state by Ω, we only consider the case when Ω is finite. The state space of the interacting particle system is

$$\Omega^n := \Omega^{\mathbb{T}^n}.$$

Configurations will be denoted

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^n} \in \Omega^n,$$

For sake of simplicity we consider discrete (integer valued) conserved quantities only. The two conserved quantities are denoted by

$$\zeta : \Omega \to \mathbb{Z},$$
$$\eta : \Omega \to \mathbb{Z},$$

we also use the notations $\zeta_j = \zeta(\omega_j)$, $\eta_j = \eta(\omega_j)$. This means that the sums $\sum_j \zeta_j$ and $\sum_j \eta_j$ are conserved by the dynamics. We assume that the conserved quantities are different and non-trivial, i.e. the functions ζ, η and the constant function 1 on Ω are linearly independent.

2.2.2 Rate function, infinitesimal generator

We consider the *rate function* $r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$. The dynamics of the system consists of elementary jumps effecting nearest neighbor spins, $(\omega_j, \omega_{j+1}) \longrightarrow (\omega'_j, \omega'_{j+1})$, performed with rate $r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$.

We require that the rate function r satisfy the following conditions.

(A) If $r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ then

$$\begin{array}{rcl} \zeta(\omega_1) + \zeta(\omega_2) &=& \zeta(\omega'_1) + \zeta(\omega'_2), \\ \eta(\omega_1) + \eta(\omega_2) &=& \eta(\omega'_1) + \eta(\omega'_2). \end{array} \tag{2.1}$$

(B) For every $Z \in n[\min\zeta, \max\zeta] \cap \mathbb{Z}$, $N \in n[\min\eta, \max\eta] \cap \mathbb{Z}$ the set

$$\Omega^n_{Z,N} := \left\{ \underline{\omega} \in \Omega^n : \sum_{j\in\mathbb{T}^n} \zeta_j = Z, \sum_{j\in\mathbb{T}^n} \eta_j = N \right\}$$

is an irreducible component of Ω^n, i.e. if $\underline{\omega}, \underline{\omega}' \in \Omega^n_{Z,N}$ then there exists a series of elementary jumps with positive rates transforming $\underline{\omega}$ into $\underline{\omega}'$.

(C) There exists a probability measure π on Ω such that it puts positive weight on every element of Ω and for any ω_1, ω_2, ω'_1, $\omega'_2 \in \Omega$

$$Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,$$

where

$$Q(\omega_1, \omega_2) := \sum_{\omega_1', \omega_2' \in \Omega} \left\{ \frac{\pi(\omega_1')\pi(\omega_2')}{\pi(\omega_1)\pi(\omega_2)} r(\omega_1', \omega_2'; \omega_1, \omega_2) - r(\omega_1, \omega_2; \omega_1', \omega_2') \right\}$$

For a precise formulation of the infinitesimal generator on Ω^n we first define the map $\Theta_j^{\omega',\omega''} : \Omega^n \to \Omega^n$ for every $\omega', \omega'' \in \Omega$, $j \in \mathbb{T}^n$:

$$\left(\Theta_j^{\omega',\omega''} \underline{\omega}\right)_i = \begin{cases} \omega' & \text{if} \quad i = j \\ \omega'' & \text{if} \quad i = j+1 \\ \omega_i & \text{if} \quad i \neq j, j+1. \end{cases}$$

The infinitesimal generator of the process defined on Ω^n is

$$L^n f(\underline{\omega}) = \sum_{j \in \mathbb{T}^n} \sum_{\omega', \omega'' \in \Omega} r(\omega_j, \omega_{j+1}; \omega', \omega'')(f(\Theta_j^{\omega',\omega''} \underline{\omega}) - f(\underline{\omega})).$$

We denote by $\mathcal{X}_t^n$ the Markov process on the state space Ω^n with infinitesimal generator L^n.

Remarks:

(1) Condition (A) implies that $\sum_j \zeta_j$ and $\sum_j \eta_j$ are indeed conserved quantities of the dynamics, while condition (B) ensures that there are no other hidden conservation laws.

(2) Condition (B) is somewhat implicit. It seems to us that it is far not trivial (if not impossible) to formulate explicit conditions involving the rate functions which would be necessary and sufficient for irreducibility. However, in the concrete examples treated in section 2.4 one can easily check that irreducibility holds.

(3) On a concrete model condition (C) is quite hard to check. Sometimes it helps to check the following stronger condition:

(C') There exists a probability measure π on Ω such that for any ω_1, ω_2, ω_1', $\omega_2' \in \Omega$

$$\pi(\omega_1)\pi(\omega_2) r(\omega_1, \omega_2; \omega_1', \omega_2') = \pi(\omega_2')\pi(\omega_1') r(\omega_2', \omega_1'; \omega_2, \omega_1).$$

Also, if we denote $R(\omega_1, \omega_2) := \sum_{\omega_1', \omega_2' \in \Omega} r(\omega_1, \omega_2; \omega_1', \omega_2')$ then for any $\omega_1, \omega_2, \omega_3 \in \Omega$

$$R(\omega_1, \omega_2) + R(\omega_2, \omega_3) + R(\omega_3, \omega_1) = R(\omega_1, \omega_3) + R(\omega_3, \omega_2) + R(\omega_2, \omega_1).$$

Of course, this implies condition (C).

(4) Condition (C) implies that the stationary measures of the process $\mathcal{X}_t^n$ are computable and have the structure required for hydrodynamic behavior. Actually, it is equivalent to the property that $\mathcal{X}_t^n$ has a stationary product measure with identical marginals. See the next subsection for details. Another consequence of this condition is Lemma 1 which turns out to be of crucial importance for hydrodynamic behavior.

(4) Conditions (A), (B) and (C) determine the measure $\pi(\omega)$ *up to an exponential distortion*, that is the probability measures satisfying these conditions are of the form (2.2) of the next subsection.

2.2.3 Stationary measures, reversed process

For every $\theta, \tau \in \mathbb{R}$ let $G(\theta, \tau)$ be the moment generating function defined below:

$$G(\theta, \tau) := \log \sum_{\omega \in \Omega} e^{\theta\zeta(\omega)+\tau\eta(\omega)}\pi(\omega).$$

In thermodynamic terms $G(\theta, \tau)$ corresponds to the Gibbs free energy, see [25]. We define the probability measures

$$\pi_{\theta,\tau}(\omega) := \pi(\omega)\exp(\theta\zeta(\omega) + \tau\eta(\omega) - G(\theta, \tau)) \tag{2.2}$$

on Ω.

Using condition (C), by very similar considerations as in [1], [2], [27] or [34] one can show that for any θ and τ the product measure

$$\pi^n_{\theta,\tau} := \prod_{j\in\mathbb{T}^n} \pi_{\theta,\tau}$$

is stationary for the Markov process on X^n_t on Ω^n with infinitesimal generator L^n. We will refer to these measures as the *canonical* measures. Since $\sum_j \zeta_j$ and $\sum_j \eta_j$ are conserved the canonical measures on Ω^n are not ergodic. The conditioned measures defined on $\Omega^n_{Z,N}$ by:

$$\pi^n_{Z,N}(\underline{\omega}) := \pi^n_{\theta,\tau}\left(\underline{\omega}\,\middle|\,\sum_j \zeta_j = Z, \sum_j \eta_j = N\right) = \frac{\pi^n_{\theta,\tau}(\underline{\omega})1\!\!1\{\underline{\omega} \in \Omega^n_{Z,N}\}}{\pi^n_{\theta,\tau}(\Omega^n_{Z,N})}$$

are also stationary and due to condition (B) satisfied by the rate functions they are also ergodic. We shall call these measures the *microcanonical measures* of our system. (It is easy to see that the measure $\pi^n_{Z,N}$ does not depend on the values of θ, τ.)

The elementary movements of the reversed stationary process are $(\omega_{j-1}, \omega_j) \longrightarrow (\omega'_{j-1}, \omega'_j)$ with rate $r(\omega_j, \omega_{j-1}; \omega'_j, \omega'_{j-1})$. The reversed generator is

$$L^{n*}f(\underline{\omega}) = \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega''\in\Omega} r(\omega_j, \omega_{j-1}; \omega'', \omega')(f(\Theta^{\omega',\omega''}_{j-1}\underline{\omega}) - f(\underline{\omega})).$$

This is the adjoint of the operator L^n with respect to all microcanonical (and canonical) measures. I.e. the reversed process is the same for any $\pi^n_{\theta,\tau}$ or $\pi^n_{Z,N}$.

2.2.4 Expectations

Expectation, variance, covariance with respect to the measures $\pi^n_{\theta,\tau}$ will be denoted by $\mathbf{E}_{\theta,\tau}(.)$, $\mathbf{Var}_{\theta,\tau}(.)$, $\mathbf{Cov}_{\theta,\tau}(.)$.

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters θ and τ:

$$\begin{aligned} u(\theta, \tau) &:= \mathbf{E}_{\theta,\tau}(\zeta) = \sum_{\omega\in\Omega} \zeta(\omega)\pi_{\theta,\tau}(\omega) = G'_\theta(\theta, \tau), \\ v(\theta, \tau) &:= \mathbf{E}_{\theta,\tau}(\eta) = \sum_{\omega\in\Omega} \eta(\omega)\pi_{\theta,\tau}(\omega) = G'_\tau(\theta, \tau). \end{aligned}$$

Elementary calculations show, that the matrix-valued function

$$\begin{pmatrix} u'_\theta & u'_\tau \\ v'_\theta & v'_\tau \end{pmatrix} = \begin{pmatrix} G''_{\theta\theta} & G''_{\theta\tau} \\ G''_{\theta\tau} & G''_{\tau\tau} \end{pmatrix} =: G''(\theta,\tau)$$

is equal to the covariance matrix $\mathbf{Cov}_{\theta,\tau}(\zeta,\eta)$ and therefore it is strictly positive definit. It follows that the function $(\theta,\tau)\mapsto(u(\theta,\tau),v(\theta,\tau))$ is invertible. We denote the inverse function by $(u,v)\mapsto(\theta(u,v),\tau(u,v))$. Denote by $(u,v)\mapsto S(u,v)$ the convex conjugate (Legendre transform) of the strictly convex function $(\theta,\tau)\mapsto G(\theta,\tau)$:

$$S(u,v) := \sup_{\theta,\tau}\big(u\theta+v\tau-G(\theta,\tau)\big), \tag{2.3}$$

and

$$\begin{aligned} \mathcal{D} &:= \{(u,v)\in\mathbb{R}_+\times\mathbb{R} : S(u,v)<\infty\} \\ &= \operatorname{co}\{(\zeta(\omega),\eta(\omega)):\omega\in\Omega\}, \end{aligned} \tag{2.4}$$

where co stands for convex hull. In probabilistic terms: $S(u,v)$ is the rate function for joint large deviations of $(\sum_j\zeta_j,\sum_j\eta_j)$. In thermodynamic terms: $S(u,v)$ corresponds to the equilibrium thermodynamic entropy, see [25]. If (u,v) is inside the physical domain $\mathcal{D}$, we have

$$\theta(u,v)=S'_u(u,v), \qquad \tau(u,v)=S'_v(u,v).$$

Let

$$\begin{pmatrix} \theta'_u & \theta'_v \\ \tau'_u & \tau'_v \end{pmatrix} = \begin{pmatrix} S''_{uu} & S''_{uv} \\ S''_{uv} & S''_{vv} \end{pmatrix} =: S''(u,v).$$

It is obvious that the matrices $G''(\theta,\tau)$ and $S''(u,v)$ are strictly positive definit and are inverse of each other:

$$G''(\theta,\tau)S''(u,v)=I, \tag{2.5}$$

where either $(\theta,\tau)=(u(\theta,\tau),v(\theta,\tau))$ or $(u,v)=(\theta(u,v),\tau(u,v))$. With slight abuse of notation we shall denote: $\pi_{\theta(u,v),\tau(u,v)} =: \pi_{u,v}$, $\pi^n_{\theta(u,v),\tau(u,v)} =: \pi^n_{u,v}$, $\mathbf{E}_{\theta(u,v),\tau(u,v)} =: \mathbf{E}_{u,v}$, etc.

We introduce the flux of the conserved quantities. The infinitesimal generator L^n acts on the conserved quantities as follows:

$$\begin{aligned} L^n\zeta_i &= -\phi(\omega_i,\omega_{i+1})+\phi(\omega_{i-1},\omega_i) =: -\phi_i+\phi_{i-1}, \\ L^n\eta_i &= -\psi(\omega_i,\omega_{i+1})+\psi(\omega_{i-1},\omega_i) =: -\psi_i+\psi_{i-1}, \end{aligned}$$

where

$$\begin{aligned} \phi(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2\in\Omega} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\zeta(\omega'_2)-\zeta(\omega_2)) \\ &= \sum_{\omega'_1,\omega'_2\in\Omega} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\zeta(\omega_1)-\zeta(\omega'_1)), \\ \psi(\omega_1,\omega_2) &:= \sum_{\omega'_1,\omega'_2\in\Omega} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\eta(\omega'_2)-\eta(\omega_2)) \\ &= \sum_{\omega'_1,\omega'_2\in\Omega} r(\omega_1,\omega_2;\omega'_1,\omega'_2)(\eta(\omega_1)-\eta(\omega'_1)). \end{aligned} \tag{2.6}$$

We shall denote the expectations of these functions with respect to the canonical measure $\pi^2_{u,v}$ by

$$
\begin{aligned}
\Phi(u,v) &:= \mathbf{E}_{u,v}(\phi) \\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2'\in\Omega}} r(\omega_1,\omega_2;\omega_1',\omega_2')(\zeta(\omega_2')-\zeta(\omega_2))\pi_{u,v}(\omega_1)\pi_{u,v}(\omega_2), \\
\Psi(u,v) &:= \mathbf{E}_{u,v}(\psi) \\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2'\in\Omega}} r(\omega_1,\omega_2;\omega_1',\omega_2')(\eta(\omega_2')-\eta(\omega_2))\pi_{u,v}(\omega_1)\pi_{u,v}(\omega_2).
\end{aligned}
\tag{2.7}
$$

The first derivative matrix of the fluxes Φ and Ψ will be denoted

$$
D(u,v) := \begin{pmatrix} \Phi_u' & \Phi_v' \\ \Psi_u' & \Psi_v' \end{pmatrix} \tag{2.8}
$$

As a general convention, if $\delta : \Omega^m \to \mathbb{R}$ is a local function then its expectation with respect to the canonical measure $\pi^m_{u,v}$ is denoted by

$$
\Delta(u,v) := \mathbf{E}_{u,v}(\delta) = \sum_{\omega_1,\ldots,\omega_m\in\Omega^m} \delta(\omega_1,\ldots,\omega_m)\pi_{u,v}(\omega_1)\cdots\pi_{u,v}(\omega_m).
$$

2.3 The hydrodynamic limit

We will show, applying Yau's relative entropy method, that under Eulerian scaling the local densities of the conserved quantities $u(t,x)$, $v(t,x)$ evolve according to the following system of partial differential equations:

$$
\begin{cases} \partial_t u + \partial_x \Phi(u,v) = 0 \\ \partial_t v + \partial_x \Psi(u,v) = 0. \end{cases} \tag{2.9}
$$

It also turns out from our proof (more precisely as a consequence of the Onsager relations proved in Lemma 1) that the *systems of conservation laws* (2.9) arising as hydrodynamic limits are necessarily *of hyperbolic type* and the equilibrium thermodynamic entropy function $(u,v)\mapsto S(u,v)$ is a (very special) globally *convex Lax entropy* for the system (2.9). (See [29] or [30] for the pde notions used.) This may be not so surprising, as it is commonly accepted fact and drops out automatically, without any computations in some particular model systems investigated so far. Nevertheless, we have not found a general statement or proof of this fact in the hydrodynamic limit literature.

2.3.1 Notations

For the proper formulation of our results we need some more notations. Let $u(t,x), v(t,x), t\in[0,T], x\in\mathbb{T}$ be a smooth solution of (2.9) (more precisely: let it be twice continuously differentiable in both variables), with $(u(0,x),v(0,x))\in \mathrm{int}(\mathcal{D})$ for $x\in\mathbb{T}$. ($\mathrm{int}(\mathcal{D})$ denotes the inside of the domain $\mathcal{D}$.) We shall use the notations

$$\begin{aligned}\theta(t,x) &:= \theta(u(t,x),v(t,x))\\ \tau(t,x) &:= \tau(u(t,x),v(t,x)).\end{aligned}$$

The *true distribution* of the Markov process $\mathcal{X}_s^n$ at macroscopic time t, i.e., at microscopic time nt is

$$\mu_t^n := \mu_0^n \exp\{ntL^n\}. \tag{2.10}$$

The true distribution will be compared to the following *time dependent reference measure* (also called local equilibrium) on Ω^n:

$$\nu_t^n := \prod_{j\in\mathbb{T}^n} \pi_{u(t,\frac{j}{n}),v(t,\frac{j}{n})}. \tag{2.11}$$

This measure is not stationary (unless u and v are constant), and the local densities of the conserved quantities are discrete approximations of the functions $u(t,x), v(t,x)$.

We shall use a stationary measure $\pi^n := \pi^n_{0,0}$ on Ω^n as an *absolute reference measure*. The Radom-Nikodym derivatives of the true distribution and the time dependent reference measure with respect to the absolute referencee measure are denoted as follows:

$$\begin{aligned}h_t^n &:= \frac{d\mu_t^n}{d\pi^n}(\underline{\omega}) = \exp\{ntL^{n*}\}h_0^n.\\ f_t^n &:= \frac{d\nu_t^n}{d\pi^n}(\underline{\omega})\\ &= \prod_{j\in\mathbb{T}^n}\exp\{\zeta(\omega_j)\theta(t,\frac{j}{n})+\eta(\omega_j)\tau(t,\frac{j}{n})-G(\theta(t,\frac{j}{n}),\tau(t,\frac{j}{n}))\}\end{aligned} \tag{2.12}$$

2.3.2 The main result

Our aim is to prove that if μ_0^n is close to ν_0^n in the sense of relative entropy, then μ_t^n stays close to ν_t^n in the same sense uniformly for $t \in [0,T]$. If we consider two different pairs of smooth solutions $(u_i(t,x), v_i(t,x))$, $i = 1,2$ of (2.9) it is a simple exercise to show that the relative entropy of the two time dependent reference measures is of order $\asymp n$. This suggests that one should prove

$$H^n(t) := H(\mu_t^n|\nu_t^n) = o(n), \tag{2.13}$$

uniformly for $t \in [0,T]$.

Theorem 1. *Consider an interacting particles system model defined as in the previous section which satisfies conditions (A), (B) and (C). Let $\Phi(u,v)$ and $\Psi(u,v)$ be defined as in (2.7).*
(i) The system of conservation laws (2.9) is hyperbolic in the domain $(u,v) \in \mathrm{int}(\mathcal{D})$. Furthermore, the equilibrium thermodynamic entropy $(u,v) \mapsto S(u,v)$ is a globally convex Lax entropy for the system (2.9).

(ii) Let $[0,T]\times\mathbb{T}\ni(t,x)\mapsto(u(t,x),v(t,x))$ *be a smooth solution of (2.9) with* $(u(0,x),v(0,x))\in\text{int}(\mathcal{D})$ *for* $x\in\mathbb{T}$. *Let* μ_t^n *and* ν_t^n *be the measures defined in (2.10), respectively, (2.11). Then, if (2.13) holds for* $t=0$ *then it will hold uniformly for* $t\in[0,T]$.

Remark: Part (i) of the Theorem is a commonly accepted fact. In some particular models investigated it simply drops out without any computation. However, we do not know about any explicit formulation (or proof) of the *general fact* stated here.

From part (ii) of the Theorem, by applying the entropy inequality in a standard way (comparing the true measure μ_t^n with the local equilibrium reference measure ν_t^n) one gets the following corollary:

Corollary 1. *Under the conditions of the Theorem, for any* $t\in[0,T]$, *the following limits hold as* $n\to\infty$:
(i) For any smooth test function $g:\mathbb{T}\to\mathbb{R}$

$$\frac{1}{n}\sum_{j\in\mathbb{T}^n}g(j/n)\zeta_j(t)\xrightarrow{\mathbf{P}}\int_{\mathbb{T}}g(x)u(t,x)\,dx,$$
$$\frac{1}{n}\sum_{j\in\mathbb{T}^n}g(j/n)\eta_j(t)\xrightarrow{\mathbf{P}}\int_{\mathbb{T}}g(x)v(t,x)\,dx.$$

(ii) The asymptotics of the relative entropy of the true distribution μ_t^n *with respect to the absolute reference measure* $\pi^n_{u_0,v_0}$ *is*

$$n^{-1}H(\mu_t^n\,|\,\pi^n_{u_0,v_0})\to\int_{\mathbb{T}}\big(S(u(t,x),v(t,x))-S(u_0,v_0)\big)\,dx, \tag{2.14}$$

where $S(u,v)$ *is the thermodynamic entropy defined in (2.3).*

Remarks.
(1) Note that since $S(u,v)$ is Lax entropy of the pde (2.9) the right hand side of (2.14) does not change in time as long as the solution $(u(t,x),v(t,x))$ of (2.9) is smooth, and starts to decrease when the first shock appears. This means that the relative entropy $H(\mu_t^n\,|\,\pi^n_{u_0,v_0})$ decreases by $o(n)$ before the appearance of the first shock in the system.
(2) A result formally similar to (2.14) *for one component systems* (in particular simple exclusion process) was established in [14] for all times - even after the appearance of discontinuities in the solution of the hydrodynamic pde. In this paper it is also proved that for the simple exclusion the relative entropy with respect to the local equilibrium measure conditioned on the number of particles remains $o(n)$ even after appearance of shocks.

2.4 Examples: deposition models

If we fix the size of the spin space, then we have finitely many equations from the conditions on the rate function, thus we can get a finite-parameter family of models. The smallest value of

$|\Omega|$, for which there exists a proper model is 3, since we need to have two different non-trivial conserved quantities. We present two concrete examples: one with $|\Omega| = 3$, one with $|\Omega| = 4$. A third example with $|\Omega| = \infty$, to which Theorem 1 applies with some modifications, is described in [38].

Our concrete examples are *deposition models.* $\eta : \Omega \to \mathbb{N}$, and $\zeta : \Omega \to \mathbb{Z}$. η_j, respectively, ζ_j are interpreted as particle occupation number, respectively, (negative) discrete gradient of deposition height. The dynamical driving mechanism is such that
(i) The deposition height growth is influenced by the local particle density. Typically: growth is enhanced by higher particle densities.
(ii) The particle motion is itself influenced by the deposition profile. Typically: particles are pushed in the direction of the negative gradient of the deposition height.

It is natural to assume left-right symmetry of the models. This is realized in the following way. There is an involution

$$R : \Omega \to \Omega, \qquad R \circ R = Id$$

which acts on the conserved quantities and the jump rates as follows:

$$\begin{aligned} \eta(R\omega) = \eta(\omega), \qquad \zeta(R\omega) = -\zeta(\omega), \\ r(R\omega_2, R\omega_1; R\omega_2', R\omega_1') = r(\omega_1, \omega_2; \omega_1', \omega_2'). \end{aligned} \tag{2.15}$$

Both models will also satisfy the stronger condition (C').

Correspondingly on the macroscopic level we shall use the traditional notation $\rho(t, x)$ (instead of $v(t, x)$ of the general formulation) and $u(t, x)$. The limiting partial differential equations will be also invariant under the left-right reflection symmetry:

$$(\rho(t, x), u(t, x)) \mapsto (\rho(t, -x), -u(t, -x)) \, .$$

2.4.1 A model with $|\Omega| = 3$

The state space is $\Omega = \{-1, 0, 1\}$. The left-right reflection symmetry is implemented by $R : \Omega \to \Omega$, $R\omega = -\omega$. The two conserved quantities are $\zeta(\omega) = \omega$ (the spin itself) and $\eta(\omega) = 1 - |\omega|$ (the number of zeros). (It is easy to see that up to linear combinations these are the only two conserved quantities we can define on Ω.) From condition (A) it follows that $r(\omega_1, \omega_2; \omega_1', \omega_2') > 0$ only if $\omega_1' = \omega_2$ and $\omega_2' = \omega_1$. I.e. the dynamics consists of exchanges of nearest neighbour spins. It follows that, without any restriction on the rates, the first part of condition (C') is satisfied with any probability measure π on Ω. Our natural parametrization is

$$\pi_{\rho,u}(0) = \rho, \quad \pi_{\rho,u}(\pm 1) = \frac{1 - \rho \pm u}{2},$$

with the parameter range $\{(\rho, u) : \rho \in [0, 1],\ u \in [-1, 1],\ \rho + |u| \leq 1\}$.

The second part of condition (C') is fulfilled if and only if

$$\begin{aligned} &r(1, -1; -1, 1) - r(-1, 1; 1, -1) \\ &\qquad = r(1, 0; 0, 1) - r(0, 1; 1, 0) + r(0, -1; -1, 0) - r(-1, 0; 0, -1) \end{aligned}$$

holds. The reflection symmetry condition (2.15) reads

$$r(1,0;0,1) = r(0,-1;-1,0), \qquad r(0,1;1,0) = r(-1,0;0,-1).$$

These conditions leave us with

$$\begin{aligned} r(1,-1;-1,1) &= a, & r(-1,1;1,-1) &= 2c+a, \\ r(0,-1;-1,0) &= b, & r(-1,0;0,-1) &= c+b, \\ r(1,0,0,1) &= b, & r(0,1,1,0) &= c+b, \end{aligned}$$

where $a, b \geq 0$ and $c \geq \max\{-b, -a/2\}$ are free parameters. Without loss of generality we may choose $c \geq 0$ (otherwise, rename $\tilde{\omega} := -\omega$). It is easy to check that condition (B) is satisfied if and only if $(a+2c)(b+c) > 0$. We are not interested in the $c = 0$ case, since that defines the reversible process which would imply diffusive rather than hyperbolic (Eulerian) scaling. By fixing an appropriate time scale we choose $c = 1$. It is easy to compute the microscopic fluxes ϕ_j and ψ_j given by formula (2.6):

$$\begin{aligned} \phi_j &= \frac{1}{2}(\omega_j - \omega_{j+1})\left((\omega_j - 1)(1+\omega_{j+1}) - 2a\omega_j\omega_{j+1} + 2b\,(1+\omega_j\omega_{j+1})\right), \\ \psi_j &= b(\omega_{j+1}^2 - \omega_j^2) + \frac{1}{2}(1-\omega_j)(1+\omega_{j+1})(\omega_j + \omega_{j+1}) \end{aligned}$$

The macroscopic fluxes are computed with formula (2.7. Inserted in (2.9) this leads to the hydrodynamic equation:

$$\left\{ \begin{array}{lcl} \partial_t \rho + \partial_x(\rho u) &=& 0 \\ \partial_t u + \partial_x(\rho + u^2) &=& 0. \end{array} \right. \tag{2.16}$$

This system is known in the pde community as *Leroux's equation.* The system has some very special features: it belongs to the so-called Temple class and it was much investigated. For details see [29]. Validity of this pde in the hydrodynamic limit *up to the occurrence of shocks* follows from our general Theorem.

Remark: It is an easy exercise to see that all models with $|\Omega| = 3$ satisfying the general conditions (A, B, C), without the extra assumption of left-right reflection symmetry, are essentially equivalent, in the sense that in the hydrodynamic limit they lead to pde-s which can be transformed to (2.16) by linear combinations of the functions involved.

2.4.2 A finite bricklayer model

In the following example we give a finite version of the infinite bricklayers model introduced in [38]. Let $\Omega = \{0,1\} \times \{-1,1\}$. The elements of Ω will be denoted $\omega =: (\eta, \zeta)$. Left-right reflection symmetry is implemented as $R : \Omega \to \Omega$, $R(\eta, \zeta) = (\eta, -\zeta)$. The conserved quantities are $\zeta(\omega) = \zeta$ and $\eta(\omega) = \eta$. Condition (A) leaves twenty (possibly) non-zero rates. Due to the left-right reflection symmetry conditions eight pairs of rates are equal. Using the notation

$$r(\omega_1, \omega_2; \omega_1', \omega_2') = r\left({}^{\eta_1}_{\zeta_1}, {}^{\eta_2}_{\zeta_2}; {}^{\eta_1'}_{\zeta_1'}, {}^{\eta_2'}_{\zeta_2'},\right)$$

in the following table we list the (possibly) non-zero rates, parameterized by twelve nonnegative parameters.

$$
\begin{array}{ll}
r\left(\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{0}{+};\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{0}{-}\right)=a, & r\left(\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{0}{-};\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{0}{+}\right)=b,\\
r\left(\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{1}{+};\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{1}{-}\right)=c, & r\left(\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{1}{-};\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{1}{+}\right)=d,\\
r\left(\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{+};\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{-}\right)=e, & r\left(\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{+};\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{-}\right)=e,\\
r\left(\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{-};\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{+}\right)=f, & r\left(\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{-};\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{+}\right)=f,\\
r\left(\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{-};\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{-}\right)=p, & r\left(\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{+};\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{+}\right)=p,\\
r\left(\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{+};\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{+}\right)=q, & r\left(\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{-};\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{-}\right)=q,\\
r\left(\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{-};\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{-}\right)=r, & r\left(\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{-};\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{-}\right)=r,\\
r\left(\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{+};\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{+}\right)=s, & r\left(\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{+};\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{+}\right)=s,\\
r\left(\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{+};\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{-}\right)=x, & r\left(\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{+};\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{-}\right)=x,\\
r\left(\genfrac{}{}{0pt}{1}{0}{+},\genfrac{}{}{0pt}{1}{1}{-};\genfrac{}{}{0pt}{1}{1}{-},\genfrac{}{}{0pt}{1}{0}{+}\right)=y, & r\left(\genfrac{}{}{0pt}{1}{1}{+},\genfrac{}{}{0pt}{1}{0}{-};\genfrac{}{}{0pt}{1}{0}{-},\genfrac{}{}{0pt}{1}{1}{+}\right)=y.
\end{array}
$$

All the other jump rates are zero.

It is easy to check that the first part of condition (C') imposes

$$r=s \tag{2.17}$$

and no other restriction. It also follows that the measures $\pi_{\rho,u}$ are of the product form

$$\pi_{\rho,u}(\eta,\zeta)=\big(\eta\rho+(1-\eta)(1-\rho)\big)\frac{1+\zeta u}{2},\qquad \eta=0,1,\quad \zeta=+,-, \tag{2.18}$$

with the parameters $\rho\in(0,1)$, $u\in(-1,+1)$.

Another straightforward computation shows that the second part of condition (C') reads

$$\begin{array}{c} c+f+p+y=d+e+q+x\\ a+f+q+y=b+e+p+x \end{array} \tag{2.19}$$

So, we are left with a nine-parameter family of models. Given the formulas (2.6) we compute the fluxes ϕ_j and ψ_j. Using the conditions (2.17) and (2.19) eventually we get

$$
\begin{aligned}
2\phi_j \;=\;& (b-a)+(p-q)(\eta_j+\eta_{j+1})-(b+a)(\zeta_{j+1}-\zeta_j)\\
&+(a+b-e-f-x-y)(\eta_j+\eta_{j+1})(\zeta_{j+1}-\zeta_j)+(a-b)\zeta_{j+1}\zeta_j\\
&-(a+b+c+d-2e-2f-2x-2y)\eta_j\eta_{j+1}(\zeta_{j+1}-\zeta_j)\\
&-(p-q)(\eta_j+\eta_{j+1})\zeta_{j+1}\zeta_j\\
4\psi_j \;=\;& -(p+q+r+s+x+y)(\eta_{j+1}-\eta_j)\\
&+(p-q)(\eta_j+\eta_{j+1})(\zeta_j+\zeta_{j+1})+(y-x)(\eta_{j+1}-\eta_j)(\zeta_{j+1}-\zeta_j)\\
&-2(p-q)\eta_j\eta_{j+1}(\zeta_j+\zeta_{j+1})\\
&-(p+q-r-s-x-y)(\eta_{j+1}-\eta_j)\zeta_{j+1}\zeta_j
\end{aligned}
$$

The macroscopic fluxes are again explicitly computable. From (2.7) and (2.18) we get

$$\begin{aligned}\Phi(\rho,u) &= \big((p-q)\rho-(a-b)/2\big)\big(1-u^2\big),\\ \Psi(\rho,u) &= (p-q)\rho(1-\rho)u.\end{aligned}$$

Without loss of generality we may assume $p-q \geq 0$ (otherwise rename the microscopic variables $\tilde{\eta}_j := \eta_j$, $\tilde{\zeta}_j := -\zeta_j$). Further on, $p = q$ leads to diffusive rather then hyperbolic (Eulerian) scaling of the particle density, so we are interested in the $p > q$ cases only. By setting the appropriate time scale we can choose $p - q = 1$ and denote $\gamma := \frac{a-b}{2(p-q)}$. So, eventually we get the system of pdes

$$\left\{\begin{array}{l}\partial_t\rho+\partial_x\big(\rho(1-\rho)u\big)=0\\ \partial_t u+\partial_x\big((\rho-\gamma)(1-u^2)\big)=0\end{array}\right. \tag{2.20}$$

In [23] another family of four-state models with two conserved quantities, the so-called *two channel traffic models* are analyzed. These models also satisfy conditions (A), (B) and (C). As a consequence our general Theorem is applicable to the two channel traffic models, too.

About the relation of our four state deposition models (treated in this subsection) and the two channel traffic models treated in [23]: Due to the different symmetry conditions imposed — we impose the left-right reflection symmetry described in the first paragraph of this section, while in [23] symmetry between the two traffic channels is imposed — the two families of models do not intersect. The one parameter family of partial differential equations derived in [23] essentially differs form our partial differential equations (2.20). (Actually there is no parameter value for which the two pde-s are equivalent.) However, the two families of models show many similarities and do have common generalizations.

2.5 Sketch of proof

The present section is divided into four subsections. In subsection 2.5.1 we present the first steps of the 'relative entropy method' applied. As there is no real novelty in this part, we only list the main steps *without the computational details* which are essentially the same as in the original work [41] of Yau or in Chapter 6 of [13] or in [6].

It turns out that for a general two (or more) component system some identity relating the macroscopic fluxes Φ and Ψ is essentially needed for completing the proof. These relations are reminiscent of Onsager's reciprocity relations of nonequilibrium thermodynamics, see e.g. Chapter 10.D of [25]. However an essential difference is worth noting: while the traditional Onsager relations are derived under the condition of reversibility of the microscopic dynamics, in our case conditions (A) and (C) are involved which do not imply reversibility by any means. Seemingly, these relations were not explicitly noted so far in the context of mathematically rigorous hydrodynamic limits. This omission is probably explained by the fact that in the concrete models investigated so far these identities just dropped out without any computations.

In subsection 2.5.2 we prove that under the conditions (A) and (C) these identities hold in general. We shall refer to these identities as *Onsager relations*. It also follows from these identities that the systems of partial differential equations arising as hydrodynamic limits under Eulerian scaling are indeed of hyperbolic type and the thermodynamic equilibrium entropy $S(u, v)$ is globally convex Lax entropy of the hydrodynamic equations, as it is commonly assumed. In subsection 2.5.3 we formulate the consequences of the Onsager relations which are crucial for the further steps of the proof of the hydrodynamic limit.

Finally, in subsection 2.5.4 we sketch the last steps of the proof. Here again we follow the standard steps of the relative entropy method, so we omit all computational details, referring only to the main stations of the proof. For the computational details of subsections 2.5.1 and 2.5.4 we refer the reader to Chapter 6. of [13] or to [6]. However, we warn the reader that the omitted details (in particular the last two steps: the control of the block replacement and the one-block estimate) are rather sophisticated and mathematically deep.

2.5.1 First transformations

In order to obtain the main estimate (2.13) our aim is to get a Grömwall type inequality: we will prove that for every $t \in [0, T]$

$$H^n(t) - H^n(0) \leq C \int_0^t H^n(s) ds + o(n), \tag{2.21}$$

where the error term is uniform in $t \in [0, T]$. Because it is assumed that $H^n(0) = o(n)$, the Theorem follows.

For proving (2.21) we try to bound (from above) $\partial_t H^n(s)$ by $\text{const} \cdot H^n(s) + o(n)$, uniformly for $s \in [0, T]$. We start form the inequality (2.22) which is derived in [13] under very general conditions, valid in our case.

$$\partial_t H^n(t) \leq n \int_{\Omega^n} \frac{L^{n*} f_t^n}{f_t^n} d\mu_t^n - \int_{\Omega^n} \frac{\partial_t f_t^n}{f_t^n} d\mu_t^n. \tag{2.22}$$

Next we transform the two terms appearing on the right hand side of (2.22). Equation (2.23) follows form the smoothness of the functions $\theta(t, x)$ and $\tau(t, x)$ and from the entropy inequality applied to the measures μ_t^n compared with the absolute reference measure π^n.

Since Ω is finite and π puts positive weight on every element it is easy to show that $H(\mu_0^n | \pi^n) < Cn$ with an absolute constant C. The measure π^n is stationary, thus we have the same bound for $H(\mu_t^n | \pi^n)$ uniformly for $0 \leq t \leq T$.*

$$\begin{aligned} n \int_{\Omega^n} \frac{L^{n*} f_t^n}{f_t^n} d\mu_t^n \quad = \quad & - \sum_{j \in \mathbb{T}^n} \partial_x \theta(t, j/n) \int_{\Omega^n} \phi_j d\mu_t^n \\ & - \sum_{j \in \mathbb{T}^n} \partial_x \tau(t, j/n) \int_{\Omega^n} \psi_j d\mu_t^n \\ & + \mathcal{O}(1) \end{aligned} \tag{2.23}$$

*In the original paper [34] $H(\mu_0^n | \pi^n) < Cn$ was assumed in the theorem, but as we can see, this is unneeded.

Equation (2.24) follows from direct computation of the time derivative of the function f_t^n.

$$\int_{\Omega^n} \frac{\partial_t f_t^n}{f_t^n} d\mu_t^n = \sum_{j\in\mathbb{T}^n} \partial_t\theta(t,j/n) \int_{\Omega^n} (\zeta_j - u(t,j/n)) d\mu_t^n + \sum_{j\in\mathbb{T}^n} \partial_t\tau(t,j/n) \int_{\Omega^n} (\eta_j - v(t,j/n)) d\mu_t^n \tag{2.24}$$

Next we replace the local variables ϕ_j and ψ_j in (2.23), respectively, ζ_j and η_j in (2.24) by their block averages defined as follows: if δ_j is a local microscopic variable its block average is defined as

$$\delta_j^l := \frac{\delta_j + \cdots + \delta_{j+l-1}}{l}.$$

In the following two block-replacements we use again the smoothness of the functions $\theta(t,x)$ and $\tau(t,x)$ and the entropy inequality applied to the measures μ_t^n compared with the absolute reference measure π^n.

$$n\int_{\Omega^n} \frac{L^{n*} f_t^n}{f_t^n} d\mu_t^n = -\sum_{j\in\mathbb{T}^n} \partial_x\theta(t,j/n) \int_{\Omega^n} \phi_j^l d\mu_t^n - \sum_{j\in\mathbb{T}^n} \partial_x\tau(t,j/n) \int_{\Omega^n} \psi_j^l d\mu_t^n + \mathcal{O}(l) \tag{2.25}$$

$$\int_{\Omega^n} \frac{\partial_t f_t^n}{f_t^n} d\mu_t^n = \sum_{j\in\mathbb{T}^n} \partial_t\theta(t,j/n) \int_{\Omega^n} (\zeta_j^l - u(t,j/n)) d\mu_t^n + \sum_{j\in\mathbb{T}^n} \partial_t\tau(t,j/n) \int_{\Omega^n} (\eta_j^l - v(t,j/n)) d\mu_t^n + \mathcal{O}(l) \tag{2.26}$$

The last transformation of this first, preparatory part is replacing in (2.25) the block averages ϕ_j^l, respectively, ψ_j^l by their equilibrium averages computed at the empirical densities, $\Phi(\zeta_j^l, \eta_j^l)$, respectively, $\Psi(\zeta_j^l, \eta_j^l)$. The error terms appearing in the third and fourth lines of the right hand side of (2.27) are the most important error terms to be controlled by the so called *one block estimate* towards the end of the proof.

$$\begin{aligned} n\int_{\Omega^n} \frac{L^{n*} f_t^n}{f_t^n} d\mu_t^n = & -\sum_{j\in\mathbb{T}^n} \partial_x\theta(t,j/n) \int_{\Omega^n} \Phi(\zeta_j^l, \eta_j^l) d\mu_t^n \\ & -\sum_{j\in\mathbb{T}^n} \partial_x\tau(t,j/n) \int_{\Omega^n} \Psi(\zeta_j^l, \eta_j^l) d\mu_t^n \\ & -\sum_{j\in\mathbb{T}^n} \partial_x\theta(t,j/n) \int_{\Omega^n} \left(\phi_j^l - \Phi(\zeta_j^l, \eta_j^l)\right) d\mu_t^n \\ & -\sum_{j\in\mathbb{T}^n} \partial_x\tau(t,j/n) \int_{\Omega^n} \left(\psi_j^l - \Psi(\zeta_j^l, \eta_j^l)\right) d\mu_t^n \\ & +\mathcal{O}(l) \end{aligned} \tag{2.27}$$

Before going on with the standard steps of the relative entropy proof we need to make a detour.

2.5.2 An Onsager type identity

Lemma 1. *Suppose we have a particle system with two conserved quantities and rates satisfying conditions (A) and (C). Then there exists a potential function* $(\theta, \tau) \mapsto U(\theta, \tau)$ *such that*

$$\begin{aligned} \Phi(\theta, \tau) &:= \Phi(u(\theta, \tau), v(\theta, \tau)) = U'_\theta, \\ \Psi(\theta, \tau) &:= \Psi(u(\theta, \tau), v(\theta, \tau)) = U'_\tau, \end{aligned} \tag{2.28}$$

or, equivalently

$$\Phi'_\tau = \Psi'_\theta. \tag{2.29}$$

Proof. We prove (2.29). Throughout the forthcoming proof we adopt the notation $\zeta_j := \zeta(\omega_j)$, $\zeta'_j := \zeta(\omega'_j)$, etc.

From the definitions

$$\pi_{\theta,\tau}(\omega_1)\pi_{\theta,\tau}(\omega_2) = \exp\{\theta(\zeta_1 + \zeta_2) + \tau(\eta_1 + \eta_2) - 2G(\theta, \tau)\}\pi(\omega_1)\pi(\omega_2),$$

and

$$\begin{aligned} &\left(\pi_{\theta,\tau}(\omega_1)\pi_{\theta,\tau}(\omega_2)\right)'_\theta = \\ &\quad \pi(\omega_1)\pi(\omega_2)e^{\theta(\zeta_1+\zeta_2)+\tau(\eta_1+\eta_2)-2G(\theta,\tau)}\left\{(\zeta_1 + \zeta_2) - 2u(\theta, \tau)\right\} = \\ &\quad \frac{\pi(\omega_1)\pi(\omega_2)}{Z(\theta, \tau)^3} \sum_{\omega_3 \in \Omega} \pi(\omega_3)(\zeta_1 + \zeta_2 - 2\zeta_3)e^{\theta(\zeta_1+\zeta_2+\zeta_3)+\tau(\eta_1+\eta_2+\eta_3)}, \end{aligned}$$

where $Z(\theta, \tau) = \exp\{G(\theta, \tau)\}$. Similarly,

$$\begin{aligned} &\left(\pi_{\theta,\tau}(\omega_1)\pi_{\theta,\tau}(\omega_2)\right)'_\tau = \\ &\quad \frac{\pi(\omega_1)\pi(\omega_2)}{Z(\theta, \tau)^3} \sum_{\omega_3 \in \Omega} \pi(\omega_3)(\eta_1 + \eta_2 - 2\eta_3)e^{\theta(\zeta_1+\zeta_2+\zeta_3)+\tau(\eta_1+\eta_2+\eta_3)}. \end{aligned}$$

Hence

$$\begin{aligned} \Phi'_\tau(\theta, \tau) &= \frac{1}{Z(\theta, \tau)^3} \sum_{\omega_1,\omega_2,\omega_3 \in \Omega} \pi(\omega_1)\pi(\omega_2)\pi(\omega_3)e^{\theta(\zeta_1+\zeta_2+\zeta_3)+\tau(\eta_1+\eta_2+\eta_3)} \\ &\qquad \times (\eta_1 + \eta_2 - 2\eta_3) \sum_{\omega'_1,\omega'_2 \in \Omega} r(\omega_1, \omega_2, \omega'_1, \omega'_2)(\zeta'_2 - \zeta_2), \\ \Psi'_\theta(\theta, \tau) &= \frac{1}{Z(\theta, \tau)^3} \sum_{\omega_1,\omega_2,\omega_3 \in \Omega} \pi(\omega_1)\pi(\omega_2)\pi(\omega_3)e^{\theta(\zeta_1+\zeta_2+\zeta_3)+\tau(\eta_1+\eta_2+\eta_3)} \\ &\qquad \times (\zeta_1 + \zeta_2 - 2\zeta_3) \sum_{\omega'_1,\omega'_2 \in \Omega} r(\omega_1, \omega_2, \omega'_1, \omega'_2)(\eta'_2 - \eta_2), \end{aligned}$$

For the proof of the lemma it is enough to prove for any $Z \in [3\min\zeta, 3\max\zeta]$ and $N \in [3\min\eta, 3\max\eta]$ the following expression equals to 0:

$$\sum_{\substack{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2'\in\Omega:\\ \zeta_1+\zeta_2+\zeta'=Z\\ \eta_1+\eta_2+\eta'=N}} \pi(\omega_1)\pi(\omega_2)\pi(\omega_3)r(\omega_1,\omega_2;\omega_1',\omega_2') \times \big((\eta_1+\eta_2-2\eta_3)(\zeta_2'-\zeta_2)-(\zeta_1+\zeta_2-2\zeta_3)(\eta_2'-\eta_2)\big) \tag{2.30}$$

From condition (A) imposed on the rate functions it follows that in all nonzero terms of the above sum one can replace $\eta_1+\eta_2$ by $\eta_1'+\eta_2'$ and $\eta_2'-\eta_2$ by $\eta_1-\eta_1'$, and similarly for the ζ-s. After straightforward computations (2.30) becomes

$$\sum_{\substack{\omega_1,\omega_2,\omega_3,\omega_1',\omega_2'\in\Omega:\\ \zeta_1+\zeta_2+\zeta'=Z\\ \eta_1+\eta_2+\eta'=N}} \pi(\omega_1)\pi(\omega_2)\pi(\omega_3)r(\omega_1,\omega_2;\omega_1',\omega_2') \times \big(\Delta(\omega_1,\omega_2,\omega_3)-\Delta(\omega_1',\omega_2',\omega_3)\big) \tag{2.31}$$

where $\Delta : \Omega\times\Omega\times\Omega \to Z$ is defined as follows

$$\Delta(\omega_1,\omega_2,\omega_3) := \zeta_1(\eta_2-\eta_3)+\zeta_2(\eta_3-\eta_1)+\zeta_3(\eta_1-\eta_2).$$

Note that Δ is *antisymmetric* regarding permutation of its variables. Next, after rearranging the sum, from the definition of the function Q in condition (C) expression (2.31) becomes

$$-\sum_{\substack{\omega_1,\omega_2,\omega_3\in\Omega:\\ \zeta_1+\zeta_2+\zeta'=Z\\ \eta_1+\eta_2+\eta'=N}} \pi(\omega_3)\pi(\omega_1)\pi(\omega_2)Q(\omega_1,\omega_2)\Delta(\omega_1,\omega_2,\omega_3).$$

Finally, from the antisymmetry of the function Δ and condition (C) imposed on the function Q it follows indeed that this last sum equals zero. □

2.5.3 Consequences of the Onsager relations

Relation (2.29) is the same as saying that the matrix $D(u(\theta,\tau),v(\theta,\tau))\cdot G''(\theta,\tau)$ is symmetric. Using (2.5) this also reads as

$$S''(u,v)\cdot D(u,v) = \big(S''(u,v)\cdot D(u,v)\big)^{\dagger}. \tag{2.32}$$

This relation implies that only *hyperbolic* two-by-two systems of conservation laws (2.9) can arise as hydrodynamic limits. Indeed, as the following elementary argument shows relation (2.32) can hold with a positive definite matrix S'' only if $D(u,v)$ can be diagonalized (in the real sense), which is exactly the condition of hyperbolicity of the system (2.9). Indeed, since S'' is positive definite, we can write

$$D = (S'')^{-1/2}\left((S'')^{-1/2}(S''D)(S'')^{-1/2}\right)(S'')^{1/2}, \tag{2.33}$$

which means that D is similar to the real symmetric matrix $(S'')^{-1/2}(S''D)(S'')^{-1/2}$, and from this the (real) diagonalizability of D follows. Furthermore, (2.32) is spelled out as

$$S''_{uu}\Phi'_v + S''_{uv}\Psi'_v = S''_{vu}\Phi'_u + S''_{vv}\Psi'_u, \tag{2.34}$$

which is readily recognized as the partial differential equation defining the *Lax entropies* of the system (2.9). The function

$$F(u,v) := \Phi(u,v)\theta(u,v) + \Psi(u,v)\tau(u,v) - U(\theta(u,v),\tau(u,v))$$

is the corresponding (macroscopic) entropy-flux. See [29] or [30] for the pde notions involved. Thus, part (i) of the Theorem is proved.

Now we turn to two further consequences of Lemma 1 which turn out to be of crucial importance in the hydrodynamic behavior.

First, the time derivatives of θ and τ are expressed. From the pde (2.9) it follows that

$$\partial_t\theta = -\theta'_u\Phi'_u\partial_x u - \theta'_u\Phi'_v\partial_x v - \theta'_v\Psi'_u\partial_x u - \theta'_v\Psi'_v\partial_x v.$$

Using the identity (2.34) we replace

$$\theta'_u\Phi'_v = \theta'_v\Phi'_u + \tau'_v\Psi'_u - \tau'_u\Psi'_v$$

in the second term of the right hand side. Using also the straightforward identities $u'_\tau = v'_\theta$ and $\theta'_v = \tau'_u$ (see subsection 2.2.4) we finally get

$$\partial_t\theta = \Phi'_u\partial_x\theta + \Psi'_u\partial_x\tau, \tag{2.35}$$

and by identical considerations

$$\partial_t\tau = \Phi'_v\partial_x\theta + \Psi'_v\partial_x\tau. \tag{2.36}$$

Second, due to identity (2.28),

$$\sum_{j\in\mathbb{T}^n}\Big(\partial_x\theta(j/n)\Phi(u(j/n),v(j/n)) + \partial_x\tau(j/n)\Psi(u(j/n),v(j/n))\Big)$$
$$= \sum_{j\in\mathbb{T}^n}\partial_x U(u(j/n),v(j/n)) = \mathcal{O}(1). \tag{2.37}$$

2.5.4 End of proof

Now we return to proving (2.21). Denote

$$\mathcal{D}\Phi(u,v;\tilde u,\tilde v) := \Phi(\tilde u,\tilde v) - \Phi(u,v) - \Phi'_u(u,v)(\tilde u - u) - \Phi'_v(u,v)(\tilde v - v)$$

and similarly for $\mathcal{D}\Psi(u,v;\tilde{u},\tilde{v})$. Applying (2.35), (2.36) and (2.37), from (2.26) and (2.27) we obtain

$$\begin{aligned}\int_{\Omega^n} \frac{\partial_t f_t^n - nL^{n*} f_t^n}{f_t^n} d\mu_t^n = & \\ & \sum_{j\in\mathbb{T}^n} \partial_x\theta(t,j/n) \int_{\Omega^n} \mathcal{D}\Phi(u(t,j/n),v(t,j/n);\zeta_j^l,\eta_j^l) d\mu_t^n \\ & + \sum_{j\in\mathbb{T}^n} \partial_x\tau(t,j/n) \int_{\Omega^n} \mathcal{D}\Psi(u(t,j/n),v(t,j/n);\zeta_j^l,\eta_j^l) d\mu_t^n \\ & + \sum_{j\in\mathbb{T}^n} \partial_x\theta(t,j/n) \int_{\Omega^n} \left(\phi_j^l - \Phi(\zeta_j^l,\eta_j^l)\right) d\mu_t^n \\ & + \sum_{j\in\mathbb{T}^n} \partial_x\tau(t,j/n) \int_{\Omega^n} \left(\psi_j^l - \Psi(\zeta_j^l,\eta_j^l)\right) d\mu_t^n \\ & + \mathcal{O}(l)\end{aligned} \tag{2.38}$$

The first two terms on the right hand side of (2.38) are estimated by the entropy inequality, comparing the measure μ_t^n with the *local equilibrium measure* ν_t^n:

$$\begin{aligned}\sum_{j\in\mathbb{T}^n} \int_{\Omega^n} \Big(& \left|\mathcal{D}\Phi(u(t,j/n),v(t,j/n);\zeta_j^l,\eta_j^l)\right| \\ & + \left|\mathcal{D}\Psi(u(t,j/n),v(t,j/n);\zeta_j^l,\eta_j^l)\right|\Big) d\mu_t^n \\ & \qquad \le CH^n(t) + \mathcal{O}(nl^{-1}).\end{aligned} \tag{2.39}$$

The last two terms in (2.38) are estimated only *integrated against time*. Applying the so-called one block estimate (see e.g. Chapter 5 of [13]) one gets

$$\begin{aligned}&\lim_{l\to\infty}\lim_{n\to\infty} n^{-1} \sum_{j\in\mathbb{T}^n} \int_0^t ds \int_{\Omega^n} \left|\phi_j^l - \Phi(\zeta_j^l,\eta_j^l)\right| d\mu_t^n = 0, \\ &\lim_{l\to\infty}\lim_{n\to\infty} n^{-1} \sum_{j\in\mathbb{T}^n} \int_0^t ds \int_{\Omega^n} \left|\psi_j^l - \Psi(\zeta_j^l,\eta_j^l)\right| d\mu_t^n = 0.\end{aligned} \tag{2.40}$$

This is the only part of the proof where condition (B) is used, which ensures ergodicity of the Markov process $\mathcal{X}_t^n$ on the 'hyperplanes' $\Omega_{Z,N}^n$.

Finally, inserting (2.39) and (2.40) in (2.38), via (2.22) we obtain (2.21) and thus the part (ii) of the Theorem is also proved.

2.6 Particle systems with several conserved variables

As noted in the introduction, the results described in the previous sections are also valid for particle systems with more than 2 conserved quantities.

Before we formulate the general results we have to summarize some notations . Let $n \geq 2$ be fixed integer, and $\boldsymbol{\zeta} = (\zeta^1, \zeta^2, \ldots, \zeta^n) : \Omega \to \mathbb{R}^n$ the vector of conserved quantities. Throughout the present section bold face symbols will denote n-vectors.

We require the rate function to satisfy similar conditions as listed in subsection 2.2.2 (in place of conditions (A) and (B) we need the suitable generalizations). For every $\boldsymbol{\theta} \in \mathbb{R}^n$ we can define momentum generating function $G(\boldsymbol{\theta})$ as

$$G(\boldsymbol{\theta}) := \log \sum_{\omega \in \Omega} e^{\boldsymbol{\theta} \cdot \boldsymbol{\zeta}(\omega)} \pi(\omega),$$

and the probability measures

$$\begin{aligned} \pi_{\boldsymbol{\theta}}(\omega) &:= \pi(\omega) \exp(\boldsymbol{\theta} \cdot \boldsymbol{\zeta}(\omega) - G(\boldsymbol{\theta})) \\ \pi_{\boldsymbol{\theta}}^n &:= \prod_{j \in \mathbb{T}^n} \pi_{\boldsymbol{\theta}} \end{aligned}$$

on Ω, respectively, on Ω^n. We define the expectation of the conserved quantities with respect to the measure $\pi_{\boldsymbol{\theta}}^n$:

$$\boldsymbol{u}(\boldsymbol{\theta}) := \mathbf{E}_{\boldsymbol{\theta}}(\boldsymbol{\zeta}) = \nabla_{\boldsymbol{\theta}} G(\boldsymbol{\theta}).$$

One can easily show, that $\nabla_{\boldsymbol{\theta}}^2 G(\boldsymbol{\theta}) = \mathbf{Cov}_{\boldsymbol{\theta}}(\boldsymbol{u}, \boldsymbol{u})$ is positive definite. As a consequence, the function $\boldsymbol{\theta} \mapsto \boldsymbol{u}$ is invertible, and

$$\boldsymbol{\theta}(\boldsymbol{u}) = \nabla_{\boldsymbol{u}} S(\boldsymbol{u}),$$

where $S(\boldsymbol{u})$ is the convex conjugate of $G(\boldsymbol{\theta})$:

$$S(\boldsymbol{u}) := \sup_{\boldsymbol{u} \in \mathbb{R}^n} (\boldsymbol{u} \cdot \boldsymbol{\theta} - G(\boldsymbol{\theta})).$$

We introduce the flux of the vector of the conserved quantities and its expectation:

$$\begin{aligned} \boldsymbol{\phi}(\omega_1, \omega_2) &:= \sum_{\omega_1', \omega_2' \in \Omega} r(\omega_1, \omega_2; \omega_1', \omega_2')(\boldsymbol{\zeta}(\omega_2') - \boldsymbol{\zeta}(\omega_2)) \\ \boldsymbol{\Phi}(\boldsymbol{u}) &:= \mathbf{E}_{\boldsymbol{\theta}(\boldsymbol{u})} \boldsymbol{\phi} \\ &= \sum_{\substack{\omega_1, \omega_2, \\ \omega_1', \omega_2' \in \Omega}} r(\omega_1, \omega_2; \omega_1', \omega_2')(\boldsymbol{\zeta}(\omega_2') - \boldsymbol{\zeta}(\omega_2)) \pi_{\boldsymbol{u}}(\omega_1) \pi_{\boldsymbol{u}}(\omega_2). \end{aligned}$$

Now we are able to formulate results of the previous sections in the more general setting.

Using the arguments presented in section 2.5 one can show that under Eulerian scaling the vector of the local densities of the conserved quantities $\boldsymbol{u}(t, x)$ evolve according to the following n-component partial differential equation:

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{\Phi}(\boldsymbol{u}) = 0. \tag{2.41}$$

Lemma 1 applies for any two conserved quantities ζ^i, ζ^j $(i \neq j)$, thus if we denote the derivative matrix of the flux vector $\boldsymbol{\Phi}(\boldsymbol{u})$ by $D(\boldsymbol{u}) := \nabla_{\boldsymbol{u}} \boldsymbol{\Phi}(\boldsymbol{u})$ and the second derivative matrix of the thermodynamic entropy $S''(\boldsymbol{u}) := \nabla_{\boldsymbol{u}}^2 S(\boldsymbol{u})$, we get

$$S''(\boldsymbol{u}) \cdot D(\boldsymbol{u}) = \big(S''(\boldsymbol{u}) \cdot D(\boldsymbol{u})\big)^{\dagger} \tag{2.42}$$

Since $S''(\boldsymbol{u})$ is positive definite (2.33) implies that $D(\boldsymbol{u})$ can be diagonalized which means that the arising system of partial differential equations is *hyperbolic*. Moreover, (2.42) spelled out is

$$\frac{\partial^2 S}{\partial u_i \partial u_i} \frac{\Phi_i}{\partial u_j} + \frac{\partial^2 S}{\partial u_i \partial u_j} \frac{\Phi_j}{\partial u_j} = \frac{\partial^2 S}{\partial u_j \partial u_i} \frac{\Phi_i}{\partial u_i} + \frac{\partial^2 S}{\partial u_j \partial u_j} \frac{\Phi_j}{\partial u_i}, \tag{2.43}$$

with $1 \leq i < j \leq n$. These are exactly the $n(n-1)/2$ equations defining the Lax entropies of the hyperbolic system (2.41). It is well-known that in the case of $n \geq 3$ only very special n-component hyperbolic conservation laws possess Lax entropies. In general, the defining equations (2.43) are overdetermined. In [29] these particular systems of hyperbolic conservation laws are called of *physical* type. From the previous arguments it follows that only physical hyperbolic equations can arise as the hydrodynamic limit of an interacting particle system satisfying our conditions.

Acknowledgement: The authors wish to thank József Fritz for the many illuminating discussions on the topics of hydrodynamic limits and for pointing out Corollary (ii). We also thank János Kertész for guiding comments on Onsager's relations.

Chapter 3

Perturbation of singular equilibria of hyperbolic two-component systems: a universal hydrodynamic limit*

We consider one-dimensional, locally finite interacting particle systems with two conservation laws which under Eulerian hydrodynamic limit lead to two-by-two systems of conservation laws:

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases}$$

with $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$, where $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$. We consider the case of isolated singular (i.e. non hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0) = (0, 0)$ and assume $\mathcal{D} \subset \{\rho \geq 0\}$. (This can be achieved by a linear transformation of the conserved quantities.) We investigate the propagation of *small nonequilibrium perturbations* of the steady state of the microscopic interacting particle system, corresponding to the densities (ρ_0, u_0) of the conserved quantities. We prove that for a very rich class of systems, under proper hydrodynamic limit the propagation of these small perturbations are *universally* driven by the two-by-two system

$$\begin{cases} \partial_t \rho + \partial_x \big(\rho u\big) = 0 \\ \partial_t u + \partial_x \big(\rho + \gamma u^2\big) = 0 \end{cases}$$

where the parameter $\gamma := \frac{1}{2}\Phi_{uu}(\rho_0, u_0)$ (with a proper choice of space and time scale) is the only trace of the microscopic structure. The proof is valid for the cases with $\gamma > 1$.

The proof relies on the relative entropy method and thus, it is valid only in the regime of smooth solutions of the pde. But there are essentially new elements: in order to control the fluctuations of the terms with Poissonian (rather than Gaussian) decay coming from the low density approximations we have to apply refined pde estimates. In particular Lax entropies of these pde systems play a *not merely technical* key role in the main part of the proof.

*This chapter contains the submitted paper [36].

3.1 Introduction

3.1.1 The PDE to be derived and some facts about it

We consider the pde

$$\begin{cases} \partial_t \rho + \partial_x \big(\rho u\big) = 0 \\ \partial_t u + \partial_x \big(\rho + \gamma u^2\big) = 0 \end{cases} \tag{3.1}$$

where $\rho = \rho(t,x) \in [0,\infty)$, $u = u(t,x) \in (-\infty,\infty)$ are density, respectively, velocity field and $\gamma \in \mathbb{R}$ is a fixed parameter. For any fixed γ this is a *hyperbolic system of conservation laws* in the domain $(\rho, u) \in \mathbb{R}_+ \times \mathbb{R}$.

Phenomenologically, the pde describes a deposition/domain growth – or, in biological term: chemotaxis – mechanism: $\rho(t,x)$ is the density of population performing the deposition and $h(t,x)$ is the height of the deposition. Let

$$u(t,x) := -\partial_x h(t,x).$$

The physics of the phenomenon is contained in the following two rules:

(a) The velocity field of the population is proportional to the *negative gradient of the height* of the deposition. That is, the population is pushed towards the local decrease of the deposition height. This rule, together with the conservation of total mass of the population leads to the continuity equation (the first equation in our system).

(b) The deposition rate is

$$\partial_t h = \rho + \gamma\big(\partial_x h\big)^2.$$

The first term on the right hand side is just saying that deposition is done additively by the population. The second term is a self-generating deposition, introduced and phenomenologically motivated by Kardar-Parisi-Zhang [12] and commonly accepted in the literature. Differentiating this last equation with respect to the space variable x results in the second equation of our system.

The pde (3.1) is invariant under the scaling:

$$\widetilde{\rho}(t,x) := A^{2\beta}\rho(A^{1+\beta}t, Ax), \qquad \widetilde{u}(t,x) := A^{\beta}u(A^{1+\beta}t, Ax),$$

where $A > 0$ and $\beta \in \mathbb{R}$ are arbitrarily fixed. The choice $\beta = 0$ gives the straightforward hyperbolic scale invariance, valid for any system of conservation laws. More interesting is the $\beta = 1/2$ case. This is the natural scale invariance of the system, since the physical variables (density and velocity fields) change *covariantly* under this scaling. This is the (presumed, but never rigorously proved) asymptotic scale invariance of the Kardar-Parisi-Zhang deposition phenomena. The nontrivial scale invariance of the pde (3.1) suggests its *universality* in some sense. Our main result indeed states its validity in a very wide context.

It is also clear that the pde is invariant under the left-right reflection symmetry $x \mapsto -x$:

The parameter γ of the pde (3.1) is of crucial importance: different values of γ lead to completely different behavior. Here are listed some particular cases which arose in the past in various contexts:

— The pde (3.1) with $\gamma = 0$ arose in the context of the 'true self-repelling motion' constructed by Tóth and Werner in [37]. For a survey of this case see also [38]. The same equation, with viscosity terms added, appear in mathematical biology under the name of (negative) chemotaxis equations (see e.g. [24], [22], [19])).

— Taking $\gamma = 1/2$ we get the 'shallow water equation'. See [5], [20]. This is the only value of the parameter γ when $m = \rho u$ is conserved and as a consequence the pde (3.1) can be interpreted as gas dynamics equation.

— With $\gamma = 1$ the pde is called 'Leroux's equation' which is of Temple class and for this reason much investigated. For many details about this equation see [29]. In the recent paper [8] Leroux's system has been derived as hydrodynamic limit under Eulerian scaling for a two-component lattice gas, going even beyond the appearance of shocks.

The main facts about the pde (3.1) are presented in Section 3.11. Here we only mention that

1. For any $\gamma \in \mathbb{R}$ the system (3.1) is strictly *hyperbolic* in $(\rho, u) \in (0, \infty) \times \mathbb{R}$, with hyperbolicity marginally lost at $(\rho, u) = (0, 0)$ for $\gamma \neq 1/2$ and at $\rho = 0$ for $\gamma = 1/2$. This follows from straightforward computations.

2. The *Riemann invariants* (or characteristic coordinates) are explicitly computed in section 3.11, for a first impression see Figure 3.3 of the Appendix where the level lines of the Riemann invariants are shown. It turns out that the picture changes qualitatively at the critical values $\gamma = 1/2$, $\gamma = 3/4$ and $\gamma = 1$. It is of crucial importance for our later problem that the level curves, expressed as $u \mapsto \rho(u)$ are convex for $\gamma < 1$, linear for $\gamma = 1$ and concave for $\gamma > 1$.

3. For any $\gamma \geq 0$ the system (3.1) is *genuinely nonlinear* in $(\rho, u) \in (0, \infty) \times \mathbb{R}$, with genuine nonlinearity marginally lost at $(\rho, u) = (0, 0)$ for $\gamma \neq 0, 1/2$ and at $\rho = 0$ for $\gamma = 0, 1/2$. (For $\gamma < 0$ genuine nonlinearity is lost on the parabola $\rho = -4\gamma(2\gamma - 1)^2(\gamma + 1)^{-2}u^2$.)

4. The system is sufficiently rich in *Lax entropies.*

5. For $\gamma \geq 0$ the system (3.1) satisfies the conditions of the Lax-Chuey-Conley-Smoller *Maximum Principle* (see [17], [18], [29]).

From the Maximum Principle a very essential difference between the cases $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$ follows, which is of crucial importance for our further work. In the case $\gamma < 1$ all convex domains bounded by level curves of the Riemann invariants are *unbounded (non-compact)* and thus there is no a priori bound on the entropy solutions. Even starting with smooth initial data with compact support nothing prevents the solutions to blow up indefinitely. On the other hand, if $\gamma \geq 1$ any bounded subset of $\mathbb{R}_+ \times \mathbb{R}$ is contained in a compact convex domain bounded by level sets of the Riemann invariants, which fact yields a priori bounds on the entropy solutions, given bounded initial data.

The goal of the present paper is to derive the two-by-two hyperbolic system of conservation laws (3.1) as decent hydrodynamic limit of some systems of interacting particles with two conserved quantities.

We consider one-dimensional, locally finite interacting particle systems with two conservation laws which under *Eulerian* hydrodynamic limit lead to two-by-two systems of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0, \end{cases}$$

with $(\rho, u) \in \mathcal{D} \subset \mathbb{R}^2$, where $\mathcal{D}$ is a convex compact polygon in $\mathbb{R}^2$. The system is *typically* strictly hyperbolic in the interior of $\mathcal{D}$ with possible non-hyperbolic degeneracies on the boundary $\partial\mathcal{D}$. We consider the case of isolated singular (i.e. non hyperbolic) point on the interior of one of the edges of $\mathcal{D}$, call it $(\rho_0, u_0) = (0, 0)$ and assume $\mathcal{D} \subset \{\rho \geq 0\}$ (otherwise we apply an appropriate linear transformation on the conserved quantities) We investigate the propagation of *small nonequilibrium perturbations* of the steady state of the microscopic interacting particle system, corresponding to the densities (ρ_0, u_0) of the conserved quantities. We prove that for a very rich class of systems, under proper hydrodynamic limit the propagation of these small perturbations are *universally* driven by the system (3.1), where the parameter $\gamma := \frac{1}{2}\Phi_{uu}(\rho_0, u_0)$ (with a proper choice of space and time scale) is the only trace of the microscopic structure. The proof is valid for the cases with $\gamma > 1$.

Actually, in order to simplify some of the arguments, we impose the left-right reflection symmetry of the pde (3.1) on the systems of interacting particles on microscopic level, see condition (C) in subsection 3.2.2. But we note that the whole proof can be extended without this condition, just some arguments would be longer.

The proof essentially relies on H-T. Yau's relative entropy method and thus, it is valid only in the regime of smooth solutions of the pde (3.1).

We should emphasize here the essential new ideas of the proof. Since we consider a *low density* limit, the distribution of particle numbers in blocks of mesoscopic size will have a *Poissonian* tail. The fluctuations of the other conserved quantity will be Gaussian, as usual. It follows that when controlling the fluctuations of the empirical block averages the usual large deviation approach would lead us to the disastrous estimate $\mathbf{E}\big(\exp\{\varepsilon\, GAU \cdot POI\}\big) = \infty$. It turns out that some very special cutoff must be applied. Since the large fluctuations which are cut off can not be estimated by robust methods (i.e. by applying entropy inequality), only some cancellation due to martingales can help. This is the reason why the cutoff function must be chosen in a very special way, in terms of a particular Lax entropy of the Euler equation (3.16). In this way the proof becomes a mixture (in our opinion rather interesting mixture) of probabilistic and pde arguments. The fine properties of the limiting pde, in particular the global behavior of Riemann invariants and some particular Lax entropies, play an essential role in the proof. The radical difference between the $\gamma \geq 1$ vs. $\gamma < 1$ cases, in particular applicability vs. non-applicability of the Lax-Chuey-Conley-Smoller maximum principle, manifests itself on the microscopic, probabilistic level.

3.1.2 The structure of the paper

In Section 3.2 we define the class of models to which our main theorem applies: we formulate the conditions to be satisfied by the interacting particle systems to be considered, we compute the steady state measures and the fluxes corresponding to the conserved quantities. At the end of this section we formulate the Eulerian hydrodynamic limit, for later reference.

In Section 3.3 first we perform asymptotic analysis of the Euler equations close to the singular point considered, then we formulate our main result, Theorem 2, and its immediate consequences.

Sections 3.4 to 3.10 are devoted to the proof of Theorem 2.

In Section 3.4 we perform the necessary preliminary computations for the proof. After introducing the minimum necessary notation we apply some standard procedures in the context of relative entropy method. Empirical block averages are introduced, numerical error terms are separated and estimated. In this first estimates only straightforward numerical approximations (Taylor expansion bounds) and the most direct entropy inequality is applied.

Section 3.5 is of crucial importance: here it is shown why the traditional approach of the relative entropy method fails to apply. Here it becomes apparent that in the fluctuation bound (usually referred to as *large deviation estimate*) instead of the tame $\mathbf{E}\big(\exp\{\varepsilon\, GAU^2\}\big)$ we would run into the wild $\mathbf{E}\big(\exp\{\varepsilon\, GAU \cdot POI\}\big)$ which is, of course, infinite. It is explained here what kind of cutoff is applied: the large fluctuations cut off can not be estimated by robust methods (i.e. by applying entropy inequality). Only some cancellation due to martingales can help. This is the reason why the cutoff function must be chosen in a very particular way, in terms of a particular Lax entropy of the Euler equation. The cutoff function is constructed and its key estimates are stated. Proofs of the lemmas formulated in this section are postponed to Section 3.9. At the end of this section the outline of the further steps is presented.

In Section 3.6 all the necessary probabilistic ingredients of the forthcoming steps are gathered. These are: fixed time large deviation bounds and fixed time fluctuation bounds, the time averaged block replacement bounds (one block estimates) and the time averaged gradient bounds (two block estimates). The proof of these last two rely on Varadhan's large deviation bound cited in that section and on some probability lemmas stated and proved in section 3.10. We should mention here that these proofs of the one- and two block estimates, in particular the probability lemmas involved also contain some new (and, we hope, instructive) elements.

Sections 3.7 and 3.8 conclude the proof: the various terms arising in section 3.5 are estimated using all the tools (probabilistic and pde) developed in earlier sections. One can see that these estimates rely heavily on the fine properties of the Lax entropy used in the cutoff procedure.

As we already mentioned sections 3.9 and 3.10 are devoted to proofs of various lemmas stated in earlier parts. Section 3.9 deals with the pde estimates while Section 3.10 is probabilistic.

Finally in the Appendix (Section 3.11) we give some details about the pde (3.1). This is included for sake of completeness and in order to let the reader have some more information about these, certainly interesting, pde-s. Strictly technically speaking this Appendix is not used in the proof.

3.2 Microscopic models

Our interacting particle systems to be defined in the present section model on a microscopic level the same deposition phenomena as the pde (3.1). There will be two conserved physical quantities: the particle number $\eta_j \in \mathbb{N}$ and the (discrete) negative gradient of the deposition height $\zeta_j \in \mathbb{Z}$.
The dynamical driving mechanism is of such nature that

(i) The deposition height growth is influenced by the local particle density. Typically: growth is enhanced by higher particle densities.

(ii) The particle motion is itself influenced by the deposition profile. Typically: particles are pushed in the direction of the negative gradient of the deposition height.

The left-right reflection symmetry of the pde will be also implemented on the microscopic level. Actually, this is not really necessary in order to prove our main result, but without this assumption some of the arguments would be somewhat longer.

3.2.1 State space, conserved quantities

Throughout this paper we denote by $\mathbb{T}^n$ the discrete tori $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. We will denote the local spin state by Ω, we only consider the case when Ω is finite. The state space of the interacting particle system of size n is

$$\Omega^n := \Omega^{\mathbb{T}^n}.$$

Configurations will be denoted

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^n} \in \Omega^n,$$

For sake of simplicity we consider discrete (integer valued) conserved quantities only. The two conserved quantities are

$$\begin{aligned} &\eta : \Omega \to \mathbb{N}, \\ &\zeta : \Omega \to v_0\mathbb{Z}, \text{ or } \zeta : \Omega \to v_0\big(\mathbb{Z} + 1/2\big). \end{aligned} \tag{3.2}$$

The trivial scaling factor v_0 will be conveniently chosen later (see (3.5)). We also use the notations $\eta_j = \eta(\omega_j)$, $\zeta_j = \zeta(\omega_j)$. This means that the sums $\sum_j \eta_j$ and $\sum_j \zeta_j$ are conserved by the dynamics. We assume that the conserved quantities are different and non-trivial, i.e. the functions ζ, η and the constant function 1 on Ω are linearly independent.

The left-right reflection symmetry of the model is implemented by an involution

$$R : \Omega \to \Omega, \qquad R \circ R = Id$$

which acts on the conserved quantities as follows:

$$\eta(R\omega) = \eta(\omega), \qquad \zeta(R\omega) = -\zeta(\omega). \tag{3.3}$$

3.2.2 Rate functions, infinitesimal generators, stationary measures

Consider a (fixed) probability measure π on Ω, which is invariant under the action of the involution R, i.e. $\pi(R\omega) = \pi(\omega)$. Since eventually we consider *low densities* of η, in order to exclude trivial cases we assume that

$$\pi\big(\, \zeta = 0 \,\big|\, \eta = 0 \,\big) < 1. \tag{3.4}$$

The scaling factor v_0 in (3.2) is chosen so that

$$\mathbf{Var}\big(\zeta \,|\, \eta = 0\big) = 1. \tag{3.5}$$

This choice simplifies some formulas (fixing a recurring constant to be equal to 1, see (3.20)) but does not restrict generality.

For later use we introduce the notation

$$\rho^* := \max\{\eta(\omega) : \pi(\omega) > 0\},$$

$$u^* := \max\{\zeta(\omega) : \pi(\omega) > 0\},$$

$$u_* := \max\{\zeta(\omega) : \eta(\omega) = 0, \pi(\omega) > 0\},$$

For $\tau, \theta \in \mathbb{R}$ let $G(\tau, \theta)$ be the moment generating function defined below:

$$G(\tau, \theta) := \log \sum_{\omega \in \Omega} e^{\tau\eta(\omega) + \theta\zeta(\omega)} \pi(\omega).$$

In thermodynamic terms $G(\tau, \theta)$ corresponds to the Gibbs free energy. We define the probability measures

$$\pi_{\tau,\theta}(\omega) := \pi(\omega) \exp(\tau\eta(\omega) + \theta\zeta(\omega) - G(\tau, \theta)) \tag{3.6}$$

on Ω. We are going to define dynamics which conserve the quantities $\sum_j \eta_j$ and $\sum_j \zeta_j$, posses no other (hidden) conserved quantities and for which the product measures

$$\pi^n_{\tau,\theta} := \prod_{j \in \mathbb{T}^n} \pi_{\tau,\theta}$$

are stationary.

We need to separate a symmetric (reversible) part of the dynamics which will be speeded up sufficiently in order to enhance convergence to local equilibrium and thus help estimating some error term in the hydrodynamic limiting procedure. So we consider two *rate functions* $r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$ and $s : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$, r will define the *asymmetric* component of the dynamics, while s will define the *reversible* component. The dynamics of the system consists of elementary jumps affecting nearest neighbor spins, $(\omega_j, \omega_{j+1}) \longrightarrow (\omega'_j, \omega'_{j+1})$, performed with rate $\lambda r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1}) + \kappa s(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$, where $\lambda, \kappa > 0$ are speed-up factors, depending on the size of the system in the limiting procedure.

We require that the rate functions r and s satisfy the following conditions.

(A) *Conservation laws:* If $r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ *or* $s(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ then

$$\eta(\omega_1) + \eta(\omega_2) = \eta(\omega'_1) + \eta(\omega'_2).$$
$$\zeta(\omega_1) + \zeta(\omega_2) = \zeta(\omega'_1) + \zeta(\omega'_2),$$

(B) *Irreducibility:* For every $N \in [0, n\rho^*]$, $Z \in [-nu^*, nu^*]$ the set

$$\Omega^n_{N,Z} := \left\{ \underline{\omega} \in \Omega^n : \sum_{j \in \mathbb{T}^n} \eta_j = N, \sum_{j \in \mathbb{T}^n} \zeta_j = Z \right\}$$

is an irreducible component of Ω^n, i.e. if $\underline{\omega}, \underline{\omega}' \in \Omega^n_{N,Z}$ then there exists a series of elementary jumps with positive rates transforming $\underline{\omega}$ into $\underline{\omega}'$.

(C) *Left-right symmetry:* The jump rates are invariant under left-right reflection *and* the action of the involution R (jointly):

$$r(R\omega_2, R\omega_1; R\omega'_2, R\omega'_1) = r(\omega_1, \omega_2; \omega'_1, \omega'_2).$$
$$s(R\omega_2, R\omega_1; R\omega'_2, R\omega'_1) = s(\omega_1, \omega_2; \omega'_1, \omega'_2).$$

(D) *Stationarity of the asymmetric part:* For any $\omega_1, \omega_2, \omega_3 \in \Omega$

$$Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,$$

where

$$Q(\omega_1, \omega_2) := \sum_{\omega'_1, \omega'_2 \in \Omega} \left\{ \frac{\pi(\omega'_1)\pi(\omega'_2)}{\pi(\omega_1)\pi(\omega_2)} r(\omega'_1, \omega'_2; \omega_1, \omega_2) - r(\omega_1, \omega_2; \omega'_1, \omega'_2) \right\}.$$

(E) *Reversibility of the symmetric part:* For any $\omega_1, \omega_2, \omega'_1, \omega'_2 \in \Omega$

$$\pi(\omega_1)\pi(\omega_2)s(\omega_1, \omega_2; \omega'_1, \omega'_2) = \pi(\omega'_1)\pi(\omega'_2)s(\omega'_1, \omega'_2; \omega_1, \omega_2).$$

For a precise formulation of the infinitesimal generator on Ω^n we first define the map $\Theta_j^{\omega'\omega''}: \Omega^n \to \Omega^n$ for every $\omega', \omega'' \in \Omega$, $j \in \mathbb{T}^n$:

$$\left(\Theta_j^{\omega'\omega''}\underline{\omega}\right)_i = \begin{cases} \omega' & \text{if} \quad i = j \\ \omega'' & \text{if} \quad i = j+1 \\ \omega_i & \text{if} \quad i \neq j, j+1. \end{cases}$$

The infinitesimal generators defined by these rates will be denoted:

$$L^n f(\underline{\omega}) = \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega''\in\Omega} r(\omega_j, \omega_{j+1}; \omega', \omega'')\big(f(\Theta_j^{\omega'\omega''}\underline{\omega}) - f(\underline{\omega})\big).$$

$$K^n f(\underline{\omega}) = \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega''\in\Omega} s(\omega_j, \omega_{j+1}; \omega', \omega'')\big(f(\Theta_j^{\omega'\omega''}\underline{\omega}) - f(\underline{\omega})\big).$$

We denote by $\mathcal{X}_t^n$ the Markov process on the state space Ω^n with infinitesimal generator $G^n := \lambda(n)L^n + \kappa(n)K^n$. with speed-up factors $\lambda(n)$ and $\kappa(n)$ to be specified later

Remarks:

(1) Conditions (A) and (B) together imply that $\sum_j \eta_j$ and $\sum_j \zeta_j$ are indeed the only conserved quantities of the dynamics.

(2) Condition (C) together with (3.3) is implementation on a microscopic level of the left-right symmetry of the pde (3.1). Actually, our main result, Theorem 2, is valid without this assumption but some of the arguments would be more technical.

(3) Condition (D) implies that the product measures $\pi_{\tau,\theta}^n$ are indeed stationary for the dynamics defined by the asymmetric rates r. This is seen by applying similar computations to those of [1], [2], [27] or [35]. Mind that this is *not* a detailed balance condition for the rates r.

(4) Condition (E) is a straightforward detailed balance condition. It implies that the product measures $\pi_{\tau,\theta}^n$ are reversible for the dynamics defined by the symmetric rates s.

We will refer to the measures $\pi_{\tau,\theta}^n$ as the *canonical* measures. Since $\sum_j \zeta_j$ and $\sum_j \eta_j$ are conserved the canonical measures on Ω^n are *not* ergodic. The conditioned measures defined on $\Omega_{N,Z}^n$ by:

$$\pi_{N,Z}^n(\underline{\omega}) := \pi_{\tau,\theta}^n\big(\underline{\omega} \big| \sum_{j\in\mathbb{T}^n} \eta_j = N, \sum_{j\in\mathbb{T}^n} \zeta_j = Z, \big) = \frac{\pi_{\tau,\theta}^n(\underline{\omega})1\!\!1\{\underline{\omega} \in \Omega_{N,Z}^n\}}{\pi_{\tau,\theta}^n(\Omega_{N,Z}^n)}$$

are also stationary and due to condition (B) satisfied by the rate functions they are ergodic. We shall call these measures the *microcanonical measures* of our system. (It is easy to see that the measure $\pi_{N,Z}^n$ does not depend on the choice of thew values of τ and θ in the previous definition.)

The assumptions are by no means excessively restrictive. Here follow some concrete examples of interacting particle systems which belong to the class specified by conditions (A)-(E) and also satisfy the further conditions (F), (G), (H), (I) to be formulated later.

$\{-1,0,+1\}$*-model* The model is described and analyzed in full detail in [35] and [8]. The one spin state space is $\Omega = \{-1,0,+1\}$. The left-right reflection symmetry is implemented by $R:\Omega\to\Omega$, $R\omega=-\omega$. The dynamics consists of nearest neighbor spin exchanges and the two conserved quantities are $\eta(\omega)=1-|\omega|$ and $\zeta(\omega)=\omega$. The jump rates are

$$\begin{aligned} r(1,-1;-1,1) &= 0, & r(-1,1;1,-1) &= 2,\\ r(0,-1;-1,0) &= 0, & r(-1,0;0,-1) &= 1,\\ r(1,0,0,1) &= 0, & r(0,1,1,0) &= 1. \end{aligned}$$

and

$$s(\omega_1,\omega_2;\omega_1',\omega_2') = \begin{cases} 1 & \text{if } (\omega_1,\omega_2)=(\omega_2',\omega_1') \text{ and } \omega_1\neq\omega_2\\ 0 & \text{otherwise.}\end{cases}$$

The one dimensional marginals of the stationary measures are

$$\pi_{\rho,u}(0)=\rho, \quad \pi_{\rho,u}(\pm 1)=\frac{1-\rho\pm u}{2}$$

with the domain of variables $\mathcal{D}=\{(\rho,u)\in\mathbb{R}_+\times\mathbb{R} : \rho+|u|\le 1\}$.

Two-lane models The following family of examples are finite state space versions of the bricklayers models introduced in [38]. Let $\Omega=\{0,1,\dots,\bar{n}\}\times\{-\bar{z},-\bar{z}+1,\dots,\bar{z}-1,\bar{z}\}$, where $\bar{n}\in\mathbb{N}$ and $\bar{z}\in\{\frac{1}{2},1,\frac{3}{2},2\dots\}$. The elements of Ω will be denoted $\omega:=\binom{\eta}{\zeta}$. Naturally enough, $\sum_j\eta_j$ and $\sum_j\zeta_j$ will be the conserved quantities of the dynamics. Left-right reflection symmetry is implemented as $R:\Omega\to\Omega$, $R\binom{\eta}{\zeta}=\binom{\eta}{-\zeta}$. We allow only the following elementary changes to occur at neighboring sites $j,j+1$:

$$\binom{\eta_j}{\zeta_j},\binom{\eta_{j+1}}{\zeta_{j+1}} \to \binom{\eta_j}{\zeta_j\mp 1},\binom{\eta_{j+1}}{\zeta_{j+1}\pm 1}, \qquad \binom{\eta_j}{\zeta_j},\binom{\eta_{j+1}}{\zeta_{j+1}} \to \binom{\eta_j\mp 1}{\zeta_j},\binom{\eta_{j+1}\pm 1}{\zeta_{j+1}}$$

with appropriate rates. Beside the conditions already imposed we also assume that the one dimensional marginals of the steady state measures factorize as follows:

$$\pi(\omega)=\pi\binom{\eta}{\zeta}=p(\eta)q(\zeta).$$

The simplest case, with $\bar{n}=1$ and $\bar{z}=1/2$, that is with $\Omega=\{0,1\}\times\{-1/2,+1/2\}$, was introduced and fully analyzed in [35] and [23]. For a full description (i.e. identification of the rates which satisfy the imposed conditions, Eulerian hydrodynamic limit, etc. see those papers.) It turns out that conditions (A)-(E) impose some nontrivial combinatorial constraints on the rates which are satisfied by a finite parameter family of models. The number of free parameters increases with $\bar{n}$ and $\bar{z}$. Since the concrete expressions of the rates are not relevant for our further presentation we omit the lengthy computations.

3.2.3 Expectations

Expectation, variance, covariance with respect to the measures $\pi^n_{\tau,\theta}$ will be denoted by $\mathbf{E}_{\tau,\theta}(.)$, $\mathbf{Var}_{\tau,\theta}(.)$, $\mathbf{Cov}_{\tau,\theta}(.)$.

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters τ and θ:

$$\rho(\tau,\theta) := \mathbf{E}_{\tau,\theta}(\eta) = \sum_{\omega\in\Omega} \eta(\omega)\pi_{\tau,\theta}(\omega) = G_\tau(\tau,\theta).$$

$$u(\tau,\theta) := \mathbf{E}_{\tau,\theta}(\zeta) = \sum_{\omega\in\Omega} \zeta(\omega)\pi_{\tau,\theta}(\omega) = G_\theta(\tau,\theta),$$

Elementary calculations show, that the matrix-valued function

$$\begin{pmatrix} \rho_\tau & \rho_\theta \\ u_\tau & u_\theta \end{pmatrix} = \begin{pmatrix} G_{\tau\tau} & G_{\tau\theta} \\ G_{\theta\tau} & G_{\theta\theta} \end{pmatrix} =: G''(\tau,\theta)$$

is equal to the covariance matrix $\mathbf{Cov}_{\tau,\theta}(\eta,\zeta)$ and therefore it is strictly positive definite. It follows that the function $(\tau,\theta) \mapsto (\rho(\tau,\theta), u(\tau,\theta))$ is invertible. We denote the inverse function by $(\rho,u) \mapsto (\tau(\rho,u),\theta(\rho,u))$. Denote by $(\rho,u) \mapsto S(\rho,u)$ the convex conjugate (Legendre transform) of the strictly convex function $(\tau,\theta) \mapsto G(\tau,\theta)$:

$$S(\rho,u) := \sup_{\tau,\theta} \big(\rho\tau + u\theta - G(\tau,\theta)\big), \tag{3.7}$$

and

$$\begin{aligned} \mathcal{D} &:= \{(\rho,u) \in \mathbb{R}_+ \times \mathbb{R} : S(\rho,u) < \infty\} \\ &= \text{co}\{(\eta,\zeta) : \pi(\omega) > 0\}, \end{aligned} \tag{3.8}$$

where co stands for convex hull. The nondegeneracy condition (3.4) implies that $\partial\mathcal{D} \cap \{\rho = 0\} = \{(0,u) : |u| \le u_*\}$. For $(\rho,u) \in \mathcal{D}$ we have

$$\tau(\rho,u) = S_\rho(\rho,u), \qquad \theta(\rho,u) = S_u(\rho,u).$$

In probabilistic terms: $S(\rho,u)$ is the rate function of joint large deviations of $(\sum_j \eta_j, \sum_j \zeta_j)$. In thermodynamic terms: $S(\rho,u)$ corresponds to the equilibrium thermodynamic entropy. Let

$$\begin{pmatrix} \tau_\rho & \tau_u \\ \theta_\rho & \theta_u \end{pmatrix} = \begin{pmatrix} S_{\rho\rho} & S_{\rho u} \\ S_{u\rho} & S_{uu} \end{pmatrix} =: S''(\rho,u).$$

It is obvious that the matrices $G''(\tau,\theta)$ and $S''(\rho,u)$ are strictly positive definite and are inverse of each other:

$$G''(\tau,\theta)S''(\rho,u) = I = S''(\rho,u)G''(\tau,\theta), \tag{3.9}$$

where either $(\tau,\theta) = (\tau(\rho,u),\theta(\rho,u))$ or $(\rho,u) = (\rho(\tau,\theta),u(\tau,\theta))$. With slight abuse of notation we shall denote: $\pi_{\tau(\rho,u),\theta(\rho,u)} =: \pi_{\rho,u}$, $\pi^n_{\tau(\rho,u),\theta(\rho,u)} =: \pi^n_{\rho,u}$, $\mathbf{E}_{\tau(\rho,u),\theta(\rho,u)} =: \mathbf{E}_{\rho,u}$, etc.

As a general convention, if $\xi : \Omega^m \to \mathbb{R}$ is a local function then its expectation with respect to the canonical measure $\pi^m_{\rho,u}$ is denoted by

$$\Xi(\rho,u) := \mathbf{E}_{\rho,u}(\xi) = \sum_{\omega_1,\dots,\omega_m\in\Omega^m} \xi(\omega_1,\dots,\omega_m)\pi_{\rho,u}(\omega_1)\cdots\pi_{\rho,u}(\omega_m).$$

3.2.4 Fluxes

We introduce the fluxes of the conserved quantities. The infinitesimal generators L^n and K^n act on the conserved quantities as follows:

$$\begin{aligned} L^n\eta_i &= -\psi(\omega_i,\omega_{i+1})+\psi(\omega_{i-1},\omega_i) &&=: -\psi_i+\psi_{i-1},\\ L^n\zeta_i &= -\phi(\omega_i,\omega_{i+1})+\phi(\omega_{i-1},\omega_i) &&=: -\phi_i+\phi_{i-1},\\ K^n\eta_i &= -\psi^s(\omega_i,\omega_{i+1})+\psi^s(\omega_{i-1},\omega_i) &&=: -\psi^s_i+\psi^s_{i-1},\\ K^n\zeta_i &= -\phi^s(\omega_i,\omega_{i+1})+\phi^s(\omega_{i-1},\omega_i) &&=: -\phi^s_i+\phi^s_{i-1}, \end{aligned}$$

where

$$\begin{aligned} \psi(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} r(\omega_1,\omega_2;\omega_1',\omega_2')\big(\eta(\omega_2')-\eta(\omega_2)\big)\\ \phi(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} r(\omega_1,\omega_2;\omega_1',\omega_2')\big(\zeta(\omega_2')-\zeta(\omega_2)\big) \end{aligned} \tag{3.10}$$

$$\begin{aligned} \psi^s(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} s(\omega_1,\omega_2;\omega_1',\omega_2')\big(\eta(\omega_2')-\eta(\omega_2)\big)\\ \phi^s(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2'\in\Omega} s(\omega_1,\omega_2;\omega_1',\omega_2')\big(\zeta(\omega_2')-\zeta(\omega_2)\big) \end{aligned} \tag{3.11}$$

Note that due to the left-right symmetry and conservations, i.e. (3.3) and conditions (A) and (C), the microscopic fluxes have the following symmetries:

$$\begin{aligned} \phi(\omega_1,\omega_2) &= \phi(R\omega_2,R\omega_1),\\ \psi(\omega_1,\omega_2) &= -\psi(R\omega_2,R\omega_1). \end{aligned}$$

In order to simplify some of our further arguments (in particular, see (3.120) in subsection 3.7.3) we impose one more microscopic condition

(F) *Gradient condition on symmetric fluxes:* The microscopic fluxes of the symmetric part, defined in (3.11) satisfy the following gradient conditions

$$\begin{aligned} \psi^s(\omega_1,\omega_2) &= \kappa(\omega_1)-\kappa(\omega_2) =: \kappa_1-\kappa_2 \qquad (3.12)\\ \phi^s(\omega_1,\omega_2) &= \chi(\omega_1)-\chi(\omega_2) =: \chi_1-\chi_2. \end{aligned}$$

Remark: (1) This is a technical assumption (referring actually to the measure π) which simplifies considerably the arguments of subsection 3.9.2. The symmetric part K^n has the role of enhancing convergence to local equilibrium. Its effect is *not seen* in the limit, so in principle we can choose it conveniently. Without this assumption we would be forced to use all the non-gradient technology developed in [39] (see also [13]), which would make the paper even longer.

(2) It is easy to see that $\eta(\omega_1) = \eta(\omega_2) = 0$ implies $\psi^s(\omega_1, \omega_2) = 0$ and thus (by choosing a suitable additive constant) $\omega \mapsto \kappa(\omega)$ can be chosen so that

$$\eta(\omega) = 0 \;\Rightarrow\; \kappa(\omega) = 0. \tag{3.13}$$

The *macroscopic fluxes* are:

$$\begin{aligned}
\Psi(\rho, u) &:= \mathbf{E}_{\rho,u}(\psi) \\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2'\in\Omega}} r(\omega_1, \omega_2; \omega_1', \omega_2')\big(\eta(\omega_2') - \eta(\omega_2)\big)\pi_{\rho,u}(\omega_1)\pi_{\rho,u}(\omega_2), \\
\Phi(\rho, u) &:= \mathbf{E}_{\rho,u}(\phi) \\
&= \sum_{\substack{\omega_1,\omega_2,\\ \omega_1',\omega_2'\in\Omega}} r(\omega_1, \omega_2; \omega_1', \omega_2')\big(\zeta(\omega_2') - \zeta(\omega_2)\big)\pi_{\rho,u}(\omega_1)\pi_{\rho,u}(\omega_2).
\end{aligned} \tag{3.14}$$

These are smooth regular functions of the variables $(\rho, u) \in \mathcal{D}$. Note that due to reversibility of K^n, for any value of ρ and u

$$\mathbf{E}_{\rho,u}(\psi^s) = 0 = \mathbf{E}_{\rho,u}(\phi^s).$$

(These identities hold true without assuming condition (F).)

For later use we mention here that according to [35], the macroscopic fluxes $\Psi(\rho, u)$ and $\Phi(\rho, u)$ satisfy the following *Onsager reciprocity relation*

$$\begin{aligned}
&\Psi_u(\rho, u)\mathbf{Var}_{\rho,u}(\zeta) - \Phi_u(\rho, u)\mathbf{Cov}_{\rho,u}(\eta, \zeta) = \\
&\qquad\qquad \Phi_\rho(\rho, u)\mathbf{Var}_{\rho,u}(\eta) - \Psi_\rho(\rho, u)\mathbf{Cov}_{\rho,u}(\eta, \zeta).
\end{aligned} \tag{3.15}$$

For the concrete examples presented at the end of subsection 3.2.2 the following domains $\mathcal{D}$ and macroscopic rates are gotten:

$\{-1, 0, +1\}$-*model*:

$$\begin{aligned}
&\mathcal{D} = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} \,:\, \rho + |u| \leq 1\} \\
&\Psi(\rho, u) = \rho u \\
&\Phi(\rho, u) = \rho + u^2.
\end{aligned}$$

Two lane models with $\bar{n} = 1$:

$$\begin{aligned}
&\mathcal{D} = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} \,:\, \rho \leq 1, \;\; |u| \leq \bar{z}\} \\
&\Psi(\rho, u) = \rho(1 - \rho)\psi(u) \\
&\Phi(\rho, u) = \varphi_0(u) + \rho\varphi_1(u),
\end{aligned}$$

where $\psi(u)$ is odd, while $\varphi_0(u)$ and $\varphi_1(u)$ are even functions of u, determined by the jump rates of the model. In the simplest particular case with $\bar{z} = 1/2$

$$\Psi(\rho, u) = \rho(1-\rho)u$$
$$\Phi(\rho, u) = (\rho - \gamma)(1-u^2),$$

where $\gamma \in \mathbb{R}$ is the only model dependent parameter which appears in the macroscopic fluxes. For details see [35].

3.2.5 The hdl under Eulerian scaling

Given a system of interacting particles as defined in the previous subsections, by applying Yau's relative entropy method (see [41] or the monograph [13]), one shows that under Eulerian scaling the local densities of the conserved quantities $\rho(t,x)$, $u(t,x)$ evolve according to the system of partial differential equations:

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi(\rho, u) = 0 \end{cases} \tag{3.16}$$

where $\Psi(\rho, u)$ and $\Phi(\rho, u)$ are the macroscopic fluxes defined in (3.14).

The precise statement of the hydrodynamical limit is as follows: Consider a microscopic system which satisfies conditions (A)-(E) of subsection 3.2.2. Note that condition (F) of subsection 3.2.4 is not assumed. Let $\Psi(\rho, u)$ and $\Phi(\rho, u)$ be the macroscopic fluxes computed for this system and $\rho(t,x), u(t,x)$ $x \in \mathbb{T}$, $t \in [0,T]$ be *smooth* solution of the pde (3.16). Let the microscopic system of size n be driven by the infinitesimal generator

$$G^n = nL^n + n^{1+\delta}K^n,$$

where $\delta \in [0,1)$ is fixed. This means that the main, asymmetric part of the generator is speeded up by n and the additional symmetric part by $n^{1+\delta}$. Let μ_0^n be a probability distribution on Ω^n which is the initial distribution of the microscopic system of size n, and

$$\mu_t^n := \mu_0^n e^{tG^n}$$

the distribution of the system at (macroscopic) time t. The *local equilibrium* measure ν_t^n (itself a probability measure on Ω^n) is defined by

$$\nu_t^n := \prod_{j \in \mathbb{T}^n} \pi_{\rho(t,\frac{j}{n}), u(t,\frac{j}{n})}.$$

This measure *mimics on a microscopic scale* the macroscopic evolution driven by the pde (3.16).

We denote by $H(\mu_t^n | \pi^n)$, respectively, by $H(\mu_t^n | \nu_t^n)$ the relative entropy of the measure μ_t^n with respect to the absolute reference measure π^n, respectively, with respect to the local equilibrium measure ν_t^n.

The precise statement of the Eulerian hydrodynamic limit is the following

Theorem. *Assume conditions (A)-(E) and let* $\delta \in [0,1)$ *be fixed. If*

$$H\left(\mu_0^n \mid \nu_0^n\right) = o(n)$$

then

$$H\left(\mu_t^n \mid \nu_t^n\right) = o(n)$$

uniformly for $t \in [0,T]$.

Remark: Note that due to finiteness of the state space Ω the condition

$$H\left(\mu_0^n \,\middle|\, \pi_{1,0}^n\right) = \mathcal{O}(n)$$

holds automatically.

The Theorem follows from direct application of Yau's relative entropy method. For the proof and its direct consequences see [41], [13] or [35]. For the main consequences of this Theorem see e.g. Corollary 1 of [35].

3.3 Low density asymptotics and the main result: hydrodynamic limit under intermediate scaling

3.3.1 General properties and low density asymptotics of the macroscopic fluxes

The fluxes in the Euler equation (3.16) are regular smooth functions of in $(\rho,u) \in \overline{\mathcal{D}}$.

From the left-right symmetry of the microscopic models it follows that

$$\Phi(\rho,-u) = \Phi(\rho,u), \qquad \Psi(\rho,-u) = -\Psi(\rho,u). \tag{3.17}$$

It is also obvious that for $u \in [-u_*, u_*]$

$$\Psi(0,u) = 0. \tag{3.18}$$

We make two assumptions about the low density asymptotics of the macroscopic fluxes. Here is the first one:

(G) We assume that $\Psi_{\rho u}(0,0) \neq 0$. Actually, by possibly redefining the time scale and orientation of space, without loss of generality we assume

$$\Psi_{\rho u}(0,0) = 1. \tag{3.19}$$

From the Onsager relation (3.15) and obvious parity considerations it also follows that

$$\Phi_\rho(0,0) = \Psi_{\rho u}(0,0)\mathbf{Var}_{0,0}(\zeta) = 1. \tag{3.20}$$

Note, that here we rely on the choice (3.5) of the scaling factor v_0 in (3.2).

We denote

$$\gamma := \frac{1}{2}\Phi_{uu}(0,0). \tag{3.21}$$

Our results will hold for $\gamma > 1$ only.

From (3.17) and (3.19) it follows that

$$\Phi_u(0,u) - \Psi_\rho(0,u) = (2\gamma - 1)u + \mathcal{O}(|u|^3). \tag{3.22}$$

The second condition imposed on the low density asymptotics of the macroscopic fluxes is:

(H) For $u \in [-u_*, u_*]$, $u \neq 0$

$$\Phi_u(0,u) - \Psi_\rho(0,u) \neq 0, \tag{3.23}$$

$$\Phi_\rho(0,u) \neq 0, \quad \Psi_{\rho u}(0,u) \neq 0 \tag{3.24}$$

Remarks: (1) (G) is a very natural nondegeneracy condition: if $\Psi_{\rho u}(0,0)$ vanished then in the perturbation calculus to be performed, higher order terms would be dominant and a different scaling limit should be taken.
(2) Due to (3.17), (3.19) and (3.22) conditions (3.23), (3.24) hold anyway in a neighborhood of $u = 0$, and this would suffice, but the forthcoming arguments, in particular the proof of Lemmas 3 and 4 would be less transparent. We assume condition (H) for technical convenience only. Condition (3.23) amounts to forbidding other non-hyperbolic pointson $\overline{\partial\mathcal{D} \cap \{\rho = 0\}}$, beside the point $(\rho, u) = (0,0)$. Condition (3.24) reflects the natural monotonicity requirements (i) and (ii) formulated about the microscopic models at the beginning of Section 3.2.

We are interested in the behavior of the pde near the isolated non-hyperbolic point $(\rho, u) = (0,0)$. The asymptotic expansion for $\rho + u^2 \ll 1$ of the macroscopic fluxes and their first partial derivatives is

$$\begin{aligned}
&\Psi(\rho,u) = \rho u\big(1 + \mathcal{O}(\rho + u^2)\big), \quad \Phi(\rho,u) = (\rho + \gamma u^2)\big(1 + \mathcal{O}(\rho + u^2)\big),\\
&\Psi_\rho(\rho,u) = u\big(1 + \mathcal{O}(\rho + u^2)\big), \quad \Phi_\rho(\rho,u) = 1 + \mathcal{O}(\rho + u^2),\\
&\Psi_u(\rho,u) = \rho\big(1 + \mathcal{O}(\rho + u^2)\big), \quad \Phi_u(\rho,u) = 2\gamma u\big(1 + \mathcal{O}(\rho + u^2)\big).
\end{aligned} \tag{3.25}$$

We are looking for "small solutions" of the pde (3.16): Let $\rho_0(x)$ and $u_0(x)$ be given profiles and assume that $\rho^\varepsilon(t,x)$, $u^\varepsilon(t,x)$ is solution of the pde (3.16) with initial condition

$$\rho^\varepsilon(0,x) = \varepsilon^2 \rho_0(x), \quad u^\varepsilon(0,x) = \varepsilon\, u_0(x).$$

Then, at least formally,

$$\varepsilon^{-2}\rho^\varepsilon(\varepsilon^{-1}t,x) \to \rho(t,x), \quad \varepsilon^{-1}u^\varepsilon(\varepsilon^{-1}t,x) \to u(t,x),$$

where $\rho(t,x)$, $u(t,x)$ is solution of the pde (3.1) with initial condition

$$\rho(0,x) = \rho_0(x), \quad u(0,x) = u_0(x).$$

3.3.2 The main result

The asymptotic computations of subsection 3.3.1 suggest the scaling under which we should derive the pde (3.1) as hydrodynamic limit: fix a (small) positive β and choose the scaling

	MICRO	MACRO
space	nx	x
time	$n^{1+\beta}t$	t
particle density	$n^{-2\beta}\rho$	ρ
'slope of the wall'	$n^{-\beta}u$	u

Ideally the result should be valid for $0 < \beta < 1/2$ but we are able to prove much less than that.

Choose a model satisfying the conditions (A)-(F) of section 3.2 and conditions (G-H) of subsection 3.3.1, and let γ be given by (3.21), corresponding to the microscopic system chosen. Let the microscopic system of size n (defined on the discrete torus $\mathbb{T}^n$) evolve (on macroscopic time scale) according to the infinitesimal generator

$$G^n = n^{1+\beta}L^n + n^{1+\beta+\delta}K^n.$$

with $\beta > 0$ and some further conditions to be imposed on β and δ (see Theorem 2). Denote by μ_t^n the true distribution of the microscopic system at macroscopic time t:

$$\mu_t^n := \mu_0^n e^{tG^n},$$

where μ_0^n is the initial distribution.

We use the translation invariant product measure

$$\pi^n := \pi^n_{n^{-2\beta},0}$$

as *absolute reference measure.* Global entropy will be considered relative to this measure, Radon-Nikodym derivatives of μ_t^n and the local equilibrium measure ν_t^n to be defined below, with respect to π^n will be used.

Given a smooth solution $\big(\rho(t,x), u(t,x)\big)$, $(t,x) \in [0,T] \times \mathbb{T}$, of the pde (3.1) define the *local equilibrium measure* ν_t^n on Ω^n as follows

$$\nu_t^n := \prod_{j\in\mathbb{T}^n} \pi_{n^{-2\beta}\rho(t,\frac{j}{n}),n^{-\beta}u(t,\frac{j}{n})}. \tag{3.26}$$

This time-dependent measure *mimics on a microscopic level* the macroscopic evolution governed by the pde (3.1).

Our main result is the following

Theorem 2. *Assume that the microscopic system of interacting particles satisfies conditions (A)-(F) of subsubsections 3.2.2, 3.2.4 and the uniform log-Sobolev condition (I) of subsection*

3.6.2. Additionally, assume that the macroscopic fluxes satisfy conditions (G), (H) of subsection 3.3.1 and $\gamma > 1$. *Choose* $\beta \in (0, 1/2)$ *and* $\delta \in (1/2, 1)$ *so that*

$$2\delta - 8\beta > 1 \quad \text{and} \quad \delta + 3\beta < 1. \tag{3.27}$$

Let $\big(\rho(t,x), u(t,x)\big)$, $(t,x) \in [0,T] \times \mathbb{T}$, *be* smooth *solution of the pde (3.1), such that* $\inf_{x\in\mathbb{T}} \rho(0,x) > 0$ *and let* ν_t^n, $t \in [0,T]$ *be the corresponding local equilibrium measure defined in (3.26).*

Under these conditions, if

$$H\big(\mu_0^n \,\big|\, \nu_0^n\big) = o(n^{1-2\beta}) \tag{3.28}$$

then

$$H\big(\mu_t^n \,\big|\, \nu_t^n\big) = o(n^{1-2\beta}) \tag{3.29}$$

uniformly for $t \in [0,T]$.

Remarks:

(i) From (3.28) via the identity (3.35) and the entropy inequality it also follows that

$$H\big(\mu_0^n \,\big|\, \pi^n\big) = \mathcal{O}(n^{1-2\beta}). \tag{3.30}$$

See the beginning of subsection 3.4.2

(i) If $\gamma > 3/4$, in smooth solutions vacuum does not appear. That is $\inf_{x\in\mathbb{T}} \rho(0,x) > 0$ implies $\inf_{(t,x)\in[0,T]\times\mathbb{T}} \rho(t,x) > 0$.

(ii) Although for the $\{-1, 0, +1\}$-*model* we have $\gamma = 1$, our proof can also be extended to cover this model. Actually, in that case the proof is much simpler, since the Eulerian pde is equal to the limit pde (3.1) and thus the cutoff function (see Section 3.5) can be determined explicitly.

Corollary 2. *Assume the conditions of Theorem 2. Let* $g : \mathbb{T} \to \mathbb{R}$ *be a test function. Then for any* $t \in [0,T]$

(i)

$$n^{2\beta-1} \sum_{j\in\mathbb{T}^n} g(\frac{j}{n})\eta_j(t) \quad \stackrel{\mathbf{P}}{\to} \quad \int_{\mathbb{T}} g(x)\rho(t,x)\,dx,$$

$$n^{\beta-1} \sum_{j\in\mathbb{T}^n} g(\frac{j}{n})\zeta_j(t) \quad \stackrel{\mathbf{P}}{\to} \quad \int_{\mathbb{T}} g(x)u(t,x)\,dx.$$

(ii)

$$H\big(\mu_0^n \,\big|\, \pi^n\big) - H\big(\mu_t^n \,\big|\, \pi^n\big) = o(n^{1-2\beta}).$$

See the proof of Corollary 1 in [35].

3.4 Notations and general preparatory computations

This section completely standard in the context of the relative entropy method. So we shall be sketchy.

3.4.1 Notation

We denote

$$\begin{aligned} h^n(t) &:= n^{-(1-2\beta)} H\big(\mu_t^n \,\big|\, \nu_t^n\big). \\ s^n(t) &:= n^{-(1-2\beta)} \left(H\big(\mu_0^n \,\big|\, \pi^n\big) - H\big(\mu_t^n \,\big|\, \pi^n\big)\right). \end{aligned}$$

We know *a priori* that $t \mapsto s^n(t)$ is monotone increasing and due to (3.30)

$$s^n(t) = \mathcal{O}(1), \qquad \text{uniformly for } t \in [0, \infty). \tag{3.31}$$

In fact, from Theorem 2 it follows (see Corollary 2) that as long as the solution $\rho(t, x), u(t, x)$ of the pde (3.1) is smooth

$$s^n(t) = o(1), \qquad \text{uniformly for } t \in [0, T].$$

For $(\rho, u) \in (0, \infty) \times (-\infty, \infty)$ denote

$$\begin{aligned} \widetilde{\tau}^{(n)}(\rho, u) &:= \tau(n^{-2\beta}\rho, n^{-\beta}u) - \tau(n^{-2\beta}, 0) \\ \theta^n(\rho, u) &:= n^{\beta}\theta(n^{-2\beta}\rho, n^{-\beta}u). \end{aligned}$$

Note that, for symmetry reasons $\theta(n^{-2\beta}, 0) = 0$. Mind that τ is chemical potential rather than fugacity and for small densities the fugacity $\lambda := e^{\tau}$ scales like ρ, i.e. $\tau(n^{-2\beta}, 0) \sim -2\beta \log n$. If $\rho > 0$ and $u \in \mathbb{R}$ are fixed then $\widetilde{\tau}^{(n)}(\rho, u)$ and $\theta^n(\rho, u)$ stay of order 1, as $n \to \infty$.

Given the smooth solution $\rho(t, x), u(t, x)$, with $\rho(t, x) > 0$ we shall use the notation

$$\begin{aligned} \widetilde{\tau}^{(n)}(t, x) &:= \widetilde{\tau}^{(n)}(\rho(t, x), u(t, x)), \\ \theta^n(t, x) &:= \theta^n(\rho(t, x), u(t, x)), \\ v(t, x) &:= \log \rho(t, x). \end{aligned}$$

The following asymptotics hold uniformly in $(t, x) \in [0, T] \times \mathbb{T}$:

$$\begin{aligned} \widetilde{\tau}^{(n)}(t, x) &= v(t, x) + \mathcal{O}(n^{-2\beta}), & \theta^n(t, x) &= u(t, x) + \mathcal{O}(n^{-2\beta}) \\ \partial_x \widetilde{\tau}^{(n)}(t, x) &= \partial_x v(t, x) + \mathcal{O}(n^{-2\beta}), & \partial_x \theta^n(t, x) &= \partial_x u(t, x) + \mathcal{O}(n^{-2\beta}) \\ \partial_t \widetilde{\tau}^{(n)}(t, x) &= \partial_t v(t, x) + \mathcal{O}(n^{-2\beta}), & \partial_t \theta^n(t, x) &= \partial_t u(t, x) + \mathcal{O}(n^{-2\beta}) \end{aligned} \tag{3.32}$$

The logarithm of the Radom-Nikodym derivative of the time dependent reference measure ν_t^n with respect to the absolute referencee measure π^n is denoted by f_t^n:

$$\begin{aligned} f_t^n(\underline{\omega}) &:= \log \frac{d\nu_t^n}{d\pi^n}(\underline{\omega}) \\ &= \sum_{j\in\mathbb{T}^n} \Big\{ \widetilde{\tau}^{(n)}(t,\frac{j}{n})\eta_j + n^{-\beta}\theta^n(t,\frac{j}{n})\zeta_j \\ &\qquad -G(\widetilde{\tau}^{(n)}(t,\frac{j}{n}) + \tau(n^{-2\beta},0), n^{-\beta}\theta^n(t,\frac{j}{n})) + G(\tau(n^{-2\beta},0),0) \Big\} \end{aligned} \tag{3.33}$$

3.4.2 Preparatory computations

In order to obtain the main estimate (3.29) our aim is to get a Grönwall type inequality: we will prove that for every $t \in [0,T]$

$$h^n(t) - h^n(0) = \int_0^t \partial_t h^n(s) ds \le C \int_0^t h^n(s) ds + o(1), \tag{3.34}$$

where the error term is uniform in $t \in [0,T]$. Because it is assumed that $h^n(0) = o(1)$, the Theorem follows.

We start with the identity

$$H(\mu_t^n|\nu_t^n) - H(\mu_t^n|\pi^n) = -\int_{\Omega^n} f_t^n \, d\mu_t^n. \tag{3.35}$$

From this identity, the explicit form of the Radon-Nikodym derivative (3.33), the asymptotics (3.32), via the entropy inequality and (3.28) the a priori entropy bound (3.30) follows indeed, as remarked after the formulation of Theorem 2.

Next we differentiate (3.35) to obtain

$$\partial_t h^n(t) = -\int_{\Omega^n} \left(n^{3\beta} L^n f_t^n + n^{3\beta+\delta} K^n f_t^n + n^{-1+2\beta}\partial_t f_t^n \right) d\mu_t^n - \partial_t s^n(t). \tag{3.36}$$

Usually, an adjoint version of (3.36) is being used in form of an inequality. In our case this form is needed. We emphasize that the term $-\partial_t s^n(t)$ on the right hand side will be of crucial importance.

We compute the three terms under the integral.

$$\begin{aligned} n^{3\beta} L^n f_t^n(\underline{\omega}) &= \frac{1}{n}\sum_{j\in\mathbb{T}^n} \partial_x v(t,\frac{j}{n}) n^{3\beta}\psi_j + \frac{1}{n}\sum_{j\in\mathbb{T}^n} \partial_x u(t,\frac{j}{n}) n^{2\beta}\phi_j \\ &\quad + A_1^n(t,\underline{\omega}) + A_2^n(t,\underline{\omega}) + A_3^n(t,\underline{\omega}) + A_4^n(t,\underline{\omega}), \end{aligned} \tag{3.37}$$

where $A_i^n(t,\underline{\omega})$, $i=1,\dots,4$ are error terms which will be easy to estimate:

$$\begin{aligned}
A_1^n(t,\underline{\omega}) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n})-\partial_x v(t,\frac{j}{n})\big)n^{3\beta}\psi_j,\\
A_2^n(t,\underline{\omega}) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\partial_x\theta^n(t,\frac{j}{n})-\partial_x u(t,\frac{j}{n})\big)n^{2\beta}\phi_j,\\
A_3^n(t,\underline{\omega}) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\nabla^n\widetilde{\tau}^{(n)}(t,\frac{j}{n})-\partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n})\big)n^{3\beta}\psi_j\\
A_4^n(t,\underline{\omega}) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\nabla^n\theta^n(t,\frac{j}{n})-\partial_x\theta^n(t,\frac{j}{n})\big)n^{2\beta}\phi_j.
\end{aligned}$$

Here and in the sequel ∇^n denotes the discrete gradient:

$$\nabla^n f(x) := n\big(f(x+1/n)-f(x)\big).$$

See subsection 3.4.4 for the estimate of the error terms $A_j^n(t,\underline{\omega})$, $j=1,\dots,12$.

Next,

$$\begin{aligned}
n^{3\beta+\delta}K^n f_t^n(\underline{\omega}) &= n^{-1+3\beta+\delta}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\left(\nabla^n\right)^2\widetilde{\tau}^{(n)}(t,\frac{j}{n})\kappa_j+\left(\nabla^n\right)^2\theta^n(t,\frac{j}{n})\chi_j\big)\\
&=: A_5^n(t,\underline{\omega})
\end{aligned}\tag{3.38}$$

is itself a numerical error term. Finally

$$\begin{aligned}
n^{-1+2\beta}\partial_t f_t^n(\underline{\omega}) &= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\eta_j-\rho(t,\frac{j}{n})\big)\\
&\quad+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\zeta_j-u(t,\frac{j}{n})\big)\\
&\quad+A_6^n(t,\underline{\omega})+A_7^n(t,\underline{\omega}),
\end{aligned}\tag{3.39}$$

where

$$\begin{aligned}
A_6^n(t,\underline{\omega}) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\partial_t\tau^n(t,\frac{j}{n})-\partial_t v(t,\frac{j}{n})\big)\big(n^{2\beta}\eta_j-\rho(t,\frac{j}{n})\big),\\
A_7^n(t,\underline{\omega}) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big(\partial_t\theta^n(t,\frac{j}{n})-\partial_t u(t,\frac{j}{n})\big)\big(n^{\beta}\zeta_j-u(t,\frac{j}{n})\big).
\end{aligned}$$

are again easy-to-estimate error terms.

3.4.3 Blocks

We fix once and for all a weight function $a:\mathbb{R}\to\mathbb{R}$. It is assumed that:

(1) $a(x)>0$ for $x\in(-1,1)$ and $a(x)=0$ otherwise,

(2) it has total weight $\int a(x)\,dx = 1$,
(3) it is even: $a(-x) = a(x)$, and
(4) it is twice continuously differentiable.

We choose a *mesoscopic* block size $l = l(n)$ such that

$$1 \ll n^{(1+\delta+5\beta)/3} \ll l(n) \ll n^{\delta-\beta} \ll n. \tag{3.40}$$

This can be done due to condition (3.27) imposed on β and δ.

Given a local variable (depending on m consecutive spins)

$$\xi_i = \xi_i(\underline{\omega}) = \xi(\omega_i, \dots, \omega_{i+m-1}),$$

its *block average at macroscopic space coordinate* x is defined as

$$\widehat{\xi}^n(x) = \widehat{\xi}^n(\underline{\omega}, x) := \frac{1}{l}\sum_j a\left(\frac{nx-j}{l}\right)\xi_j. \tag{3.41}$$

Since $l = l(n)$, we do not denote explicitly dependence of the block average on the mesoscopic block size l.

Note that $x \mapsto \widehat{\xi}^n(x)$ is smooth

$$\partial_x \widehat{\xi}^n(x) = \partial_x \widehat{\xi}^n(\underline{\omega}, x) = \frac{n}{l}\frac{1}{l}\sum_j a'\left(\frac{nx-j}{l}\right)\xi_j,$$

and it is straightforward that

$$\sup_{\underline{\omega}\in\Omega^n}\sup_{x\in\mathbb{T}}\left|\partial_x\widehat{\xi}^n(\underline{\omega}, x)\right| \le C\left(\max_{\omega_1,\dots\omega_m}\xi(\omega_1,\dots,\omega_m)\right)\frac{n}{l}. \tag{3.42}$$

For a more sophisticated bound on $\left|\partial_x\widehat{\xi}^n(\underline{\omega}, x)\right|$ see (3.108).

We shall use the handy (but slightly abused) notation

$$\widehat{\xi}^n(t, x) := \widehat{\xi}^n(\mathcal{X}^n_t, x).$$

This is the empirical block average process of the local observable ξ_i.

For the scaled block average of the two conserved quantities we shall also use the notation

$$\widehat{\rho}^n(t,x) := n^{2\beta}\widehat{\eta}^n(t,x), \qquad \widehat{u}^n(t,x) := n^{\beta}\widehat{\zeta}^n(t,x).$$

Introducing block averages the main terms on the right hand side of (3.37) and (3.39) become:

$$\begin{aligned}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})n^{3\beta}\psi_j + \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})n^{2\beta}\phi_j = \\ \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})n^{3\beta}\widehat{\psi}^n(\frac{j}{n}) + \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})n^{2\beta}\widehat{\phi}^n(\frac{j}{n}) \\ + A^n_8(t,\underline{\omega}) + A^n_9(t,\underline{\omega}),\end{aligned} \tag{3.43}$$

respectively

$$\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\eta_j-\rho(t,\frac{j}{n})\big)+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\zeta_j-u(t,\frac{j}{n})\big)=$$

$$\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t v(t,\frac{j}{n})\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n})-\rho(t,\frac{j}{n})\big) \quad (3.44)$$

$$+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_t u(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n})-u(t,\frac{j}{n})\big)$$

$$+A_{10}^n(t,\underline{\omega})+A_{11}^n(t,\underline{\omega}),$$

where the error terms are

$$A_8^n(t,\underline{\omega}):=\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_x v\big(t,\frac{j}{n}\big)-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_x v\big(t,\frac{k}{n}\big)\Big)n^{3\beta}\psi_j,$$

$$A_9^n(t,\underline{\omega}):=\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_x u\big(t,\frac{j}{n}\big)-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_x u\big(t,\frac{k}{n}\big)\Big)n^{2\beta}\phi_j,$$

$$A_{10}^n(t,\underline{\omega}):=\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_t v\big(t,\frac{j}{n}\big)-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_t v\big(t,\frac{k}{n}\big)\Big)n^{2\beta}\eta_j,$$

$$A_{11}^n(t,\underline{\omega}):=\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big(\partial_t u\big(t,\frac{j}{n}\big)-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\partial_t u\big(t,\frac{k}{n}\big)\Big)n^{\beta}\zeta_j.$$

These error terms will be estimated in subsection 3.4.4.

Since $[0,T]\times\mathbb{T}\ni(t,x)\mapsto(\rho(t,x),u(t,x))$, is a *smooth* solution of the pde (3.1), we have

$$\partial_t v=-u\partial_x v-\partial_x u,\qquad \partial_t u=-\rho\partial_x v-\gamma u\partial_x u.$$

Inserting these expressions into the main terms of (3.44) eventually we obtain for the integrand in (3.36)

$$n^{3\beta}L^n f_t^n(\underline{\omega})+n^{3\beta+\delta}K^n f_t^n(\underline{\omega})+n^{-1+2\beta}\partial_t f_t^n(\underline{\omega})= \quad (3.45)$$

$$\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x v(t,\frac{j}{n})\Big\{n^{3\beta}\widehat{\psi}^n(\frac{j}{n})-\rho(t,\frac{j}{n})u(t,\frac{j}{n})$$

$$-u(t,\frac{j}{n})\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n})-\rho(t,\frac{j}{n})\big)-\rho(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n})-u(t,\frac{j}{n})\big)\Big\}$$

$$+\frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x u(t,\frac{j}{n})\Big\{n^{2\beta}\widehat{\phi}^n(\frac{j}{n})-\big(\rho(t,\frac{j}{n})+\gamma u(t,\frac{j}{n})^2\big)$$

$$-\big(n^{2\beta}\widehat{\eta}^n(\frac{j}{n})-\rho(t,\frac{j}{n})\big)-2\gamma u(t,\frac{j}{n})\big(n^{\beta}\widehat{\zeta}^n(\frac{j}{n})-u(t,\frac{j}{n})\big)\Big\}$$

$$+\sum_{k=1}^{12}A_k^n(t,\underline{\omega}),$$

where

$$
\begin{aligned}
A_{12}^n(t) &:= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\big((\partial_x v)\rho u + (\partial_x u)\big(\rho+\gamma u^2\big)\big)(t,\frac{j}{n}) \\
&= \frac{1}{n}\sum_{j\in\mathbb{T}^n}\partial_x\big(\rho u + \frac{\gamma}{3}u^3\big)(t,\frac{j}{n})
\end{aligned}
$$

3.4.4 The error terms A_k^n, $k = 1, \ldots, 12$

Lemma 2. *There exists a finite constant C, such that for any $n \in \mathbb{N}$, $t \in [0, T]$ and for any sequence of real numbers b_j, $j = 1, \ldots, n$ the following bounds hold:*

$$
\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j\eta_j\Big) \le Cn^{-2\beta}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j + \sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j^2}\right) \tag{3.46}
$$

$$
\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j\zeta_j\Big) \le Cn^{-\beta}\sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j^2} \tag{3.47}
$$

$$
\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j\psi_j\Big) \le Cn^{-2\beta}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j + \sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j^2}\right) \tag{3.48}
$$

$$
\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j\phi_j\Big) \le C\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j + n^{-\beta}\sqrt{\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j^2}\right) \tag{3.49}
$$

Proof. The proof relies on the entropy inequality

$$
\begin{aligned}
&\mathbf{E}_{\mu_t^n}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} b_j\big(\xi_j - \mathbf{E}_{\pi^n}(\xi_j)\big)\Big) \le \\
&\qquad \frac{1}{\gamma n}H(\mu_t^n|\pi^n) + \frac{1}{\gamma n}\log\mathbf{E}_{\pi^n}\Big(\exp\Big\{\gamma\sum_{j\in\mathbb{T}^n} b_j\big(\xi_j - \mathbf{E}_{\pi^n}(\xi_j)\big)\Big\}\Big),
\end{aligned} \tag{3.50}
$$

where ξ_j stands for either of η_j, ζ_j, ψ_j or ϕ_j. We note that all these variables are bounded and

$$
\begin{aligned}
&\big|\,\mathbf{E}_{\pi^n}\big(\eta_j\big)\,\big| \le Cn^{-2\beta}, &&\mathbf{Var}_{\pi^n}\big(\eta_j\big) \le Cn^{-2\beta},\\
&\big|\,\mathbf{E}_{\pi^n}\big(\zeta_j\big)\,\big| = 0, &&\mathbf{Var}_{\pi^n}\big(\zeta_j\big) \le C,\\
&\big|\,\mathbf{E}_{\pi^n}\big(\psi_j\big)\,\big| = 0, &&\mathbf{Var}_{\pi^n}\big(\psi_j\big) \le Cn^{-2\beta},\\
&\big|\,\mathbf{E}_{\pi^n}\big(\phi_j\big)\,\big| \le C, &&\mathbf{Var}_{\pi^n}\big(\phi_j\big) \le C.
\end{aligned}
$$

From these bounds and the entropy inequality (3.50) the statement of the lemma follows directly. □

Now we turn to the estimates on the error terms. We use the bounds (3.46), (3.47), (3.48) and (3.49) of Lemma 2, the asymptotics (3.32) and uniform approximation of ∂_x of smooth functions by their discrete derivative ∇^n. Straightforward computations yield

$$\mathbf{E}_{\mu_t^n}\big(A_1^n(t)\big) \le C\big(n^{-1+\beta} + n^{-\beta}\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_2^n(t)\big) \le C\big(n^{-1+2\beta} + n^{-\beta}\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_3^n(t)\big) \le Cn^{-1+\beta} = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_4^n(t)\big) \le C\big(n^{-1+2\beta} + n^{-1+\beta}\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_5^n(t)\big) \le C\big(n^{-1+\beta+\delta} + n^{-1+2\beta+\delta}\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_6^n(t)\big) \le Cn^{-2\beta} = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_7^n(t)\big) \le Cn^{-2\beta} = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_8^n(t)\big) \le C\big(n^{-1+\beta} + n^{\beta}l^{-1} + n^{-1+\beta}l\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_9^n(t)\big) \le C\big(n^{-1+2\beta} + n^{\beta}l^{-1} + n^{-1+\beta}l\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_{10}^n(t)\big) \le C\big(n^{-1} + l^{-1} + n^{-1}l\big) = o(1),$$

$$\mathbf{E}_{\mu_t^n}\big(A_{11}^n(t)\big) \le C\big(l^{-1} + n^{-1}l\big) = o(1).$$

Finally, $A_{12}^n(t)$ is a simple numerical error term (no probability involved):

$$A_{12}^n(t) \le Cn^{-1} = o(1).$$

3.4.5 Sumup

Thus, integrating (3.36), using (3.45) and the bounds of subsection 3.4.4 we obtain

$$h^n(t) = \int_0^t \mathcal{A}^n(s)\,ds + \int_0^t \mathcal{B}^n(s)\,ds - s^n(t) + o(1), \tag{3.51}$$

where

$$\mathcal{A}^n(t) := \mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \Big\{\big(\partial_x v\big)\big\{n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u)\big\}\Big\}(t, \frac{j}{n})\Big) \tag{3.52}$$

and

$$\mathcal{B}^n(t) := \mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n} \Big\{\big(\partial_x u\big)\big\{n^{2\beta}\widehat{\phi}^n - (\rho + \gamma u^2) - (\widehat{\rho}^n - \rho) - 2\gamma u(\widehat{u}^n - u)\big\}\Big\}(t, \frac{j}{n})\Big) \tag{3.53}$$

The main difficulty is caused by $\mathcal{A}^n(t)$. The term $\mathcal{B}^n(t)$ is estimated exactly as it is done in [34] for the one-component systems: since $\Phi(\rho, u) = \rho + \gamma u^2$ is linear in ρ and quadratic in u no problem is caused by the low particle density. By repeating the arguments of [34] we obtain

$$\int_0^t \mathcal{B}^n(s)ds \le C\int_0^t h^n(s)ds + o(1). \tag{3.54}$$

In the rest of the proof we concentrate on the essentially difficult term $\mathcal{A}^n(t)$.

3.5 Cutoff

We define the *rescaled macroscopic fluxes*

$$\Psi^n(\rho,u) := n^{3\beta}\Psi(n^{-2\beta}\rho, n^{-\beta}u), \qquad \Phi^n(\rho,u) := n^{2\beta}\Phi(n^{-2\beta}\rho, n^{-\beta}u). \tag{3.55}$$

defined on the scaled domain

$$\mathcal{D}^n := \{(\rho,u) : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}\}. \tag{3.56}$$

The first partial derivatives of the scaled fluxes are

$$\begin{aligned} \Psi^n_\rho(\rho,u) &= n^{\beta}\Psi_\rho(n^{-2\beta}\rho, n^{-\beta}u), & \Phi^n_\rho(\rho,u) &= \Phi_\rho(n^{-2\beta}\rho, n^{-\beta}u), \\ \Psi^n_u(\rho,u) &= n^{2\beta}\Psi_u(n^{-2\beta}\rho, n^{-\beta}u), & \Phi^n_u(\rho,u) &= n^{\beta}\Phi_u(n^{-2\beta}\rho, n^{-\beta}u). \end{aligned} \tag{3.57}$$

For any $(\rho,u) \in \mathbb{R}_+ \times \mathbb{R}$

$$\begin{aligned} \lim_{n\to\infty} \Psi^n(\rho,u) &= \rho u, & \lim_{n\to\infty} \Phi^n(\rho,u) &= \rho + \gamma u^2, \\ \lim_{n\to\infty} \Psi^n_\rho(\rho,u) &= u, & \lim_{n\to\infty} \Phi^n_\rho(\rho,u) &= 1, \\ \lim_{n\to\infty} \Psi^n_u(\rho,u) &= \rho, & \lim_{n\to\infty} \Phi^n_u(\rho,u) &= 2\gamma u. \end{aligned} \tag{3.58}$$

The convergence is uniform in compact subsets of $\mathbb{R}_+ \times \mathbb{R}$

Note that

$$\Psi^n(\widehat{\rho}^n(t,x), \widehat{u}^n(t,x)) = n^{3\beta}\Psi(\widehat{\eta}^n(t,x), \widehat{\zeta}^n(t,x)),$$
$$\Phi^n(\widehat{\rho}^n(t,x), \widehat{u}^n(t,x)) = n^{2\beta}\Phi(\widehat{\eta}^n(t,x), \widehat{\zeta}^n(t,x)).$$

3.5.1 The direct approach — why it fails?

The most natural thing is to write the summand in $\mathcal{A}^n(t)$ as

$$\begin{aligned} n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u) = \\ n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)) + (\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n) + (\widehat{\rho}^n - \rho)(\widehat{u}^n - u) \end{aligned} \tag{3.59}$$

By applying Varadhan's "one block estimate" and controlling the error terms in the Taylor expansion of Ψ, the first two terms on the right hand side can be dealt with. However, the last term causes serious problems: with proper normalization, it is distributed with respect to the local equilibrium measure ν^n_t, like a product of independent Poisson and Gaussian random variables, and thus it does *not* have a finite exponential moment. Since the robust estimates heavily rely on the entropy inequality where the finite exponential moment is needed, we have to choose another approach for estimating $\mathcal{A}^n(t)$.

Instead of writing plainly (3.59), we introduce a cutoff. We let

$$M > \sup\{\rho(t,x) \vee |u(t,x)| : (t,x) \in [0,T] \times \mathbb{T}\}.$$

The value of M will be specified by the large deviation bounds given in Proposition 2 (via Lemma 10).

Let $I^n, J^n : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded functions so that

$$\begin{aligned} &I^n + J^n = 1, \\ &I^n(\rho, u) = 1 \quad \text{for} \quad \rho \vee |u| \leq M, \\ &I^n(\rho, u) = 0 \quad \text{for 'large' } (\rho, u). \end{aligned}$$

The last property will be specified later.

We split the right hand side of (3.59) in a most natural way, according to this cutoff:

$$\begin{aligned} n^{3\beta}\widehat{\psi}^n - \rho u - u(\widehat{\rho}^n - \rho) - \rho(\widehat{u}^n - u) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.60) \\ n^{3\beta}\widehat{\psi}^n J^n(\widehat{\rho}^n, \widehat{u}^n) - (\widehat{\rho}^n u + \rho\widehat{u}^n - \rho u) J^n(\widehat{\rho}^n, \widehat{u}^n) \\ + n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n)) I^n(\widehat{\rho}^n, \widehat{u}^n) \\ + (\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n) I^n(\widehat{\rho}^n, \widehat{u}^n) + (\widehat{\rho}^n - \rho)(\widehat{u}^n - u) I^n(\widehat{\rho}^n, \widehat{u}^n) \end{aligned}$$

The second term on the right hand side is linear in the block averages, so it does not cause any problem. The third term is estimated by use of Varadhan's one block estimate. The fourth term is Taylor approximation. Finally, the last term can be handled with the entropy inequality *if the cutoff $I^n(\rho, u)$ is strong enough* to tame the tail of the Gaussian×Poisson random variable.

The main difficulty is caused by the first term on the right hand side. This term certainly can not be estimated with the robust method, i.e. with entropy inequality: we would run into the same problem we wanted to overcome by introducing the cutoff. The only way this term may be small is by some cancellation. It turns out that the desired cancellations indeed occur (in form of a martingale appearing in the space-time average) if and only if

$$J^n(\rho, u) = S^n_\rho(\rho, u), \tag{3.61}$$

where $S^n(\rho, u)$ is a particular *Lax entropy of the scaled Euler equation*

$$\begin{cases} \partial_t \rho + \partial_x \Psi^n(\rho, u) = 0 \\ \partial_t u + \partial_x \Phi^n(\rho, u) = 0, \end{cases} \tag{3.62}$$

with $\Psi^n(\rho, u)$ and $\Phi^n(\rho, u)$ defined in (3.55). That is S^n is solution of the pde

$$\Psi^n_u S^n_{\rho\rho} + \left(\Phi^n_u - \Psi^n_\rho\right) S^n_{\rho u} - \Phi^n_\rho S^n_{uu} = 0. \tag{3.63}$$

3.5.2 The cutoff function

In the present subsection we construct the cutoff function (3.61) and we state some estimates related to it. These bounds will be of paramount importance in our further proof. They are stated in the technical Lemmas 3, 4 and 5. The proof of these lemmas is pure classical pde theory and it is postponed to section 3.9.

First, in subsubsection 3.5.2, we formulate our construction and estimates in terms of Lax entropies of the *unscaled* Euler equation (3.16). Then in subsubsection 3.5.2 we rescale these estimates in order to get the necessary bounds on S^n and its derivatives.

A Lax entropy/flux pair $S(\rho,u)$, $F(\rho,u)$ of the system (3.16) is solution of the system of pdes

$$F_\rho = \Psi_\rho S_\rho + \Phi_\rho S_u, \qquad F_u = \Psi_u S_\rho + \Phi_u S_u, \tag{3.64}$$

defined on $\mathcal{D}$. In particular the Lax entropy $S(\rho,u)$ solves the pde:

$$\Psi_u S_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big) S_{\rho u} - \Phi_\rho S_{uu} = 0, \tag{3.65}$$

The linear pde (3.65) is hyperbolic in $\mathcal{D}$. One family of its characteristic curves are solutions of the following ODE, meant in the domain $\mathcal{D}$:

$$\frac{d\rho}{du} = \frac{\sqrt{\big(\Phi_u - \Psi_\rho\big)^2 + 4\Phi_\rho\Psi_u} - \big(\Phi_u - \Psi_\rho\big)}{2\Phi_\rho}, \tag{3.66}$$

The other family is obtained by reflecting u to $-u$.

First we conclude that the line segment $\mathcal{D} \cap \{u = 0\}$ is *not* characteristic for the hyperbolic pde (3.65). That is: is intersects transversally the characteristic lines defined by the differential equation (3.66). Indeed, from the Onsager relation (3.15) and obvious parity considerations it follows, that the right hand side of (3.66) restricted to $\{u = 0\}$ becomes $\big(\mathbf{Var}_{r,0}(\eta)/\mathbf{Var}_{r,0}(\zeta)\big)^{1/2}$ and this expression is obviously finite for $r \in [0, \rho^*)$. It follows that the Cauchy problem (3.65), with the following initial condition:

$$S(r,0) = s(r), \quad S_u(r,0) = 0, \qquad r \in [0,\rho^*) \tag{3.67}$$

is *well posed.*

In our concrete problem the function $s(r)$ will be chosen as follows: we fix $0 < \underline{r} < \overline{r}$, and define

$$s(r) = \begin{cases} 0 & \text{if } r \in [0, \underline{r}), \\ \dfrac{r\log(r/\underline{r}) - (r - \underline{r})}{\log(\overline{r}/\underline{r})} & \text{if } r \in [\underline{r}, \overline{r}), \\ r - \dfrac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})} & \text{if } r \in [\overline{r}, \infty). \end{cases} \tag{3.68}$$

Note that $s(r)$ and $s'(r)$ are continuous.

We first analyze the global structure of the characteristic curves. Due to the assumption (H) imposed on-, and regularity of the flux functions Φ and Ψ, there exists some $\rho_0 > 0$ such that the ODE (3.66) is regular in $\{(\rho, u) \in \mathcal{D} : \rho < \rho_0 \text{ and } (\rho, u) \neq (0,0)\}$. We shall not be concerned about what happens outside this strip. Denote by $\sigma(u;r)$ the solution of the ODE (3.66) with initial condition $\sigma(0;r) = r$.

Lemma 3. *There exist constants* $0 < C_2 < C_1 < \infty$ *and* $r_0 > 0$ *such that for any* $r \in [0, r_0]$

$$\begin{aligned} r + C_1\sqrt{r}u \leq \sigma(u;r) \leq r + C_2\sqrt{r}u \qquad & \textit{if } u \leq 0, \\ r \leq \sigma(u;r) \leq r + C_1\left(\sqrt{r}u \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u^{\frac{1}{2\gamma-1}}\right) \qquad & \textit{if } u \geq 0. \end{aligned} \tag{3.69}$$

The inequalities are valid as long as $(\sigma(u;r), u) \in \mathcal{D}$. *The map* $u \mapsto \sigma(u;r)$ *is regular and monotone increasing.*

See subsection 3.9.1 for the proof of this lemma.

For $r < r_0$ we partition the domain $\mathcal{D}$ in three parts as follows

$$\begin{aligned} \mathcal{D}_1(r) &:= \{(\rho, u) \in \mathcal{D} : \rho < \sigma(-|u|; r)\} \\ \mathcal{D}_2(r) &:= \{(\rho, u) \in \mathcal{D} : \rho > \sigma(|u|; r)\} \\ \mathcal{D}_3(r) &:= \{(\rho, u) \in \mathcal{D} : \sigma(-|u|; r) \leq \rho \leq \sigma(|u|; r)\} \\ &= \mathcal{D} \setminus \big(\mathcal{D}_1(r) \cup \mathcal{D}_2(r)\big). \end{aligned}$$

See Figure 1 for a sketch of the domains $\mathcal{D}_1(r), \mathcal{D}_2(r), \mathcal{D}_3(r)$.

From Lemma 3 it follows that

$$\begin{aligned} &\{(\rho, u) : 0 \leq \rho < r - C_1\sqrt{r}|u|\} \subset \mathcal{D}_1(r) \subset \{(\rho, u) : 0 \leq \rho < r - C_2\sqrt{r}|u|\}, \\ &\{(\rho, u) : r + C_1\sqrt{r}|u| < \rho\} \subset \mathcal{D}_2(r) \subset \{(\rho, u) : r < \rho\}, \end{aligned} \tag{3.70}$$

and for $0 \leq r \leq r' \leq r_0$

$$\mathcal{D}_1(r) \subset \mathcal{D}_1(r'), \qquad \mathcal{D}_2(r') \subset \mathcal{D}_2(r). \tag{3.71}$$

From now r_0 is *fixed for ever* and we denote

$$\widetilde{\mathcal{D}} := \mathcal{D}_1(r_0).$$

This domain is a *rectangle* in characteristic coordinates with diagonal $\widetilde{\mathcal{D}} \cap \{u = 0\}$, as opposed to $\mathcal{D}$ which may not be a full characteristic rectangle. (Actually, choosing the characteristic coordinates in a natural symmetric way, $z(\rho, u) = w(\rho, -u)$, the domain $\widetilde{\mathcal{D}}$ is a square in characteristic coordinates.) Note that $\mathcal{D}_3(r_0) \cap \{u \geq 0\}$ and $\mathcal{D}_3(r_0) \cap \{u \leq 0\}$ are also characteristic rectangles.

Next we turn to the construction of a particular family of Lax entropies which will serve for obtaining the cutoff functions needed. We fix $0 < \underline{r} < \overline{r} < r_0$. and define $S : \mathcal{D} \to \mathbb{R}$ as follows:

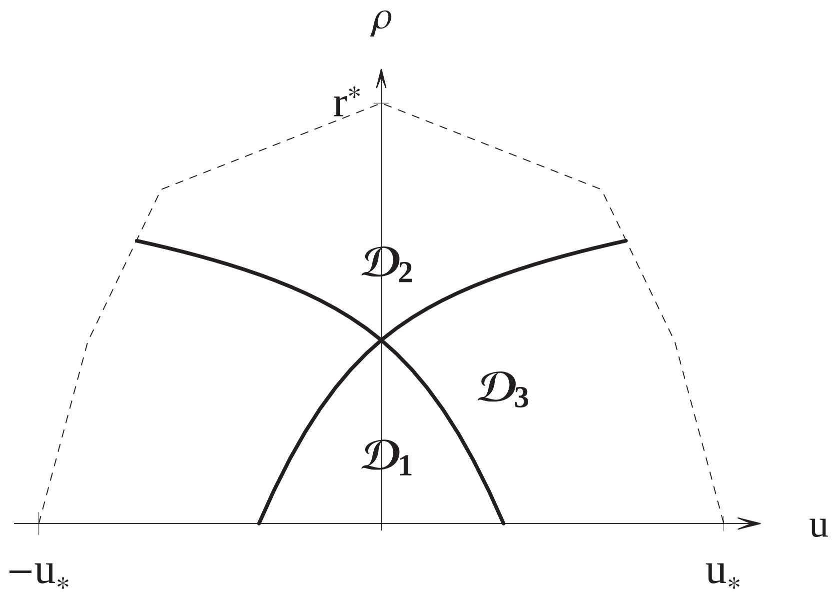


Figure 3.1: $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$

(i) In $\widetilde{\mathcal{D}}$: $S(\rho, u)$ is solution of the Cauchy problem (3.65)+(3.67) with $s(r)$ given in (3.68). Note that

$$S(\rho, u) = \begin{cases} 0 & \text{if } (\rho, u) \in \mathcal{D}_1(\underline{r}) \subset \widetilde{\mathcal{D}}, \\ \rho - \dfrac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})} & \text{if } (\rho, u) \in \mathcal{D}_2(\overline{r}) \cap \widetilde{\mathcal{D}}. \end{cases} \tag{3.72}$$

(ii) In $\mathcal{D}_2(\overline{r})$:

$$S(\rho, u) := \rho - \frac{\overline{r} - \underline{r}}{\log(\overline{r}/\underline{r})}, \qquad \text{if } (\rho, u) \in \mathcal{D}_2(\overline{r}) \tag{3.73}$$

Note that there is no contradiction: in $\widetilde{\mathcal{D}} \cap \mathcal{D}_2(\underline{r})$, (i) yields the same expression.

(iii) In $\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}$: $S(\rho, u)$ is defined as solution of the Goursat problem (3.65) with boundary conditions on the characteristic lines $\partial\widetilde{\mathcal{D}} \cap \mathcal{D}_3(\overline{r})$, respectively, $\partial\mathcal{D}_2(\overline{r}) \setminus \widetilde{\mathcal{D}}$ provided by (i), respectively, (ii).

Note that $S(\rho, u)$ is solution of the pde (3.65), *globally* in $\widetilde{\mathcal{D}}$.

We denote

$$\begin{aligned} \mathcal{D}_3(\underline{r}, \overline{r}) \quad &:= \quad \mathcal{D} \setminus \big(\mathcal{D}_1(\underline{r}) \cup \mathcal{D}_2(\overline{r})\big) \\ &= \quad \big(\mathcal{D}_1(\overline{r}) \cap \mathcal{D}_2(\underline{r})\big) \cup \big(\mathcal{D}_1(\overline{r}) \cap \mathcal{D}_3(\underline{r})\big) \\ &\qquad \cup \big(\mathcal{D}_3(\overline{r}) \cap \mathcal{D}_2(\underline{r})\big) \cup \big(\mathcal{D}_3(\overline{r}) \cap \mathcal{D}_3(\underline{r})\big). \end{aligned}$$

The following lemma provides the necessary bounds on the partial derivatives of $S(\rho, u)$ (up to second order) in the domain $\mathcal{D}_3(\underline{r}, \overline{r})$.

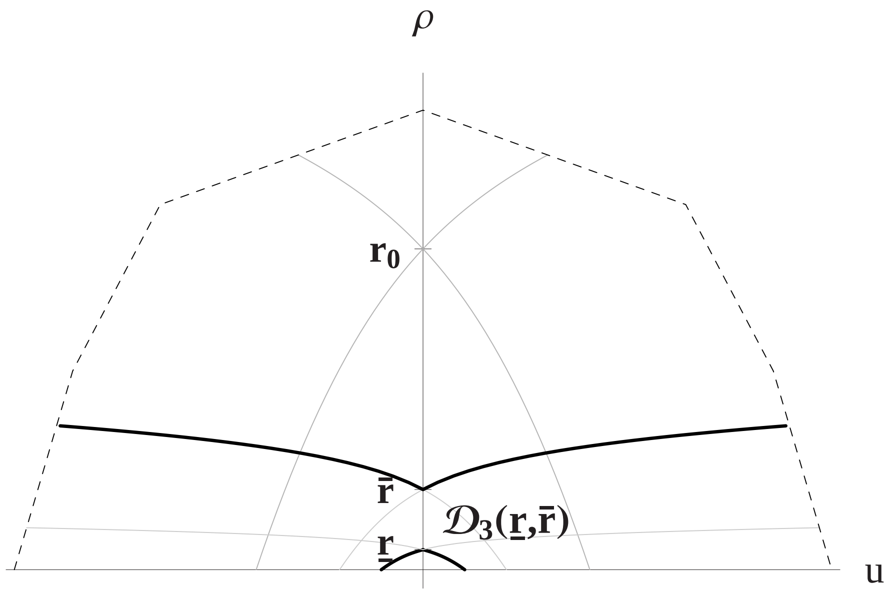


Figure 3.2: $\mathcal{D}_3(\underline{r},\overline{r})$

Lemma 4. *There exists a constant $C<\infty$ such that for any $0<\underline{r}<\overline{r}<r_0$ and $S(\rho,u)$ defined as above the following global bounds hold:*

$$|S_\rho(\rho,u)-\mathbb{1}_{\mathcal{D}_2(\overline{r})}(\rho,u)| \leq C\,\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{3.74}$$

$$|S_u(\rho,u)| \leq \frac{C(\sqrt{\overline{r}}-\sqrt{\underline{r}})}{\log(\overline{r}/\underline{r})}\,\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{3.75}$$

$$|S_{\rho\rho}(\rho,u)| \leq \frac{C}{\log(\overline{r}/\underline{r})}\,\frac{1}{\underline{r}+\rho}\,\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{3.76}$$

$$|S_{\rho u}(\rho,u)| \leq \frac{C}{\log(\overline{r}/\underline{r})}\,\frac{1}{\sqrt{\underline{r}}+\sqrt{\rho}+|u|}\,\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{3.77}$$

$$|S_{uu}(\rho,u)| \leq \frac{C}{\log(\overline{r}/\underline{r})}\,\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u), \tag{3.78}$$

This lemma is proved in subsection 3.9.2.

Beside the bounds on the partial derivatives of $S(\rho,u)$ we shall also need a bound on the function $F(\rho,u)-\Psi(\rho,u)S_\rho(\rho,u)$. From (3.64) and (3.72) it follows that

$$F(\rho,u)=\begin{cases} 0 & \text{if } (\rho,u)\in\mathcal{D}_1(\underline{r}),\\ \Psi(\rho,u) & \text{if } (\rho,u)\in\mathcal{D}_2(\overline{r}).\end{cases}$$

Lemma 5. *With the assumptions and notations of Lemma 4*

$$|F(\rho,u)-\Psi(\rho,u)S_\rho(\rho,u)| \leq \frac{C\sqrt{\overline{r}}}{\log(\overline{r}/\underline{r})}(\overline{r}+u^2)\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u). \tag{3.79}$$

See subsection 3.9.3 for the proof of this lemma.

The *scaled* functions $S^n(\rho, u)$, $F^n(\rho, u)$ are defined on the scaled domain $\mathcal{D}^n$ given in (3.56), as follows: fix $0 < \underline{r} < \overline{r} < \infty$ and define the *unscaled* Lax entropy/flux pair as in the previous section but with *downscaled* initial conditions

$$S(r, 0) = n^{-2\beta} s(n^{2\beta} r), \qquad S_u(r, 0) = 0. \qquad r \in [0, \rho^*), \tag{3.80}$$

with the function $r \mapsto s(r)$ given in (3.68). Now, define the pair of scaled functions $S^n, F^n : \mathcal{D}^n \to \mathbb{R}$ as

$$S^n(\rho, u) := n^{2\beta} S(n^{-2\beta}\rho, n^{-\beta}u), \qquad F^n(\rho, u) := n^{3\beta} F(n^{-2\beta}\rho, n^{-\beta}u). \tag{3.81}$$

It is straightforward to check that S^n, F^n form a Lax entropy/flux pair of the pde (3.62):

$$F^n_\rho = \Psi^n_\rho S^n_\rho + \Phi^n_\rho S^n_u, \qquad F^n_u = \Psi^n_u S^n_\rho + \Phi^n_u S^n_u, \tag{3.82}$$

in particular S^n solves the pde (3.63).

We partition the scaled domain

$$\mathcal{D}^n = \mathcal{D}^n_1(\underline{r}) \cup \mathcal{D}^n_2(\overline{r}) \cup \mathcal{D}^n_3(\underline{r}, \overline{r})$$

with the partition elements

$$\begin{aligned}
\mathcal{D}^n_1(\underline{r}) &:= \{(\rho, u) \in \mathcal{D}_n : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}_1(n^{-2\beta}\underline{r})\} \\
\mathcal{D}^n_2(\overline{r}) &:= \{(\rho, u) \in \mathcal{D}_n : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}_2(n^{-2\beta}\overline{r})\} \\
\mathcal{D}^n_3(\underline{r}, \overline{r}) &:= \{(\rho, u) \in \mathcal{D}_n : (n^{-2\beta}\rho, n^{-\beta}u) \in \mathcal{D}_3(n^{-2\beta}\underline{r}, n^{-2\beta}\overline{r})\}.
\end{aligned}$$

In the following Proposition we summarize our main estimates formulated in terms of the scaled objects. The statement is a mere corollary of the previous lemmas. It follows by simple scaling from (3.70) and (3.74)-(3.79).

Proposition 1. *There exists a constant $C < \infty$, such that given any $0 < \underline{r} < \overline{r} < \infty$ and the scaled Lax entropy/flux pair defined as prescribed above, the following bounds hold uniformly in n:*

$$\mathcal{D}^n_1(\underline{r}) \supset \{(\rho, u) : 0 \le \rho \le \underline{r} - C^{-1}\sqrt{\underline{r}}|u|\} \tag{3.83}$$

$$\mathcal{D}^n_3(\underline{r}, \overline{r}) \subset \{(\rho, u) : \rho \le \overline{r} + C\sqrt{\overline{r}}|u|\} \tag{3.84}$$

$$|S^n_\rho(\rho,u) - 1\!\!1_{\mathcal{D}^n_2(\overline{r})}(\rho,u)| \leq C\,1\!\!1_{\mathcal{D}^n_3(\underline{r},\overline{r})}(\rho,u), \tag{3.85}$$

$$|S^n_u(\rho,u)| \leq \frac{C(\sqrt{\overline{r}}-\sqrt{\underline{r}})}{\log(\overline{r}/\underline{r})}\,1\!\!1_{\mathcal{D}^n_3(\underline{r},\overline{r})}(\rho,u), \tag{3.86}$$

$$|S^n_{\rho\rho}(\rho,u)| \leq \frac{C}{\log(\overline{r}/\underline{r})}\,\frac{1}{\underline{r}+\rho}\,1\!\!1_{\mathcal{D}^n_3(\underline{r},\overline{r})}(\rho,u), \tag{3.87}$$

$$|S^n_{\rho u}(\rho,u)| \leq \frac{C}{\log(\overline{r}/\underline{r})}\,\frac{1}{\sqrt{\underline{r}}+\sqrt{\rho}+|u|}\,1\!\!1_{\mathcal{D}^n_3(\underline{r},\overline{r})}(\rho,u), \tag{3.88}$$

$$|S^n_{uu}(\rho,u)| \leq \frac{C}{\log(\overline{r}/\underline{r})}\,1\!\!1_{\mathcal{D}^n_3(\underline{r},\overline{r})}(\rho,u), \tag{3.89}$$

$$|F^n(\rho,u) - \Psi^n(\rho,u)S^n_\rho(\rho,u)| \leq \frac{C\sqrt{\overline{r}}}{\log(\overline{r}/\underline{r})}(\overline{r}+u^2)1\!\!1_{\mathcal{D}^n_3(\underline{r},\overline{r})}(\rho,u). \tag{3.90}$$

Due to (3.83) we can choose $\underline{r}$ large enough to ensure that for all n

$$\{(\rho,u) : \rho \vee |u| \leq M\} \subset \mathcal{D}^n_1(\underline{r}), \tag{3.91}$$

where M is specified by the large deviation bounds given in Proposition 2 (via Lemma 10).

Further on, we choose $\overline{r}$ so large that

$$\frac{C}{\log(\overline{r}/\underline{r})} < \left(100 \sup_{(t,x)\in[0,T]\times\mathbb{T}} |\log\rho(t,x)|\right)^{-1}, \tag{3.92}$$

and thus the bounds (3.87)-(3.89) of Proposition 1 are sufficient for our further purposes.

3.5.3 Outline of the further steps of proof

In section 3.6 we present the main probabilistic technical ingredients of the forthcoming proof. These are variants of entropy inequalities and of the celebrated one and two block estimates. In Section 3.7 we give an estimate for the terms with 'large' values of (ρ,u), we prove that

$$\begin{aligned}\left|\mathbf{E}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{\big(\partial_x v\big)\big(n^{3\beta}\widehat{\psi}^n\big)J^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,\frac{j}{n})\,ds\right)\right| \\ \leq \frac{1}{2}\,h^n(t) + \frac{1}{2}\,s^n(t) + C\int_0^t h^n(s)\,ds + o(1).\end{aligned} \tag{3.93}$$

In Section 3.8 we estimate the terms with 'small' values of (ρ,u), the section is divided into four subsections.

In subsection 3.8.1 we prove

$$\begin{aligned}\left|\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{\big(\partial_x v\big)\,\big(\widehat{\rho}^n u + \rho\widehat{u}^n - \rho u\big)\,J^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,\frac{j}{n})\right)\right| \\ \leq C\,h^n(s) + o(1).\end{aligned} \tag{3.94}$$

In subsection 3.8.2 we prove the one block estimate

$$\left|\mathbf{E}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{(\partial_x v)\, n^{3\beta}\big(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\, I^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,\frac{j}{n})\, ds\right)\right| = o(1). \tag{3.95}$$

In subsection 3.8.3 we control the Taylor approximation

$$\left|\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{(\partial_x v)\,\big(\Psi^n(\widehat{\rho}^n,\widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n\big)\, I^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,\frac{j}{n})\right)\right| \le C\, h^n(s) + o(1). \tag{3.96}$$

Finally, in subsection 3.8.4 we control the fluctuations

$$\left|\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{(\partial_x v)\,\big(\widehat{\rho}^n - \rho\big)\big(\widehat{u}^n - u\big)\, I^n(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,\frac{j}{n})\right)\right| \le C\, h^n(s) + o(1). \tag{3.97}$$

Having all these done, from (3.52), (3.60) and the bounds (3.93), (3.94), (3.95), (3.96), (3.97) it follows that

$$\int_0^t \mathcal{A}^n(s)ds \le \frac{1}{2}\, h^n(t) + \frac{1}{2}\, s^n(t) + C\int_0^t h^n(s)ds + o(1). \tag{3.98}$$

Finally, from (3.51), (3.54), (3.98) and noting that $s^n(t) \ge 0$ we get the desired Grönwall inequality (3.34) and the Theorem follows. Note the importance of the term $-\partial_t s^n(t)$ on the right hand side of (3.36).

3.6 Tools

3.6.1 Fixed time estimates

In the estimates with fixed time $s \in [0, T]$ we shall use the notation

$$L = L(n) := n^{-2\beta} l. \tag{3.99}$$

Note that $L \gg 1$ as $n \to \infty$.

The following general entropy estimate will be exploited all over:

Lemma 6. (Fixed time entropy inequality)
Let $l \le n$, $\mathcal{V} : \Omega^l \to \mathbb{R}$ and denote $\mathcal{V}_j(\underline{\omega}) := \mathcal{V}(\omega_j, \ldots, \omega_{j+l-1})$. Then for any $\gamma > 0$

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\mathcal{V}_j(\mathcal{X}^n_s)\Big) \le \frac{1}{\gamma}h^n(s) + \frac{1}{\gamma L}\frac{1}{n}\sum_{j\in\mathbb{T}^n}\log \mathbf{E}_{\nu^n_s}\left(\exp\{\gamma\, L\, \mathcal{V}_j\}\right). \tag{3.100}$$

This lemma is standard tool in the context of relative entropy method. For its proof we refer the reader to the original paper [41] or the monograph [13].

Proposition 2. (Fixed time large deviation bounds)
(i) For any $\varepsilon > 0$ there exists $M < \infty$ such that for any $s \in [0, T]$

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big\{\big(1+\widehat{\rho}^n+|\widehat{u}^n|\big)1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\big\}(s,\frac{j}{n})\Big) \le \varepsilon\, h^n(s)+o(1). \tag{3.101}$$

(ii) There exist $C < \infty$ and $M < \infty$ such that for any $s \in [0, T]$

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big\{\,|\widehat{u}^n|^2\,1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\big\}(s,\frac{j}{n})\Big) \le C\, h^n(s)+o(1). \tag{3.102}$$

The proof of Proposition 2 is postponed to subsection 3.10.1. It relies on the entropy inequality (3.100) of Lemma 6, the stochastic dominations formulated in Lemma 9 (see subsection 3.10.1) and standard large deviation bounds.

We shall refer to (3.101) and (3.102) as *large deviation bounds.*

Proposition 3. (Fixed time fluctuation bounds)
For any $M < \infty$ there exists a $C < \infty$ such that the following bounds hold:

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}|\widehat{u}^n-u|^2\,(s,\frac{j}{n})\Big) \le C\, h^n(s)+o(1), \tag{3.103}$$

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\big\{\,|\widehat{\rho}^n-\rho|^2\,1\!\!1_{\{\widehat{\rho}^n\le M\}}\big\}(s,\frac{j}{n})\Big) \le C\, h^n(s)+o(1). \tag{3.104}$$

The proof of Proposition 3 is postponed to subsection 3.10.2. It relies on the entropy inequality (3.100) of Lemma 6, and Gaussian fluctuation estimates.

We shall refer to (3.103) and (3.104) as *fluctuation bounds.*

3.6.2 Convergence to local equilibrium and a priori bounds

The hydrodynamic limit relies on macroscopically fast convergence to (local) equilibrium in blocks of mesoscopic size l. Fix the block size l and $(N, Z) \in \mathbb{N} \times (w_0/2)\mathbb{Z}$ with the restriction $N \in [0, l \max \eta]$, $Z \in [l \min \zeta, l \max \zeta]$ and denote

$$\begin{aligned}
\Omega^l_{N,Z} &:= \big\{\underline{\omega} \in \Omega^l \,:\, \sum_{j=1}^{l}\eta_j = N, \sum_{j=1}^{l}\zeta_j = Z\big\},\\
\pi^l_{N,Z}(\underline{\omega}) &:= \pi^l_{\lambda,\theta}(\underline{\omega}\,|\,\sum_{j=1}^{l}\eta_j = N. \sum_{j=1}^{l}\zeta_j = Z),
\end{aligned}$$

Expectation with respect to the measure $\pi^l_{N,Z}$ is denoted by $\mathbf{E}^l_{N,Z}\big(\,\cdot\,\big)$. For $f:\Omega^l_{N,Z}\to\mathbb{R}$ let

$$K^l_{N,Z}f(\underline{\omega}) := \sum_{j=1}^{l-1}\sum_{\omega',\omega''} s(\omega_j,\omega_{j+1};\omega',\omega'')\big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big),$$

$$D^l_{N,Z}(f) := \frac{1}{2}\sum_{j=1}^{l-1}\mathbf{E}^l_{N,Z}\left(\sum_{\omega',\omega''} s(\omega_j,\omega_{j+1};\omega',\omega'')\big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big)^2\right).$$

In plain words: $\Omega^l_{N,Z}$ is the hyperplane of configurations $\underline{\omega}\in\Omega^l$ with fixed values of the conserved quantities, $\pi^l_{N,Z}$ is the *microcanonical distribution* on this hyperplane, $K^l_{N,Z}$ is the symmetric infinitesimal generator restricted to the hyperplane $\Omega^l_{N,Z}$, and finally $D^l_{N,Z}$ is the Dirichlet form associated to $K^l_{N,Z}$. Note, that $K^l_{N,Z}$ is defined with *free boundary conditions.*

The convergence to local equilibrium is *quantitatively controlled* by the following uniform logarithmic Sobolev estimate, assumed to hold:

(I) *Logarithmic Sobolev inequality:* There exists a finite constant $\aleph$ such that for any $l\in\mathbb{N}$, $(N,Z)\in\mathbb{N}\times(w_0/2)\mathbb{Z}$ with the restriction $N\in[0,l\max\eta]$, $Z\in[l\min\zeta, l\max\zeta]$, and any $h:\Omega^l_{N,Z}\to\mathbb{R}_+$ with $\mathbf{E}^l_{N,Z}(h)=1$ the following bound holds:

$$\mathbf{E}^l_{N,Z}\big(h\log h\big)\le \aleph\, l^2 D^l_{N,Z}\left(\sqrt{h}\right). \tag{3.105}$$

Remark: The uniform logarithmic Sobolev inequality (3.105) is expected to hold for a very wide range of locally finite interacting particle systems, though we do not know about a fully general proof. In [42] the logarithmic Sobolev inequality is proved for symmetric K-exclusion processes. This implies that (3.105) holds for the two lane models defined in subsection 3.2. In [8] Yau's method of proving logarithmic Sobolev inequality is applied and the logarithmic Sobolev inequality is stated for random stirring models with arbitrary number of colors. In particular, (3.105) follows for the $\{-1,0,+1\}$-model defined in subsection 3.2.

The following large deviation bound goes back to Varadhan [39]. See also the monographs [13] and [6].

Lemma 7. (Time-averaged entropy inequality, local equilibrium)
Let $l\le n$, $\mathcal{V}:\Omega^l\to\mathbb{R}_+$ and denote $\mathcal{V}_j(\underline{\omega}):=\mathcal{V}(\omega_j,\dots,\omega_{j+l-1})$. Then for any $\gamma>0$

$$\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\int_0^t \mathcal{V}_j(\mathcal{X}^n_s)\,ds\right)\le \tag{3.106}$$
$$\frac{\aleph\, l^3}{2\,\gamma\, n^{1+3\beta+\delta}}\Big(s^n(t)+\frac{2\,n^{1+3\beta+\delta}\,t}{\aleph l^3}\max_{N,Z}\log\mathbf{E}^l_{N,Z}\big(\exp\{\gamma\mathcal{V}\}\big)\Big).$$

Remarks: (1) Since

$$\frac{n^{1+3\beta+\delta}}{l^3}=o(1),$$

in order to apply efficiently Lemma 7 one has to chose $\gamma = \gamma(n)$ so that

$$\mathbf{E}^l_{N,Z}\big(\exp\{\gamma\mathcal{V}\}\big) = \mathcal{O}(1),$$

uniformly in the block size $l = l(n) \in \mathbb{N}$, and in $N \in [0, l\max\eta]$ and $Z \in [l\min\zeta, l\max\zeta]$.
(2) Assuming only uniform bound of size $\sim \big(\aleph l^2\big)^{-1}$ on the spectral gap of $K^l_{N,Z}$ (rather than the stronger logarithmic Sobolev inequality (3.105)) and using Rayleigh-Schrödinger perturbation (see Appendix 3 of [13]) we would get

$$\mathbf{E}\left(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\int_0^t \mathcal{V}_j(\mathcal{X}^n_s)\,ds\right) \le$$

$$\frac{\aleph\, l^3\,\|\mathcal{V}\|_\infty}{2\,n^{1+3\beta+\delta}}s^n(t) + t\,\|\mathcal{V}\|_\infty\Big(\frac{\max_{N,Z}\mathbf{E}^l_{N,Z}(\mathcal{V})}{\|\mathcal{V}\|_\infty} + \frac{\max_{N,Z}\mathbf{Var}^l_{N,Z}(\mathcal{V})}{\|\mathcal{V}\|^2_\infty}\Big),$$

which would not be sufficient for our needs.
(3) The proof of the bound (3.106) explicitly relies on the logarithmic Sobolev inequality (3.105). It appears in [43] and it is reproduced in several places, see e.g. [6], [7]. We do not repeat it here.

The main probabilistic ingredients of our proof are summarized in Proposition 4 which is consequence of Lemma 7. These are variants of the celebrated *one block estimate*, respectively, *two blocks estimate* of Varadhan and co-authors.

Proposition 4. (Time-averaged block replacement and gradient bounds)
Given a local variable $\xi : \Omega^m \to \mathbb{R}$ there exists a constant C such that the following bounds hold:
(i)

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\big|\{\widehat{\xi}^n - \Xi(\widehat{\eta}^n,\widehat{\zeta}^n)\}(s,x)\big|^2\,dx\,ds\right) \le C\,\frac{l^2}{n^{1+3\beta+\delta}}\,\big(s^n(t)+o(1)\big). \tag{3.107}$$

(ii)

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\big|\partial_x\widehat{\xi}^n(s,x)\big|^2\,dx\,ds\right) \le Cn^{1-3\beta-\delta}\big(s^n(t)+o(1)\big). \tag{3.108}$$

(iii) Further on, if $\xi : \Omega \to \mathbb{R}$ (that is: it depends on a single spin) and $\xi(\omega) = 0$ whenever $\eta(\omega) = 0$ then the following stronger version of the gradient bound holds:

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\frac{\big|\partial_x\widehat{\xi}^n(s,x)\big|^2}{\widehat{\eta}^n(s,x)}\,dx\,ds\right) \le Cn^{1-3\beta-\delta}\big(s^n(t)+o(1)\big). \tag{3.109}$$

The proof of Proposition 4 is postponed to subsection 3.10.3. It relies on the large deviation bound (3.106) and some elementary probability estimates stated in Lemma 13 (see subsection 3.10.3).

We shall refer to (3.107), respectively, (3.108) and (3.109) as the *block replacement bounds*, respectively, the *gradient bounds*.

We shall apply (3.107) to $\xi = \phi$ and $\xi = \psi$. From (3.108) it follows that

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\partial_x \widehat{u}^n(s,x)\right|^2 dx\, ds\right) \le Cn^{1-\beta-\delta}\big(s^n(t) + o(1)\big), \tag{3.110}$$

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\partial_x \widehat{\rho}^n(s,x)\right|^2 dx\, ds\right) \le Cn^{1+\beta-\delta}\big(s^n(t) + o(1)\big). \tag{3.111}$$

Using (3.109) the last bound is improved to

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \frac{\left|\partial_x \widehat{\rho}^n(s,x)\right|^2}{\widehat{\rho}^n(s,x)} dx\, ds\right) \le Cn^{1-\beta-\delta}\big(s^n(t) + o(1)\big). \tag{3.112}$$

The bound (3.109) will also be applied to $\xi = \kappa$ (see (3.12) and (3.13)) to get

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \frac{\left|n^{2\beta}\partial_x \widehat{\kappa}^n(s,x)\right|^2}{\widehat{\rho}^n(s,x)} dx\, ds\right) \le Cn^{1-\beta-\delta}\big(s^n(t) + o(1)\big). \tag{3.113}$$

3.7 Control of the large values of (ρ, u): proof of (3.93)

3.7.1 Preparations

In the present section we prove (3.93). First we replace $\frac{1}{n}\sum_{j\in\mathbb{T}^n}\cdots$ by $\int_{\mathbb{T}}\cdots dx$. Note that given a smooth function $F : \mathbb{T} \to \mathbb{R}$

$$\left|\frac{1}{n}\sum_{j\in\mathbb{T}^n} F(\frac{j}{n}) - \int_{\mathbb{T}} F(x)\, dx\right| \le \frac{1}{n}\left(\int_{\mathbb{T}} |\partial_x F(x)|^2\, dx\right)^{1/2}. \tag{3.114}$$

Hence it follows that

$$\mathbf{E}\left(\int_0^t \frac{1}{n}\sum_{j\in\mathbb{T}^n}\Big\{\big(\partial_x v\big)\,\big(n^{3\beta}\widehat{\psi}^n\big)\, J^n(\widehat{\rho}^n, \widehat{u}^n)\Big\}(s, \frac{j}{n})\, ds\right) = \tag{3.115}$$

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \Big\{\big(\partial_x v\big)\,\big(n^{3\beta}\widehat{\psi}^n\big)\, J^n(\widehat{\rho}^n, \widehat{u}^n)\Big\}(s,x)\, dx\, ds\right) + A_{13}^n,$$

where A_{13}^n is again a simple numerical error term:

$$\begin{aligned} |A_{13}^n| \;\le\;& Cn^{3\beta-1}\Bigg\{1 + \sqrt{\mathbf{E}\left(\int_0^s \int_{\mathbb{T}} \left|\partial_x \widehat{\psi}^n(s,x)\right|^2\, dx\, ds\right)} + \sqrt{\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} |\partial_x \widehat{\rho}^n(s,x)|^2\, dx\, ds\right)} \\ & + \sqrt{\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} |\partial_x \widehat{u}^n(s,x)|^2\, dx\, ds\right)}\Bigg\} \\ =\;& \mathcal{O}(n^{5\beta} l^{-1}) = o(1). \end{aligned} \tag{3.116}$$

In the last step we use the most straightforward gradient bound (3.42). (Using the gradient bound (3.108) we could obtain the much better upper bound

$$|A_{13}^n| = \mathcal{O}(n^{(-1-\delta+7\beta)/2}) = o(n^{5\beta} l^{-1}),$$

but we do not need this sharper estimate at this stage.)

So, we have to prove that the first term on the right hand side of (3.115) is negligible. Recall that $J^n = S^n_\rho$. We start with the application of the martingale identity:

$$\mathbf{E}\left(\int_{\mathbb{T}}\left\{\{vS^n(\widehat{\rho}^n,\widehat{u}^n)\}(t,x)-\{vS^n(\widehat{\rho}^n,\widehat{u}^n)\}(0,x)-\int_0^t\{(\partial_t v)S^n(\widehat{\rho}^n,\widehat{u}^n)\}(s,x)\,ds\right\}\,dx\right)=$$
$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}v(s,x)\left(n^{1+\beta}L^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))\right)(\mathcal{X}^n_s)\,dx\,ds\right)$$
$$+\,\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}v(s,x)\left(n^{1+\beta+\delta}K^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))\right)(\mathcal{X}^n_s)\,dx\,ds\right) \tag{3.117}$$

3.7.2 The left hand side of (3.117)

From (3.85), (3.86), and (3.91), we conclude that

$$|S^n(\rho,u)|\le C\left(\rho+|u|\right)1\!\!1_{\{\rho\vee|u|>M\}}.$$

Hence, using the large deviation bound (3.101) it follows that, by choosing M sufficiently large we obtain

$$\mathbf{E}\Big(\frac{1}{n}\sum_{j\in\mathbb{T}^n}\left|S^n(\widehat{\rho}^n,\widehat{u}^n)(s,\frac{j}{n})\right|\Big)\le\varepsilon\,h^n(s)+o(1).$$

Hence, applying again (3.114) we get

$$|\,\text{l.h.s. of (3.117)}\,|\le\frac{1}{2}\,h^n(t)+C\int_0^t h^n(s)\,ds+o(1). \tag{3.118}$$

Remark: Note that this is the point where M and thus the lower edge of the cutoff is fixed. Also note the importance of the factor 1/2 in fromt of $h^n(t)$ on the right hand side.

3.7.3 The right hand side of (3.117): first computations

First we compute how the infinitesimal generators $n^{1+\beta}L^n$ and $n^{1+\beta+\delta}K^n$ act on the function $\underline{\omega}\mapsto S^n(\widehat{\rho}^n(x),\widehat{u}^n(x))$:

$$n^{1+\beta}L^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))= \tag{3.119}$$
$$\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{3\beta}\partial_x\widehat{\psi}^n\big)+S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x\widehat{\phi}^n\big)\Big\}(x)+A^n_{14}(x),$$

$$n^{1+\beta+\delta}K^nS^n(\widehat{\rho}^n(x),\widehat{u}^n(x))= \tag{3.120}$$
$$n^{-1+\beta+\delta}\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial^2_x\widehat{\kappa}^n\big)+S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial^2_x\widehat{\chi}^n\big)\Big\}(x)+A^n_{15}(x),$$

where $A^n_{14}(x)$ and $A^n_{15}(x)$ are the following *numerical error terms*:

$$
\begin{aligned}
A^n_{14}(x) = A^n_{14}(\underline{\omega}, x) := n^{1+\beta} \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega'\in\Omega} & r(\omega_j, \omega_{j+1}; \omega', \omega') \times \\
& \Big\{ S^n\big(\widehat{\rho}^n(x) + \frac{n^{2\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\eta' - \eta_j), \\
& \qquad \widehat{u}^n(x) + \frac{n^{\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\zeta' - \zeta_j)\big) \\
& -S^n\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \\
& -S^n_\rho\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{2\beta}}{l^2} a'\big(\frac{nx-j}{l}\big)(\eta' - \eta_j) \\
& -S^n_u\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{\beta}}{l^2} a'\big(\frac{nx-j}{l}\big)(\zeta' - \zeta_j)\Big\}.
\end{aligned}
$$

$$
\begin{aligned}
A^n_{15}(x) = A^n_{15}(\underline{\omega}, x) := n^{1+\beta+\delta} \sum_{j\in\mathbb{T}^n} \sum_{\omega',\omega'\in\Omega} & s(\omega_j, \omega_{j+1}; \omega', \omega') \times \\
& \Big\{ S^n\big(\widehat{\rho}^n(x) + \frac{n^{2\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\eta' - \eta_j), \\
& \qquad \widehat{u}^n(x) + \frac{n^{\beta}}{l}\big(a\big(\frac{nx-j}{l}\big) - a\big(\frac{nx-j-1}{l}\big)\big)(\zeta' - \zeta_j)\big) \\
& -S^n\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \\
& -S^n_\rho\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{2\beta}}{l^2} a'\big(\frac{nx-j}{l}\big)(\eta' - \eta_j) \\
& -S^n_u\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big)\frac{n^{\beta}}{l^2} a'\big(\frac{nx-j}{l}\big)(\zeta' - \zeta_j)\Big\} \\
+ S^n_\rho\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \times & \\
\frac{n^{1+3\beta+\delta}}{l^2} \sum_{j\in\mathbb{T}^n} & \Big\{ a'\big(\frac{nx-j}{l}\big)(\kappa_{j+1} - \kappa_j) + \frac{1}{l} a''\big(\frac{nx-j}{l}\big)\kappa_j \Big\} \\
+ S^n_u\big(\widehat{\rho}^n(x), \widehat{u}^n(x)\big) \times & \\
\frac{n^{1+2\beta+\delta}}{l^2} \sum_{j\in\mathbb{T}^n} & \Big\{ a'\big(\frac{nx-j}{l}\big)(\chi_{j+1} - \chi_j) + \frac{1}{l} a''\big(\frac{nx-j}{l}\big)\chi_j \Big\}.
\end{aligned}
$$

These error terms are easily estimated: using the fact that the second partial derivatives of S^n are uniformly bounded and ζ and η are bounded, by simple Taylor expansion after tedious but otherwise straightforward computations we find:

$$\sup_{\underline{\omega}\in\Omega^n} \sup_{x\in\mathbb{T}} |A^n_{14}(\underline{\omega}, x)| \le C n^{1+3\beta} l^{-2} = o(1), \tag{3.121}$$

$$\sup_{\underline{\omega}\in\Omega^n} \sup_{x\in\mathbb{T}} |A^n_{15}(\underline{\omega}, x)| \le C n^{1+5\beta+\delta} l^{-3} = o(1). \tag{3.122}$$

No probabilistic arguments are involved in these bounds. The global (averaged and integrated) error introduced by these terms will be of the same order.

Next we do some further transformations on the main terms coming from the right hand sides of (3.119) and (3.120). Performing integrations by part, introducing the macroscopic fluxes and using (3.82) we obtain:

$$
\begin{aligned}
-\int_{\mathbb{T}} v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\partial_x\big(n^{3\beta}\widehat{\psi}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\partial_x\big(n^{2\beta}\widehat{\phi}^n\big)\Big\}(x)\,dx = \\
\int_{\mathbb{T}} \partial_x v(x)\Big\{\big(n^{3\beta}\widehat{\psi}^n\big)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} \partial_x v(x)\Big\{F^n(\widehat{\rho}^n,\widehat{u}^n) - \Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} \partial_x v(x)\Big\{n^{2\beta}S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} v(x)\Big\{\ n^{3\beta}S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big) \\
+n^{3\beta}S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\psi}^n - \Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big) \\
+n^{2\beta}S^n_{u\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big) \\
+n^{2\beta}S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\phi}^n - \Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(x)\,dx
\end{aligned}
$$

Note that, since $J^n = S^n_\rho$, the first term on the right hand side is exactly the expression in the main term on the right hand side of (3.115). Estimating the other terms on the right hand side of (3.123) is the object of the next subsection.

Now we turn to the main term on the right hand side of (3.120). Here, straightforward integration by parts yields

$$
\begin{aligned}
-\int_{\mathbb{T}} v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x^2\widehat{\kappa}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x^2\widehat{\chi}^n\big)\Big\}(x)\,dx = \\
\int_{\mathbb{T}} \partial_x v(x)\Big\{S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big) + S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big\}(x)\,dx \\
+\int_{\mathbb{T}} v(x)\Big\{\quad S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big) \\
+\,S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\Big(\big(\partial_x\widehat{u}^n\big)\big(n^{2\beta}\partial_x\widehat{\kappa}^n\big) + \big(\partial_x\widehat{\rho}^n\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big) \\
+\,S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(n^{\beta}\partial_x\widehat{\chi}^n\big)\Big\}(x)\,dx
\end{aligned}
\tag{3.123}
$$

We will estimate the terms emerging from the right hand side in the next subsection.

3.7.4 The right hand side of (3.117): bounds

We note that

$$\big|F^n(\widehat{\rho}^n,\widehat{u}^n)-\Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\big|\le C\big(1+|\widehat{u}^n|^2\big)1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}},$$

see (3.90) and (3.91). Hence, applying the large deviation bounds (3.101) and (3.102) we obtain

$$\mathbf{E}\left(\int_{\mathbb{T}}\Big|\Big\{F^n(\widehat{\rho}^n,\widehat{u}^n)-\Psi^n(\widehat{\rho}^n,\widehat{u}^n)S^n_\rho(\widehat{\rho}^n,\widehat{u}^n)\Big\}(s,x)\Big|\,dx\right)\le C\,h^n(s)+o(1). \tag{3.124}$$

We use

$$|S^n_u(\widehat{\rho}^n,\widehat{u}^n)|\le C,$$

see (3.86) and the first block replacement bound (3.107) to obtain:

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_u(\widehat{\rho}^n,\widehat{u}^n)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{(-1-\delta+\beta)/2}=o(1). \tag{3.125}$$

Next we use

$$\big|S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big|\le\frac{C}{\log(\overline{r}/\underline{r})}\frac{1}{\underline{r}+\widehat{\rho}^n},\qquad \big|S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big|\le\frac{C}{\log(\overline{r}/\underline{r})}\frac{1}{\sqrt{\underline{r}}+\sqrt{\widehat{\rho}^n}}$$

$$|S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)|\le\frac{C}{\log(\overline{r}/\underline{r})}, \tag{3.126}$$

see (3.87), (3.88), respectively, (3.89), and note that here *we do not exploit* the fact that the constant factors on the right hand side are actually small. These, together with the block replacement bounds (3.107), the gradient bounds (3.110), (3.112) and the bound (3.31) on the relative entropy $s^n(t)$ yield the following four estimates:

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{3\beta}S^n_{\rho\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{\beta-\delta}=o(1),$$

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{3\beta}S^n_{\rho u}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\psi}^n-\Psi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{\beta-\delta}=o(1),$$

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_{u\rho}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{\rho}^n\big)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{-\delta}=o(1), \tag{3.127}$$

$$\mathbf{E}\left(\int_0^t\int_{\mathbb{T}}\Big|\Big\{n^{2\beta}S^n_{uu}(\widehat{\rho}^n,\widehat{u}^n)\big(\partial_x\widehat{u}^n\big)\big(\widehat{\phi}^n-\Phi(\widehat{\eta}^n,\widehat{\zeta}^n)\big)\Big\}(s,x)\Big|\,dx\,ds\right)\le C\,l\,n^{-\delta}=o(1).$$

Using

$$\left|S^n_\rho(\widehat{\rho}^n, \widehat{u}^n)\right| \le C, \qquad |S^n_u(\widehat{\rho}^n, \widehat{u}^n)| \le C,$$

see (3.85) and (3.86), and the gradient bounds (3.110), (3.111) we obtain the following two bounds:

$$\begin{aligned}
n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S^n_\rho(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{\rho}^n\big)\right\}(s,x)\right| \, dx \, ds\right) \\
\le C\, n^{(-1+\delta+3\beta)/2} = o(1), \\
n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S^n_u(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{u}^n\big)\right\}(s,x)\right| \, dx \, ds\right) \\
\le C\, n^{(-1+\delta+\beta)/2} = o(1).
\end{aligned} \tag{3.128}$$

The following bounds are of paramount importance and they are sharp. We use (3.126) again and note that here we exploit it in its *full power*: the constant factor on the right hand side is small. These and the gradient bounds (3.110) and (3.112) yield the following three bounds:

$$\begin{aligned}
n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S^n_{\rho\rho}(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{\rho}^n\big)\big(n^{2\beta}\partial_x \widehat{\kappa}^n\big)\right\}(s,x)\right| \, dx \, ds\right) \\
\le c\, s^n(t) + o(1), \\
n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S^n_{\rho u}(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{u}^n\big)\big(n^{2\beta}\partial_x \widehat{\kappa}^n\big)\right\}(s,x)\right| \, dx \, ds\right) \\
\le c\, s^n(t) + o(1), \\
n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S^n_{\rho u}(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{\rho}^n\big)\big(n^{\beta}\partial_x \widehat{\chi}^n\big)\right\}(s,x)\right| \, dx \, ds\right) \\
\le c\, s^n(t) + o(1), \\
n^{-1+\beta+\delta}\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\left\{S^n_{uu}(\widehat{\rho}^n, \widehat{u}^n)\big(\partial_x \widehat{u}^n\big)\big(n^{\beta}\partial_x \widehat{\chi}^n\big)\right\}(s,x)\right| \, dx \, ds\right) \\
\le c\, s^n(t) + o(1).
\end{aligned} \tag{3.129}$$

The ratio $\overline{r}/\underline{r}$ is chosen so large that

$$c \sup_{(t,x)\in[0,T]\times\mathbb{T}} |v(t,x)| < \frac{1}{2}. \tag{3.130}$$

3.7.5 Sumup

The identities (3.119), (3.120), (3.123), (3.123) and the bounds (3.121), (3.122), (3.124), (3.125), (3.127), (3.128), (3.129) yield

$$\left|\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \Big\{ (\partial_x v) \left(n^{3\beta}\widehat{\psi}^n\right) S^n_\rho(\widehat{\rho}^n, \widehat{u}^n) \Big\}(s,x)\, dx\, ds\right) - \Big(\text{r.h.s. of (3.117)}\Big)\right| \\ \leq \frac{1}{2}\, s^n(t) + c\int_0^t h^n(s)\, ds + o(1). \qquad (3.131)$$

Finally, from (3.115), (3.116), (3.117), (3.118) and (3.131) we obtain (3.93).

3.8 Control of the small values of (ρ, u): proof of the bounds (3.94) to (3.97)

3.8.1 Proof of (3.94)

We exploit the straightforward inequality

$$|J^n(\widehat{\rho}^n, \widehat{u}^n)| = \left|S^n_\rho(\widehat{\rho}^n, \widehat{u}^n)\right| \leq C\, 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}},$$

see (3.85) and (3.91), and boundedness of the functions $\rho(t,x)$, $u(t,x)$, $\partial_x v(t,x)$. Thus, applying the large deviation bound (3.101) we readily obtain (3.94).

3.8.2 Proof of (3.95)

This is very similar to what has been done in various parts of subsection 3.7.4. We use the block replacement bound (3.107) and the bound

$$|I^n(\widehat{\rho}^n, \widehat{u}^n)| = \left|1 - S^n_\rho(\widehat{\rho}^n, \widehat{u}^n)\right| \leq C \qquad (3.132)$$

which follows from (3.85). We readily obtain

$$\mathbf{E}\left(\int_0^t \int_{\mathbb{T}} \left|\Big\{ n^{3\beta}(\widehat{\psi}^n - \Psi(\widehat{\eta}^n, \widehat{\zeta}^n))\, |I^n(\widehat{\rho}^n, \widehat{u}^n)| \Big\}(s,x)\right|\, ds\, dx\right) \\ \leq C\, l\, n^{(-1-\delta+3\beta)/2} = o(1),$$

which proves (3.95).

3.8.3 Proof of (3.96)

We write

$$I^n(\widehat{\rho}^n, \widehat{u}^n) = 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| \leq M\}} + 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| > M\}} I^n(\widehat{\rho}^n, \widehat{u}^n), \qquad (3.133)$$

and note that, by Taylor expansion of the function $(\rho, u) \mapsto \Psi(\rho, u)$

$$|\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n \widehat{u}^n|\, 1\!\!1_{\{\widehat{\rho}^n \vee |\widehat{u}^n| \leq M\}} \leq C\, n^{-2\beta}.$$

On the other hand

$$|\Psi^n(\widehat{\rho}^n, \widehat{u}^n)| \leq C\,\widehat{\rho}^n\;|\widehat{u}^n|$$

and

$$\widehat{\rho}^n\;|I^n(\widehat{\rho}^n, \widehat{u}^n)| \leq C(1 + |\widehat{u}^n|), \tag{3.134}$$

see (3.70) and (3.85). Thus

$$|\Psi^n(\widehat{\rho}^n, \widehat{u}^n) - \widehat{\rho}^n\widehat{u}^n|\,|I^n(\widehat{\rho}^n, \widehat{u}^n)| \leq C\left(n^{-2\beta} + \big(\,|\widehat{u}^n| + |\widehat{u}^n|^2\,\big)\,1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\right).$$

From this, using the large deviation bounds (3.101) and (3.102) we obtain (3.96).

3.8.4 Proof of (3.97)

We use again (3.133) and (3.134) and get

$$\begin{aligned}\big|(\widehat{\rho}^n - \rho)\big(\widehat{u}^n - u\big)\,I^n(\widehat{\rho}^n, \widehat{u}^n)\big| \leq \quad & \big|(\widehat{\rho}^n - \rho)\big(\widehat{u}^n - u\big)\big|\,1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|\leq M\}} \\ & + C\left(1 + |\widehat{u}^n| + |\widehat{u}^n|^2\right)1\!\!1_{\{\widehat{\rho}^n\vee|\widehat{u}^n|>M\}}\end{aligned}$$

Now the fluctuation bounds (3.103), (3.104), and the large deviation bounds (3.101), (3.102) together yield (3.97).

3.9 Construction of the cutoff function: proofs

3.9.1 Proof of Lemma 3

Proof. We sketch the proof for $u \geq 0$ and leave the very similar $u \leq 0$ case for the reader. Let $\rho_1 > 0$, $u_1 > 0$ be so chosen that for $(\rho, u) \in [0, \rho_1] \times [0, u_1]$ the following bounds hold with a fixed $c > 0$:

$$\begin{aligned}&|\Phi_u(\rho, u) - \Psi_\rho(\rho, u) - (2\gamma - 1)u| \leq cu(u^2 + \rho), \\ &|\Phi_\rho(\rho, u) - 1| \leq c(u^2 + \rho), \\ &|\Psi_u(\rho, u) - \rho| \leq c\rho(u^2 + \rho),\end{aligned}$$

and

$$\Phi_u(\rho, u) - \Psi_\rho(\rho, u) \neq 0 \quad \text{ for } (\rho, u) \neq (0, 0).$$

This can be done due to the $(\rho, u) \to (0, 0)$ asymptotics of the macroscopic fluxes Φ and Ψ.

It follows that as long as $(\sigma(u; r), u) \in [0, \rho_1] \times [0, u_1]$

$$\frac{d\rho}{du} \leq \frac{2\rho(1 + c'(\rho + u^2))}{\sqrt{(2\gamma - 1)^2 u^2 + 4\rho} + (2\gamma - 1)u}.$$

This implies

$$\sigma(u;r) \le r + C\left(\sqrt{r}u \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u^{\frac{1}{2\gamma-1}}\right)$$

with a positive C, as long as $(\sigma(u;r), u) \in [0,\rho_1] \times [0,u_1]$.

From our assumptions it also follows that for $\rho \le \rho_1$ and $u > u_1$

$$\frac{d\rho}{du} \le b\,\rho,$$

where

$$b := \sup_{\substack{\rho<\rho_1 \\ u>u_1}} \frac{2\Psi_u(\rho,u)}{\rho\left(\Phi_u(\rho,u) - \Psi_\rho(\rho,u)\right)} < \infty.$$

Hence it follows that for $u \ge u_1$

$$\sigma(u;r) \le \sigma(u_1;r)\exp\{b(u-u_1)\}$$

as long as $\sigma(u;r) \le \rho_1$.

Putting these two arguments together the upper bound

$$\sigma(u;r) \le r + C_1\left(\sqrt{r}u \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u^{\frac{1}{2\gamma-1}}\right), \quad u \ge 0, \quad r < r_0,$$

follows with

$$r_0 := \sup\left\{r : \left(r + C\left(\sqrt{r}u_1 \wedge r^{\frac{4\gamma-3}{4\gamma-2}} u_1^{\frac{1}{2\gamma-1}}\right)\right)\exp\{b(u^*-u_1)\} \le \rho_1\right\},$$

and

$$C_1 = \frac{r_0^{\frac{1}{4\gamma-2}}\left(\exp\{b(u^*-u_1)\}-1\right)}{u_1} + C\exp\{b(u^*-u_1)\}.$$

□

3.9.2 Proof of Lemma 4

Note first that given the bounds (3.25) and condition (H) of subsection 3.3.1, (3.78) follows directly from (3.77), (3.76) and (3.65). So, we shall concentrate on (3.74)-(3.77) only.

By differentiating in the pde (3.65) and applying straightforward transformations we obtain the following differential equations for S_ρ, S_u, $S_{\rho\rho}$, and $S_{\rho u}$, respectively:

$$\begin{aligned}\Psi_u(S_\rho)_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)(S_\rho)_{\rho u} - \Phi_\rho(S_\rho)_{uu} + \\ +\Phi_\rho\left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho (S_\rho)_\rho + \Phi_\rho\left(\frac{\Phi_u-\Psi_\rho}{\Phi_\rho}\right)_\rho (S_\rho)_u = 0,\end{aligned} \tag{3.135}$$

$$\begin{aligned}\Psi_u(S_u)_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)(S_u)_{\rho u} - \Phi_\rho(S_u)_{uu} + \\ +\Psi_u\left(\frac{\Phi_u-\Psi_\rho}{\Psi_u}\right)_u (S_u)_\rho - \Psi_u\left(\frac{\Phi_\rho}{\Psi_u}\right)_u (S_u)_u = 0,\end{aligned} \tag{3.136}$$

$$\begin{aligned}\Psi_u(S_{\rho\rho})_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)(S_{\rho\rho})_{\rho u} - \Phi_\rho(S_{\rho\rho})_{uu} + \\ +2\Phi_\rho\left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho (S_{\rho\rho})_\rho + 2\Phi_\rho\left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho (S_{\rho\rho})_u - \Phi_\rho\left(\frac{\Psi_u}{\Phi_\rho}\right)_{\rho\rho} S_{\rho\rho} \\ = -\Phi_\rho\left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_{\rho\rho} S_{\rho u},\end{aligned} \tag{3.137}$$

$$\begin{aligned}\Psi_u(S_{\rho u})_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big)(S_{\rho u})_{\rho u} - \Phi_\rho(S_{\rho u})_{uu} + \\ +\left\{\Psi_u\left(\frac{\Phi_u - \Psi_\rho}{\Psi_u}\right)_u + \Phi_\rho\left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho\right\}(S_{\rho u})_\rho + \\ +\left\{\Phi_\rho\left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho - \Psi_u\left(\frac{\Phi_\rho}{\Psi_u}\right)_u\right\}(S_{\rho u})_u + \\ +\left\{\Psi_u\left(\frac{\Phi_\rho}{\Psi_u}\right)_u\left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_\rho + \Phi_\rho\left(\frac{\Phi_u - \Psi_\rho}{\Phi_\rho}\right)_{\rho u}\right\} S_{\rho u} \\ = -\left\{\Psi_u\left(\frac{\Phi_\rho}{\Psi_u}\right)_u\left(\frac{\Psi_u}{\Phi_\rho}\right)_\rho + \Phi_\rho\left(\frac{\Psi_u}{\Phi_\rho}\right)_{\rho u}\right\} S_{\rho\rho}.\end{aligned} \tag{3.138}$$

Because of the conditions outlined in subsection 3.3.1 all the coefficients are smooth functions, if ρ is small enough.

The respective initial conditions are

$$S_\rho(\rho,0) = \frac{\log(r/\underline{r})}{\log(\overline{r}/\underline{r})}\mathbb{1}_{\{\rho\in[\underline{r},\overline{r}]\}} + \mathbb{1}_{\{\rho\in[\overline{r},r_0]\}}, \quad (S_\rho)_u(\rho,0) = 0, \tag{3.139}$$

$$S_u(\rho,0) = 0, \quad (S_u)_u(\rho,0) = \frac{1}{\log(\overline{r}/\underline{r})}\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)}\mathbb{1}_{\{\rho\in[\underline{r},\overline{r}]\}}, \tag{3.140}$$

$$S_{\rho\rho}(\rho,0) = \frac{\rho^{-1}}{\log(\overline{r}/\underline{r})}\mathbb{1}_{\{\rho\in[\underline{r},\overline{r}]\}}, \quad (S_{\rho\rho})_u(\rho,0) = 0, \tag{3.141}$$

$$\begin{aligned}&S_{\rho u}(\rho,0) = 0,\\ &(S_{\rho u})_u(\rho,0) = \frac{1}{\log(\overline{r}/\underline{r})}\left(\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)}\right)_\rho + \frac{1}{\log(\overline{r}/\underline{r})}\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)}\left\{\delta(\rho-\underline{r}) - \delta(\rho-\overline{r})\right\}.\end{aligned} \tag{3.142}$$

Observe, that because of the asymptotics (3.25) we have

$$\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)} = 1 + \mathcal{O}(\rho), \quad \left(\frac{\Psi_u(\rho,0)}{\rho\,\Phi_\rho(\rho,0)}\right)_\rho(\rho,0) = \mathcal{O}(1). \tag{3.143}$$

In order to understand the pdes (3.135)-(3.138) first we analyze in general the pde

$$\Psi_u f_{\rho\rho} + \big(\Phi_u - \Psi_\rho\big) f_{\rho u} - \Phi_\rho f_{uu} + A f_\rho + B f_u + G f = 0, \tag{3.144}$$

the functions $\{A\,,\,B\,,\,G\} = \{A\,,\,B\,,\,G\}(\rho,u)$ being given on the left hand side of the pdes (3.135)-(3.138). It is easy to check that A and G are even, B is odd with respect to u and also that $\kappa := A(0,0)$ is $1, 2\gamma-1, 2, 2\gamma$, respectively, in the four cases.

We solve in $\widetilde{\mathcal{D}}$ the Cauchy problem (3.144) with the initial condition

$$f(\rho, 0) = s(\rho), \qquad f_u(\rho, 0) = t(\rho), \qquad \rho \in [0, r_0]. \tag{3.145}$$

The functions $s(\rho)$ and $t(\rho)$ will be identified with the various expressions in (3.139)-(3.142). Then, in $\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}$ we solve the Goursat problem (3.144) with boundary conditions

$$f(\rho, u) = \begin{cases} f(r_1, 0) & \text{on} \quad \partial\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}}, \\ \text{given by the solution of the previous Cauchy problem} & \text{on} \quad \partial\widetilde{\mathcal{D}} \cap \mathcal{D}_3(\overline{r}). \end{cases} \tag{3.146}$$

Mind that $f(r_1, 0) = 1$ in the case of (3.135) and $f(r_1, 0) = 0$ in the cases (3.136)-(3.138).

The pde (3.144) is hyperbolic in the domains considered. Its Jacobian matrix is

$$D = D(\rho, u) := \begin{pmatrix} \Psi_\rho & \Psi_u \\ \Phi_\rho & \Phi_u \end{pmatrix}. \tag{3.147}$$

The eigenvalues of $D(\rho, u)$ are

$$\left.\begin{matrix} \lambda \\ \mu \end{matrix}\right\} = \pm\frac{1}{2}\left\{\sqrt{\left(\Phi_u - \Psi_\rho\right)^2 + 4\Phi_\rho\Psi_u} \mp \left(\Phi_u - \Psi_\rho\right)\right\} + \Phi_u \tag{3.148}$$

Mind that from the Onsager relation (3.15) it follows that for any $(\rho, u) \in \mathcal{D}$ $\left(\Phi_u - \Psi_\rho\right)^2 + 4\Phi_\rho\Psi_u \geq 0$.

The characteristic coordinates (or Riemann invariants) $w = w(\rho, u)$, $z = z(\rho, u)$ of the pde (3.144) are determined, up to a functional relation

$$\widetilde{w}(\rho, u) = g(w(\rho, u)), \qquad \widetilde{z}(\rho, u) = h(z(\rho, u)), \tag{3.149}$$

by the eigenvalue equations

$$(w_\rho, w_u)D = \lambda\,(w_\rho, w_u), \qquad (z_\rho, z_u)D = \mu\,(z_\rho, z_u). \tag{3.150}$$

Due to the gauge invariance (3.149) we can choose the characteristic coordinates w and z so that

$$w(0, u) = u1\!\!1_{\{u>0\}}, \qquad z(0, u) = -u1\!\!1_{\{u<0\}}.$$

This choice determines uniquely the characteristic coordinates. Observe, that as a corollary of Lemma 3 we have

$$c_1\left(\sqrt{\rho} + u\right) \;<\; w(\rho, u) < c_2\left(\sqrt{\rho} + u\right), \tag{3.151}$$

$$\rho \;<\; c\,z(\rho, u)^{\frac{4\gamma-3}{2\gamma-1}}\,w(\rho, u)^{\frac{1}{2\gamma-1}}. \tag{3.152}$$

We denote

$$w^\circ := w(r_0, 0) = z(r_0, 0) =: z^\circ$$

$$\underline{w} := w(\underline{r}, 0) = z(\underline{r}, 0) =: \underline{z}$$

$$\overline{w} := w(\overline{r}, 0) = z(\overline{r}, 0) =: \overline{z}$$

In characteristic coordinates

$$\begin{aligned}
&\widetilde{\mathcal{D}} = [0, w^\circ] \times [0, z^\circ],\\
&\widetilde{\mathcal{D}} \cap \{u = 0\} = [0, w^\circ] \times [0, z^\circ] \cap \{w = z\},\\
&\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}} = [w^\circ, u_*] \times [0, \overline{z}] \cup [0, \overline{w}] \times [z^\circ, u_*],\\
&\partial\mathcal{D}_3(\overline{r}) \setminus \widetilde{\mathcal{D}} = \{(w, \overline{z}) : w \in [w^\circ, u_*]\} \cup \{(\overline{w}, z) : z \in [z^\circ, u_*]\}\\
&\partial\widetilde{\mathcal{D}} \cap \mathcal{D}_3(\overline{r}) = \{(w^\circ, z) : z \in [0, z^\circ]\} \cup \{(w, z^\circ) : w \in [0, w^\circ]\}.
\end{aligned}$$

The pde (3.144) written in characteristic coordinates reads

$$f_{wz} + \alpha f_w + \beta f_z + \nu f = 0, \tag{3.153}$$

where

$$\begin{aligned}
\alpha = \alpha(w, z) &:= \frac{\lambda_z - A u_z + B\rho_z}{\lambda - \mu}\\
\beta = \beta(w, z) &:= -\frac{\mu_w - A u_w + B\rho_w}{\lambda - \mu}\\
\nu = \nu(w, z) &:= \frac{G(\rho_w u_z - \rho_z u_w)}{\lambda - \mu}
\end{aligned} \tag{3.154}$$

(Now all the functions on the right are understood as functions of (w, z).) In characteristic coordinates the initial conditions of the Cauchy problem in $\widetilde{\mathcal{D}}$ are

$$\begin{aligned}
&f(v, v) = s(\rho(v, v)) =: \widetilde{s}(v),\\
&\big(f_w - f_z\big)(v, v) = 2u_w(v, v)\, t(\rho(v, v)) =: \widetilde{t}(v).
\end{aligned} \tag{3.155}$$

The Cauchy problem (3.153)+(3.155) in the domain $\widetilde{\mathcal{D}} \cap \{z \le w\}$ is solved by

$$\begin{aligned}
f(w_0, z_0) \quad = \quad & \frac{1}{2}\varphi(z_0, z_0)\widetilde{s}(z_0) + \frac{1}{2}\varphi(w_0, w_0)\widetilde{s}(w_0)\\
&+\frac{1}{2}\int_{z_0}^{w_0} \varphi(v, v)\widetilde{t}(v)dv + \frac{1}{2}\int_{z_0}^{w_0} \big(\varphi_w - \varphi_z\big)(v, v)\widetilde{s}(v)dv\\
&+\int_{z_0}^{w_0} \varphi(v, v)(\beta(v, v) - \alpha(v, v))\widetilde{s}(v)dv
\end{aligned} \tag{3.156}$$

where the *Riemann function* φ is a solution of the adjoint Goursat problem:

$$\varphi_{wz} - (\alpha\varphi)_w - (\beta\varphi)_z + \nu\varphi = 0, \tag{3.157}$$

with boundary conditions:

$$\begin{cases} \varphi(w_0, t) = \exp \int_{z_0}^t \alpha(w_0, v)dv, & t \in [z_0, w_0], \\ \varphi(s, z_0) = \exp \int_{w_0}^s \beta(v, z_0)dv, & s \in [z_0, w_0]. \end{cases} \tag{3.158}$$

Actually, the Riemann function depends also on (w_0, z_0): $\varphi(w, z) = \varphi(w_0, z_0; w, z)$. In order to avoid heavy typography we omit explicit notation of this dependence. Note that in our cases (because of the left-right reflection symmetry of the respective pde), we will have $\beta(v, v) = \alpha(v, v)$, thus on the right hand side of (3.156) the last term cancels.
Also, if we consider the non-homogeneous pde

$$h_{wz} + \alpha h_w + \beta h_z + \nu h = g$$

with the same initial conditions then it is solved by

$$h(w_0, z_0) = f(w_0, z_0) + \int\int_{\triangle_{w_0,z_0}} g(s, t)\varphi(s, t)dsdt, \tag{3.159}$$

where $\triangle_{w_0,z_0}$ is the triangle with vertices $(z_0, z_0), (w_0, w_0), (w_0, z_0)$.

For details see any advanced textbook on partial differential equations, e.g. [11], [9], [5], [30].

In order to estimate f we will give a uniform estimate on the Riemann function φ.

Proposition 5 (Bounds on the Riemann function). *Let φ be the Riemann function associated to the equation (3.144) (where the coefficients A, B, G are given on the left hand side of (3.135), (3.136), (3.137) or (3.138)) and $w_0 > z_0 > 0$. Then the following bounds hold uniformly for $(w_0, z_0) \in \widetilde{\mathcal{D}}$ with $z_0 < t < s < w_0$ and $(s, t) \in \widetilde{\mathcal{D}}$:*

$$\begin{aligned} |\varphi(s, t)| &< c\left(\frac{s}{w_0}\right)^{\frac{\kappa-1}{2\gamma-1}} \\ |(\partial_w\varphi - \partial_z\varphi)(s, s)| &< c\frac{1}{w_0}\left(\frac{s}{w_0}\right)^{\frac{\kappa-1}{2\gamma-1}-1}. \end{aligned} \tag{3.160}$$

Using Proposition 5 with (3.156), the initial conditions (3.139)-(3.142) and with Lemma 3 we can estimate f in $\widetilde{\mathcal{D}}$ which gives (3.74) and (3.75) in this domain.

The equations (3.137) and (3.138) are not closed for $S_{\rho\rho}$ and $S_{\rho u}$, respectively, with the previous method one can only prove the required estimates for the solution of the respective homogeneous pdes. However, with (3.159) it is easy to show that these estimates can be extended for $S_{\rho\rho}$ and $S_{\rho u}$, too.

As an example, we show how to get the bound (3.76) for the *homogeneous* solution f of (3.137) with initial conditions (3.141).

For the initial conditions we have the following bounds (using (3.151),(3.152)):

$$\begin{aligned} |\widetilde{s}(v)| &< \frac{c}{\log(\overline{r}/\underline{r})v^2}\mathbb{1}_{\{v\in[\underline{z},\overline{z}]\}}, \\ \left|\widetilde{t}(v)\right| &= 0. \end{aligned}$$

From Proposition 5 we have that for any $z_0 < w_0$

$$|\varphi(v,v)| < c\left(\frac{v}{w_0}\right)^{\frac{1}{2\gamma-1}}, \quad |\partial_w\varphi(v,v) - \partial_z\varphi(v,v)| < \frac{c}{w_0}\left(\frac{v}{w_0}\right)^{\frac{1}{2\gamma-1}-1}.$$

Together with (3.156) we get

$$|f(w_0,z_0)| < \frac{c}{\log(\overline{r}/\underline{r})} w_0^{-\frac{1}{2\gamma-1}} \underline{z}^{2-\frac{1}{2\gamma-1}} \mathbb{1}_{\{z_0\in[\underline{z},\overline{z}],\underline{z}\le w_0\}}, \tag{3.161}$$

for any $z_0 < w_0$. This means

$$|f(w_0,z_0)| < \frac{c}{\log(\overline{r}/\underline{r})}\underline{z}^2 \mathbb{1}_{\{z_0\in[\underline{z},\overline{z}],\underline{z}\le w_0\}},$$

which (using (3.151)) translates to the (ρ,u) coordinates the following way:

$$|f(\rho,u)| < \frac{c}{\log(\overline{r}/\underline{r})}\frac{1}{\underline{r}}\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u).$$

(We only get this for $u \geq 0$, but from the symmetry of the pde this is also true for $u < 0$.) Also from (3.161) we get

$$|f(w_0,z_0)| < \frac{c}{\log(\overline{r}/\underline{r})} w_0^{-\frac{1}{2\gamma-1}} z_0^{2-\frac{1}{2\gamma-1}} \mathbb{1}_{\{z_0\in[\underline{z},\overline{z}],\underline{z}\le w_0\}},$$

which (using (3.152)) gives

$$|f(\rho,u)| < \frac{c}{\log(\overline{r}/\underline{r})}\frac{1}{\rho}\mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho,u).$$

Putting together the two bounds on f we get the required estimate of (3.76).

Now back to the proof of Proposition 5. The following lemma will be a basic tool for our estimates:

Lemma 8 (Goursat estimate). *Suppose the functions $A(w,z), B(w,z), C(w,z)$ are defined on $[x_1,x_2]\times[y_1,y_2]$ where $0 \le y_1 < y_2 \le x_1 < x_2 \le \infty$ and have the following properties:*

$$\begin{aligned} \sup_{z\in[y_1,y_2]} \int_{x_1}^{x_2} |B(s,z)|\,ds &< \frac{1}{6}, \\ \sup_{w\in[x_1,x_2]} \int_{y_1}^{y_2} |A(w,t)|\,dt &< \frac{1}{6}, \\ \int_{x_1}^{x_2}\int_{y_1}^{y_2} |C(s,t)|\,dsdt &< \frac{1}{6}. \end{aligned} \tag{3.162}$$

Let U be a solution of

$$U_{wz} - (AU)_w - (BU)_z + CU = 0 \tag{3.163}$$

on the rectangle $[x_1, x_2] \times [y_1, y_2]$ and let

$$\sup_{y_1 \le z \le y_2} |U(x_1, z)| + \sup_{x_1 \le w \le x_2} |U(w, y_2)| = M.$$

Then

$$\sup_{(w,z) \in [x_1,x_2] \times [y_1,y_2]} |U(w, z)| \le 5M.$$

Proof. Denote $U(w, y_2) = f(w)$, $U(x_1, z) = g(z)$. Denote

$$\begin{aligned} \widetilde{f}(w) &= f(w) - \int_{x_1}^{w} B(s, y_2) f(s) ds, \\ \widetilde{g}(z) &= g(z) - \int_{y_2}^{z} A(x_1, t) g(t) dt. \end{aligned}$$

Then U satisfies the following integral equation (for every $(w, z) \in [x_1, x_2] \times [y_1, y_2]$):

$$\begin{aligned} U(w, z) &= \widetilde{f}(w) + \widetilde{g}(z) - U(x_1, y_2) + \int_{y_2}^{w} A(w, t) U(w, t) dt + \int_{x_1}^{z} B(s, z) U(s, z) ds \\ &\quad - \int_{x_1}^{w} \int_{y_2}^{z} C(s, t) U(s, t) ds dt. \end{aligned} \tag{3.164}$$

Taking absolute values after some trivial estimates we get:

$$|U(w, z)| \le \frac{13}{6} M + \frac{1}{2} \sup_{(s,t) \in [x_1, w] \times [z, y_2]} |U(s, t)|,$$

from that the needed bound follows immediately. □

Remark. *Observe, that if we have some additional estimates on A_z, B_w and C then differentiating (3.164) with respect to w or z and using the Grönwall inequality, one could also get bounds on $|U_w|$ and $|U_z|$ in $[x_1, x_2] \times [y_1, y_2]$.*

Proof of Proposition 5.
(i) We first prove the proposition in the case when $\Psi(\rho, u) = \rho u$ and $\Phi(\rho, u) = \rho + \gamma u^2$. In that case the coefficient functions in (3.144) take the following form: $A = \kappa$, $B = C = 0$. Also, the Riemann invariants can be computed explicitly:

$$\begin{aligned} w(\rho, u) &:= \left(\frac{\sqrt{(2\gamma-1)u^2+4\rho}+(2\gamma-1)u}{4\gamma-2} \right)^{\frac{2\gamma-1}{4\gamma-3}} \left(\sqrt{(2\gamma-1)u^2+4\rho} - (2\gamma-2)u \right)^{\frac{2\gamma-2}{4\gamma-3}} \\ z(\rho, u) &:= \left(\frac{\sqrt{(2\gamma-1)u^2+4\rho}-(2\gamma-1)u}{4\gamma-2} \right)^{\frac{2\gamma-1}{4\gamma-3}} \left(\sqrt{(2\gamma-1)u^2+4\rho} + (2\gamma-2)u \right)^{\frac{2\gamma-2}{4\gamma-3}}. \end{aligned} \tag{3.165}$$

One can easily check that the equations (3.150) hold. We define

$$\beta_0(w) := \beta(w, 0) \quad \text{and} \quad U(w, z) := \varphi(w, z) \exp\left(- \int_{w_0}^{w} \beta_0(s) ds \right) \tag{3.166}$$

(for $0 \le z < w$). From the definitions one can calculate that

$$\beta_0(w) = \frac{\kappa - 1}{2\gamma - 1} \frac{u_w(w,0)}{u(w,0)} = \frac{\kappa - 1}{2\gamma - 1} w^{-1}.$$

Thus, it is enough to prove, that

$$|U(s,t)| < C, \quad |\partial_w U(s,s)| < C\frac{1}{s}, \quad |\partial_z U(s,s)| < C \tag{3.167}$$

for $z_0 \le t \le s \le w_0$ uniformly, with a constant depending only on κ. To show (3.167) we will apply Lemma 8.

From (3.157), (3.158) and (3.166) we get that

$$U_{wz} - (\alpha U)_w - ((\beta - \beta_0)U)_z - \alpha\beta_0 U = 0, \tag{3.168}$$

and

$$\begin{aligned} U(w_0, z) &= \exp\left(\int_{z_0}^{z} \alpha(w_0, t)dt\right), \\ U(w, z_0) &= \exp\left(\int_{w_0}^{w} (\beta(s, z_0) - \beta(s, 0))ds\right). \end{aligned}$$

Using the explicit formulas for the Riemann-invariants one can get estimates for the integrals needed for Lemma 8. Suppose $[x_1, x_2] \times [y_1, y_2] \subseteq [v, w_0] \times [v, z_0]$ for some $z_0 < v < w_0$. Then it can be shown that for $z_0 \le z \le w \le w_0$

$$\begin{aligned} \left|\int_{y_1}^{y_2} \alpha(w,t)dt\right| &< c\frac{x_2 - x_1}{w}, \\ \left|\int_{x_1}^{x_2} \beta(s,z) - \beta(s,0)ds\right| &< cz\left(\frac{1}{x_1} - \frac{1}{x_2}\right), \\ \left|\int_{x_1}^{x_2}\int_{y_1}^{y_2} (\alpha\beta_0)(s,t)ds\,dt\right| &< c\left(\frac{1}{x_1} - \frac{1}{x_2}\right)(y_2 - y_1), \end{aligned} \tag{3.169}$$

where c only depends on κ. From the first and second inequality it follows that $|U(w_0, z)|$ and $|U(w, z_0)|$ can be bounded by a constant depending only on κ. Now fix s, t with $z_0 < t < s < w_0$. Using (3.169) one can partition $[v, w_0] \times [z_0, v]$ into smaller rectangles in a way that the number of rectangles only depends on the value of κ and on each small rectangle the conditions of Lemma 8 hold (with $A = \alpha$, $B = \beta - \beta_0$, $C = -\alpha\beta_0$). Applying successively Lemma 8 for the small rectangles (starting with the vertex (w_0, z_0)) one gets $|U(s,t)| < c(\kappa)$. Using the remark after Lemma 8 one can also get the results of (3.167) for the partial derivatives of U.

(ii) In the general case we do not know the explicit forms of the coefficients in (3.144) only their asymptotics:

$$\begin{aligned} A(\rho, u) &= \kappa(1 + \mathcal{O}(\rho + u^2)) \\ B(\rho, u) &= c_1 u(1 + \mathcal{O}(\rho + u^2)) \\ C(\rho, u) &= c_2(1 + \mathcal{O}(\rho + u^2)). \end{aligned}$$

We also do not have explicit formulas for the Riemann-invariants, but because $\Psi(\rho,u)=\rho u(1+\mathcal{O}(\rho+u^2))$ and $\Phi(\rho,u)=(\rho+\gamma u^2)(1+\mathcal{O}(\rho+u^2))$ if $\rho,|u|\ll 1$ the level-lines will approximate the respective level-lines of the system examined in (i). We will follow the steps of the proof for the specific case. We define β_0 and U as in (3.166). From the asymptotics we have $\beta_0(w)=\frac{\kappa-1}{2\gamma-1}w^{-1}+\mathcal{O}(1)$ which means it is again enough to prove (3.167). We have the following equation for U:

$$U_{wz}-(\alpha U)_w-((\beta-\beta_0)U)_z+(\nu-\alpha\beta_0)U=0, \tag{3.170}$$

with

$$\begin{aligned} U(w_0,z) &= \exp\left(\int_{z_0}^{z}\alpha(w_0,t)dt\right), \\ U(w,z_0) &= \exp\left(\int_{w_0}^{w}(\beta(s,z_0)-\beta(s,0))ds\right). \end{aligned}$$

If we can prove similar bounds for the integrals of coefficients as in (3.169) then using Lemma 8 the required estimates follow. From (3.154) we have that

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\nu(s,t)ds\,dt=\int\int_{\Delta}\frac{G}{\lambda-\mu}d\rho\,du$$

where Δ is the domain corresponding to the $[x_1,x_2]\times[y_1,y_2]$ rectangle on the (ρ,u) plane. If r_0 is small enough, then for $(\rho,u)\in\widetilde{\mathcal{D}}$

$$\left|\frac{C}{\lambda-\mu}\right|<c\frac{1}{\sqrt{u^2+\rho}}$$

and since the right-hand side is integrable this gives uniform bounds on the previous integral. To get bounds on the integrals

$$\int_{y_1}^{y_2}\alpha(w,t)dt,\ \int_{x_1}^{x_2}\beta(s,z)-\beta(s,0)ds,\ \int_{x_1}^{x_2}\int_{y_1}^{y_2}(\alpha\beta_0)(s,t)ds\,dt,$$

we observe that because of the asymptotics described earlier if $r_0\to 0$ these integrals will be uniformly close to the respective integrals of the (i) case. Thus, again if r_0 is small enough (but fixed!) then the arguments of (i) may be repeated. □

We have proved that the bounds (3.74)-(3.77) hold in $\widetilde{\mathcal{D}}$ if r_0 is small enough, now we have to extend this to the domain $\mathcal{D}_3(\bar{r})\setminus\widetilde{\mathcal{D}}$ for the solution of the respective Goursat-problems with boundary conditions (3.146). The solution of these Goursat-problems can be expressed by integral equations similar to (3.164) (see [11], [9]). Thus if we have some estimates for the solution on the boundary (which we have from (3.146) and the previous estimates of the Cauchy problem) then these may be extended (up to a constant multiplier) if we have uniform bounds on the integrals of the respective coefficients.

If $\bar{r}$ is small enough then for $(\rho,u)\in\mathcal{D}_3(\bar{r})\setminus\widetilde{\mathcal{D}}$ we have

$$|\Phi_\rho(\rho,u)-\Psi_u(\rho,u)|>c>0$$

which also implies that in this domain $\lambda - \mu > c' > 0$. From this it follows, that we can fix a small enough $\overline{r}_0 > 0$ such that in the domain $\mathcal{D}_3(\overline{r}_0) \setminus \widetilde{\mathcal{D}}$ all the respective coefficients are well-defined smooth functions. Moreover, since this domain is compact, they are all bounded with a fixed constant which means that we have uniform bounds on the respective integrals. Using similar arguments as in the estimate of the solution of the Cauchy-problem one can get the bounds (3.74)-(3.77) also in this domain which completes the proof of Lemma 4.

3.9.3 Proof of Lemma 5

Proof. Note that

$$\big(F - \Psi S_\rho\big)_\rho = \Phi_\rho S_u - \Psi S_{\rho\rho},$$
$$\big(F - \Psi S_\rho\big)_u = \Phi_u S_u - \Psi S_{\rho u}.$$

Also, there exists a constant $C < \infty$ such that for any $(\rho, u) \in \mathcal{D}$

$$|\Psi(\rho, u)| \leq C\rho|u|, \quad |\Phi_\rho(\rho, u)| \leq C, \quad |\Phi_u(\rho, u)| \leq C|u|.$$

From these and the bounds (3.75), (3.76), (3.77) of Lemma 4 it follows that

$$\big|\big(F - \Psi S_\rho\big)_\rho\big|(\rho, u) \leq \frac{C}{\log(\overline{r}/\underline{r})}\big(\sqrt{\overline{r}} + |u|\big)\, \mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho, u),$$
$$\big|\big(F - \Psi S_\rho\big)_u\big|(\rho, u) \leq \frac{C}{\log(\overline{r}/\underline{r})}\big(\rho + \sqrt{\overline{r}}|u|\big)\, \mathbb{1}_{\mathcal{D}_3(\underline{r},\overline{r})}(\rho, u).$$

Integrating these and using (3.70), the bound (3.79) follows. □

3.10 Proof of the "Tools"

3.10.1 Proof of the large deviation bounds (Proposition 2)

Recall the definition (3.99) of L. The following lemma follows from simple coupling arguments.

Lemma 9. (Stochastic dominations)
There exists a constant C depending only on $\max_{(s,x)\in[0,T]\times\mathbb{T}} \rho(s, x)$ and $\max_{(s,x)\in[0,T]\times\mathbb{T}} |u(s, x)|$ such that for any fixed $(s, x) \in [0, T] \times \mathbb{T}$ the following stochastic dominations hold:

$$\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(x) > z\Big) \leq \mathbf{P}\Big(\mathrm{POI}(L) > (z/C)L\Big), \tag{3.171}$$

$$\mathbf{P}_{\nu_s^n}\Big(\,|\widehat{u}^n(x)| > z\Big) \leq \mathbf{P}\Big(\,|\mathrm{GAU}| > \big((z/C) - 1\big)\sqrt{L}\Big), \tag{3.172}$$

where $\mathrm{POI}(L)$ *is a Poissonian random variable with expectation L, and* GAU *is a standard Gaussian random variable.*

Lemma 10. (Large deviation bounds)
(i) For any $\gamma < \infty$ there exists $M < \infty$, such that for any n, $j \in \mathbb{T}^n$ and $s \in [0, T]$

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \widehat{\rho}^n(\frac{j}{n}) \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) \leq 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \widehat{\rho}^n(\frac{j}{n}) \mathbb{1}_{\{|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) \leq 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \big|\widehat{u}^n(\frac{j}{n})\big| \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) \leq 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \big|\widehat{u}^n(\frac{j}{n})\big| \mathbb{1}_{\{|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) \leq 1.
\end{aligned} \tag{3.173}$$

(ii) For any $\gamma \in (0, 1/(8C^2))$ there exists $M < \infty$, such that for any n, $j \in \mathbb{T}^n$ and $s \in [0, T]$

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \big|\widehat{u}^n(\frac{j}{n})\big|^2 \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) \leq 1,\\
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \big|\widehat{u}^n(\frac{j}{n})\big|^2 \mathbb{1}_{\{|\widehat{u}^n(\frac{j}{n})|>M\}}\big\}\Big) \leq 1.
\end{aligned} \tag{3.174}$$

Proof. (i) We prove the first bound of (3.173), the other ones are done very similarly.

Let Z_L be a $POI(L)$-distributed random variable. Using the stochastic domination (3.171) we obtain

$$\begin{aligned}
\log \mathbf{E}_{\nu_s^n}&\Big(\exp\big\{\gamma L \widehat{\rho}^n(\frac{j}{n}) \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big)\\
&\leq \log\Big(1 + \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L \widehat{\rho}^n(\frac{j}{n})\big\} \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\Big)\Big)\\
&\leq \sqrt{\mathbf{E}_{\nu_s^n}\Big(\exp\big\{2\gamma L \widehat{\rho}^n(\frac{j}{n})\big\}\Big)} \sqrt{\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(\frac{j}{n}) > M\Big)}\\
&\leq \sqrt{\mathbf{E}\Big(\exp\big\{\gamma C Z_L\big\}\Big)} \sqrt{\mathbf{P}\Big((C/M) Z_L > L\Big)}\\
&\leq \exp\big\{\frac{L}{2}\big((e^{(\alpha C)/M} - 1) + (e^{\gamma C} - 1) - \alpha\big)\big\},
\end{aligned}$$

where α is arbitrary positive number. In the last step Markov's inequality is being used. Now, choosing $\alpha > \exp(2\gamma C)$ and $M > (C\alpha)/(\ln 2)$ we obtain (3.173).

(ii) Again, we prove the first bound in (3.174). The other one is done in an identical way.

Let again Z_L be a $POI(L)$-distributed and X be a standard Gaussian random variable.

Using the stochastic dominations (3.171) and (3.172) we obtain

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2 \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\big\}\Big) \\
&\quad \le \log\Big(1+\mathbf{E}_{\nu_s^n}\Big(\exp\big\{\gamma L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2\big\} \mathbb{1}_{\{\widehat{\rho}^n(\frac{j}{n})>M\}}\Big)\Big) \\
&\quad \le \sqrt{\mathbf{E}_{\nu_s^n}\Big(\exp\big\{2\gamma L\big|\widehat{u}^n(\tfrac{j}{n})\big|^2\big\}\Big)}\sqrt{\mathbf{P}_{\nu_s^n}\Big(\widehat{\rho}^n(\tfrac{j}{n})>M\Big)} \\
&\quad \le \sqrt{\mathbf{E}\Big(\exp\big\{4\gamma C^2\big(X^2+L\big)\big\}\Big)}\sqrt{\mathbf{P}\Big(Z_L>(M/C)L\Big)} \\
&\quad \le \big(1-8\gamma C^2\big)^{-1/4}\exp\big\{\tfrac{L}{2}\big(4\gamma C^2+(e^{(\alpha C)/M}-1)-\alpha\big)\big\},
\end{aligned}$$

where α is arbitrary positive number. Given $\gamma < 1/(8C^2)$, we choose α sufficiently large and $M > (C\alpha)/(\ln 2)$ to obtain (3.174). □

Now we turn to the proof of Proposition 2:

Proof. The bounds (3.101), respectively, (3.102) follow directly from the entropy inequality (3.100) of Lemma 6 and the bounds (3.173), respectively, (3.174) of Lemma 10. Recall that $L \gg 1$, as $n \to \infty$. □

3.10.2 Proof of the fluctuation bounds (Proposition 3)

Within this proof we need the notation

$$\begin{aligned}
\widetilde{u}^n(s,x) &:= \frac{n^\beta}{l}\sum_k a\big(\frac{nx-k}{l}\big)\Big(\zeta_k - n^{-\beta}u\big(s,\frac{k}{n}\big)\Big) = \widehat{u}^n(x)-\mathbf{E}_{\nu_s^n}\big(\widehat{u}^n(x)\big), \\
\widetilde{\rho}^n(s,x) &:= \frac{n^{2\beta}}{l}\sum_k a\big(\frac{nx-k}{l}\big)\Big(\eta_k - n^{-2\beta}\rho\big(s,\frac{k}{n}\big)\Big) = \widehat{\rho}^n(x)-\mathbf{E}_{\nu_s^n}\big(\widehat{\rho}^n(x)\big).
\end{aligned}$$

Since

$$\begin{aligned}
&\Big|\,\big|\,\widehat{u}^n\big(s,\frac{j}{n}\big)-u\big(s,\frac{j}{n}\big)\,\big| - \big|\,\widetilde{u}^n\big(s,\frac{j}{n}\big)\,\big|\,\Big| \\
&\quad\le \Big|\,1-\frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\,\Big|\,\Big|\,u\big(s,\frac{j}{n}\big)\,\Big| + \frac{1}{l}\sum_k a\big(\frac{j-k}{l}\big)\,\Big|\,u\big(s,\frac{j}{n}\big)-u\big(s,\frac{k}{n}\big)\,\Big| \\
&\quad\le C\left(\frac{1}{l}+\frac{l}{n}\right) = o(1),
\end{aligned}$$

and, similarly

$$\begin{aligned}
&\Big| \, | \, \widetilde{\rho}^n\big(s, \frac{j}{n}\big) - \rho\big(s, \frac{j}{n}\big) \, | - | \, \widetilde{\rho}^n\big(s, \frac{j}{n}\big) \, | \, \Big| \\
&\quad \leq | \, 1 - \frac{1}{l} \sum_k a\big(\frac{j-k}{l}\big) \, | \, | \, \rho\big(s, \frac{j}{n}\big) \, | + \frac{1}{l} \sum_k a\big(\frac{j-k}{l}\big) \, | \, \rho\big(s, \frac{j}{n}\big) - \rho\big(s, \frac{k}{n}\big) \, | \\
&\quad \leq C \left(\frac{1}{l} + \frac{l}{n} \right) = o(1),
\end{aligned}$$

we have to prove

$$\mathbf{E}\Big(\frac{1}{n} \sum_{j \in \mathbb{T}^n} | \, \widetilde{u}^n\big(s, \frac{j}{n}\big) \, |^2 \Big) \leq C \, h^n(s) + o(1), \tag{3.175}$$

respectively,

$$\mathbf{E}\Big(\frac{1}{n} \sum_{j \in \mathbb{T}^n} | \, \widetilde{\rho}^n\big(s, \frac{j}{n}\big) \, |^2 \, \mathbb{1}_{\{ | \, \widetilde{\rho}^n(s, \frac{j}{n}) \, | \leq M \}} \Big) \leq C \, h^n(s) + o(1). \tag{3.176}$$

Lemma 11. *(i) There exists $\gamma > 0$ (sufficiently small) such that for all n, $j \in \mathbb{T}^n$ and $s \in [0, T]$*

$$\log \mathbf{E}_{\nu_s^n} \Big(\exp \big\{ \gamma L \, | \, \widetilde{u}^n\big(s, \frac{j}{n}\big) \, |^2 \big\} \Big) \leq 1 \tag{3.177}$$

(ii) For any $M < \infty$ there exists $\gamma > 0$ (sufficiently small) such that for all n, $j \in \mathbb{T}^n$ and $s \in [0, T]$

$$\log \mathbf{E}_{\nu_s^n} \Big(\exp \big\{ \gamma L \, | \, \widetilde{\rho}^n\big(s, \frac{j}{n}\big) \, |^2 \mathbb{1}_{\{ | \, \widetilde{\rho}^n(s, \frac{j}{n}) \, | \leq M \}} \big\} \Big) \leq 1. \tag{3.178}$$

Proof. (i) Let X be a standard Gaussian random variable, which is independent of all other random variables appearing in this paper, and denote by $\langle \ldots \rangle$ expectation with respect to X.

$$\begin{aligned}
&\log \mathbf{E}_{\nu_s^n} \Big(\exp \big\{ \gamma L \, | \, \widetilde{u}^n\big(s, \frac{j}{n}\big) \, |^2 \big\} \Big) \qquad (3.179) \\
&\quad = \log \mathbf{E}_{\nu_s^n} \Big(\exp \big\{ \frac{\gamma}{l} \, | \, \sum_k a\big(\frac{j-k}{l}\big) \big(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)\big) \, |^2 \big\} \Big) \\
&\quad = \log \Big\langle \mathbf{E}_{\nu_s^n} \Big(\exp \big\{ X \sqrt{\frac{2\gamma}{l}} \sum_k a\big(\frac{j-k}{l}\big) \big(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)\big) \big\} \Big) \Big\rangle.
\end{aligned}$$

Now, note that the random variables $\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)$, $k \in \mathbb{T}^n$, are uniformly bounded and under the distribution $\mathbf{P}_{\nu_s^n}$ they are independent and have zero mean. Hence there exists a finite constant C such that for any collection of real numbers λ_k, $k \in \mathbb{T}^n$

$$\mathbf{E}_{\nu_s^n} \Big(\exp \big\{ \sum_k \lambda_k \big(\zeta_k - \mathbf{E}_{\nu_s^n}(\zeta_k)\big) \big\} \Big) \leq \exp \big\{ C \sum_k \lambda_k^2 \big\}.$$

Further on, there exists a finite constant C such that for any l

$$\frac{1}{l} \sum_k | \, a\big(\frac{k}{l}\big) \, |^2 \leq C. \tag{3.180}$$

From these it follows that for some finite constant C,

$$\text{r.h.s. of (3.179)} \ \leq \log \langle \exp \{ C\,\gamma\, X^2 \} \rangle\,.$$

Choosing γ sufficiently small in this last inequality we obtain (3.177).

(ii) Note first that, given $M<\infty$ fixed, there exists a zero mean bounded random variable Y such that for any $r\in\mathbb{R}$

$$r^2 1\!\!1_{\{\,|\,r\,|\,\leq M\,\}} \leq \log \mathbf{E}\Big(\exp\{\,r\,Y\,\}\Big).$$

Let $Y_1, Y_2, \ldots$ be i.i.d. copies of Y which are also independent of all other random variables appearing in this paper, and denote by $\langle \ldots \rangle$ expectation with respect to these. Then we have

$$\log \mathbf{E}_{\nu_s^n}\Big(\exp\{\gamma L\,|\,\widetilde{\rho}^n(s,\frac{j}{n})\,|^2 1\!\!1_{\{\,|\,\widetilde{\rho}^n(s,\frac{j}{n})\,|\,\leq M\}}\}\Big) \tag{3.181}$$
$$\leq \log\Big\langle \mathbf{E}_{\nu_s^n}\Big(\exp\{\frac{\sum_{p=1}^{\lceil\gamma L\rceil} Y_p}{L}\sum_k a(\frac{j-k}{l})(\eta_k-\mathbf{E}_{\nu_s^n}(\eta_k))\}\Big)\Big\rangle.$$

Next note that for any $\overline{\lambda}<\infty$ there exists a constant $C<\infty$ such that for any $n\in\mathbb{N}$, any $s\in[0,T]$ and any collection of real numbers $\lambda_k\in[-\overline{\lambda},\overline{\lambda}]$, $k\in\mathbb{T}^n$

$$\mathbf{E}_{\nu_s^n}\Big(\exp\{\sum_k \lambda_k\,(\eta_k-\mathbf{E}_{\nu_s^n}(\eta_k))\}\Big)\leq \exp\{Cn^{-2\beta}\sum_k\lambda_k^2\}.$$

Hence, using again (3.180),

$$\text{r.h.s. of (3.181)} \ \leq \log\Big\langle\exp\{C\gamma\,\Big((Y_1+\ldots+Y_{\lceil\gamma L\rceil})/\sqrt{\lceil\gamma L\rceil}\Big)^2\}\Big\rangle.$$

Now, since the i.i.d. random variables $Y_1, Y_2, \ldots$ are bounded and have zero mean, choosing γ sufficiently small this last expression can be made arbitrarily small, uniformly in L. Hence (3.178). □

Now back to the proof of Proposition 3.

Proof. From (3.100) and (3.177), respectively, from (3.100) and (3.178) we deduce (3.175), respectively, (3.176). Finally, these two bounds and the arguments at the beginning of the present subsection imply (3.103), respectively, (3.104). □

3.10.3 Proof of the block replacement and gradient bounds (Proposition 4)

An elementary probability lemma

Let (Ω,π) be a finite probability space and ω_i, $i\in\mathbb{Z}$ i.i.d. Ω-valued random variables with distribution π. Further on let

$$\boldsymbol{\zeta}:\Omega\to\mathbb{R}^d,\qquad \boldsymbol{\zeta}_i:=\boldsymbol{\zeta}(\omega_i),$$
$$\xi:\Omega^m\to\mathbb{R},\qquad \xi_i:=\xi(\omega_i\ldots,\omega_{i+m-1}).$$

For $\boldsymbol{x} \in \mathrm{co}(\mathrm{Ran}(\boldsymbol{\zeta}))$ denote

$$\Xi(\boldsymbol{x}) := \frac{\mathbf{E}_\pi\big(\xi_1 \exp\{\sum_{i=1}^m \boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_i\}\big)}{\mathbf{E}_\pi\big(\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)^m},$$

where $\mathrm{co}(\cdot)$ stands for 'convex hull' and $\boldsymbol{\lambda} \in \mathbb{R}^d$ is chosen so that

$$\frac{\mathbf{E}_\pi\big(\boldsymbol{\zeta}_1 \exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)}{\mathbf{E}_\pi\big(\exp\{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_1\}\big)} = \boldsymbol{x}.$$

For $l \in \mathbb{N}$ we denote *plain* block averages by

$$\overline{\boldsymbol{\zeta}}_l := \frac{1}{l}\sum_{j=1}^{l} \boldsymbol{\zeta}_j.$$

Finally, let $b : [0,1] \to \mathbb{R}$ be a fixed smooth function and denote

$$M(b) := \int_0^1 b(s)\,ds, \qquad V(b) := M(b^2) - M(b)^2.$$

We also define the block averages *weighted by b*

$$\langle b\,,\,\boldsymbol{\zeta}\rangle_l := \frac{1}{l}\sum_{j=0}^{l} b(j/l)\boldsymbol{\zeta}_j, \quad \langle b\,,\,\xi\rangle_l := \frac{1}{l}\sum_{j=0}^{l} b(j/l)\xi_j,$$

The following lemma relies on elementary probability arguments:

Lemma 12. (Microcanonical exponential moments of block averages)
There exists a constant $C < \infty$, depending only on m, on the joint distribution of $(\xi_i, \boldsymbol{\zeta}_i)$ and on the function b, such that the following bounds hold uniformly in $l \in \mathbb{N}$ and $\boldsymbol{x} \in (\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta}))/l$:
(i) If $M(b) = 0$, then

$$\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l}\langle b\,,\,\xi\rangle_l\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(\gamma^2 + \gamma/\sqrt{l})\}. \tag{3.182}$$

(ii) If $M(b) = 1$ then

$$\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l}\big(\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big)\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(\gamma^2 + \gamma/\sqrt{l})\}. \tag{3.183}$$

Proof. We prove the lemma with $m = 1$, that is with $(\xi_i)_{i=1}^l$ independent rather than m-dependent. The m-dependent case follows by applying Jensen's inequality in a rather straightforward way.
(i) In order to simplify the argument we make the assumption that the function $s \mapsto b(s)$ is odd:

$$b(1-s) = -b(s). \tag{3.184}$$

The same argument works if the function $s \mapsto b(s)$ can be rearranged (by permutation of finitely many subintervals of $[0,1]$) into a piecewise continuous odd function. This case is sufficient for our purposes. The proof of the fully general case — which goes through induction on l — is more tedious and it is left as a fun exercise for the reader.

Assuming (3.184) we have

$$\sqrt{l}\langle b\,,\,\xi\rangle_l = l^{-1/2}\sum_{j=0}^{[l/2]} b(j/l)(\xi_j - \xi_{l-j})$$

and hence

$$\begin{aligned}
&\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l}\langle b\,,\,\xi\rangle_l\big\}\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big)\\
&\qquad = \mathbf{E}\Big(\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l}\langle b\,,\,\xi\rangle_l\big\}\,\Big|\,\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}:\ j = 0,\dots,l\Big)\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big)\\
&\qquad = \mathbf{E}\Big(\prod_{j=0}^{[l/2]}\mathbf{E}\Big(\exp\big\{\gamma l^{-1/2} b(j/l)(\xi_j - \xi_{l-j})\big\}\,\Big|\,\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}\Big)\,\Big|\,\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big)\\
&\qquad \le \exp\Big\{C\gamma^2\sum_{j=1}^{[l/2]} l^{-1} b(j/l)^2\Big\}\\
&\qquad = \exp\big\{C\gamma^2(V(b) + \mathcal{O}(1/l))\big\}.
\end{aligned}$$

In the second step we use the fact that the pairs (ξ_j, ξ_{l-j}), $j = 0,\dots,[l/2]$ are independent, given $\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}$, $j = 0,\dots,[l/2]$. In the third step we note that the variables ξ_j are bounded and $\mathbf{E}\big(\xi_j - \xi_{l-j}\big|\boldsymbol{\zeta}_j + \boldsymbol{\zeta}_{l-j}\big) = 0$.

(ii) Beside $\Xi(\boldsymbol{x})$ we also introduce the functions

$$\Xi_l : \big(\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta})\big)/l \to \mathbb{R}, \qquad \Xi_l(\boldsymbol{x}) := \mathbf{E}\big(\xi_1\big|\overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\big).$$

We shall exploit the following facts

(1) The functions $\Xi(\boldsymbol{x})$ and $\Xi_l(\boldsymbol{x})$ are uniformly bounded. This follows from the boundedness of ξ_j.

(2) The function $\boldsymbol{x} \mapsto \Xi(\boldsymbol{x})$ is smooth with bounded first two derivatives. This follows from direct computations.

(3) There exists a finite constant C, such that

$$|\Xi_l(\boldsymbol{x}) - \Xi(\boldsymbol{x})| \le C l^{-1}.$$

This follows from the so-called equivalence of ensembles (see e.g. Appendix 2 of [13]).

We write

$$\begin{aligned}
\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l) \;=\;& \big(\langle b\,,\,\xi\rangle_l - \overline{\xi}_l\big) + \big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big) \\
&+ \big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi(\overline{\boldsymbol{\zeta}}_l)\big) + \big(\Xi(\overline{\boldsymbol{\zeta}}_l) - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l)\big)\,.
\end{aligned} \qquad (3.185)$$

By applying Jensen's inequality we conclude that we have to bound the exponential moments of type (3.183), separately for the four terms.

Bounding the first and last terms reduces directly to (3.182), the third term is uniformly $\mathcal{O}(l^{-1})$, so we only have to bound the exponential moments of the second term in (3.185). This is done by induction on l. Let $C(l)$ be the best constant such that for any $\gamma \in \mathbb{R}$

$$\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l}(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l))\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Big) \le \exp\{C(l)\gamma^2\}.$$

We prove that $C(l)$ stays bounded as $l \to \infty$.

The following identity holds

$$\begin{aligned}\sqrt{l+1}\big(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big) \quad = \quad & \frac{l}{\sqrt{l+1}}\big(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l)\big) \\ & + \frac{1}{\sqrt{l+1}}\big(\xi_{l+1} - \Xi_1(\boldsymbol{\zeta}_{l+1})\big) \\ & + \frac{l}{\sqrt{l+1}}\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big) \\ & + \frac{1}{\sqrt{l+1}}\big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)\end{aligned}$$

Thus

$$\begin{aligned}&\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l+1}(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_{l+1} = \boldsymbol{x}\Big) = \\ &\quad \mathbf{E}\Big(\mathbf{E}\Big(\exp\big\{\gamma\sqrt{l+1}(\overline{\xi}_{l+1} - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l\,,\, \boldsymbol{\zeta}_{l+1}\Big) \,\Big|\, \overline{\boldsymbol{\zeta}}_{l+1} = \boldsymbol{x}\Big) = \\ &\quad \mathbf{E}\Bigg(\qquad \mathbf{E}\Big(\exp\big\{\frac{\gamma l}{\sqrt{l+1}}(\overline{\xi}_l - \Xi_l(\overline{\boldsymbol{\zeta}}_l))\big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l\Big) \times \\ &\qquad \mathbf{E}\Big(\exp\big\{\frac{\gamma}{\sqrt{l+1}}(\overline{\xi}_{l+1} - \Xi_1(\overline{\boldsymbol{\zeta}}_{l+1}))\big\} \,\Big|\, \boldsymbol{\zeta}_{l+1}\Big) \times \\ &\qquad \exp\big\{\frac{\gamma l}{\sqrt{l+1}}(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\big\} \times \\ &\qquad \exp\big\{\frac{\gamma}{\sqrt{l+1}}(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}))\big\} \,\Bigg|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x}\Bigg).\end{aligned}$$

The terms

$$\big(\overline{\xi}_{l+1} - \Xi_1(\overline{\boldsymbol{\zeta}}_{l+1})\big), \quad l\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big), \quad \big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1})\big)$$

are uniformly bounded and

$$\begin{aligned}\mathbf{E}\big(\overline{\xi}_{l+1} - \Xi_1(\overline{\boldsymbol{\zeta}}_{l+1}) \,\big|\, \boldsymbol{\zeta}_{l+1}\big) \quad &= \quad 0, \\ \mathbf{E}\big(\Xi_l(\overline{\boldsymbol{\zeta}}_l) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}) \,\big|\, \overline{\boldsymbol{\zeta}}_{l+1}\big) \quad &= \quad 0, \\ \mathbf{E}\big(\Xi_1(\boldsymbol{\zeta}_{l+1}) - \Xi_{l+1}(\overline{\boldsymbol{\zeta}}_{l+1}) \,\big|\, \overline{\boldsymbol{\zeta}}_{l+1}\big) \quad &= \quad 0.\end{aligned}$$

Using the induction hypothesis it follows that there exists a finite constant B such that

$$C(l+1) \le \frac{l}{l+1} C(l) + \frac{1}{l+1} B.$$

Hence, $\limsup_{l\to\infty} C(l) \le B$ and the lemma follows. □

Lemma 13. (Microcanonical Gaussian bounds)
There exists a $\gamma_0 > 0$, depending only on m, on the joint distribution of $(\xi_i, \boldsymbol{\zeta}_i)$ and on the function b, such that the following bounds hold uniformly in $l \in \mathbb{N}$ and $\boldsymbol{x} \in (\mathrm{Ran}(\boldsymbol{\zeta}) + \cdots + \mathrm{Ran}(\boldsymbol{\zeta}))/l$:
(i) If $M(b) = 0$, then

$$\log \mathbf{E}\Big(\exp\big\{ \gamma_0 l \langle b\,,\,\xi\rangle_l^2 \big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x} \Big) \le 1. \tag{3.186}$$

(ii) If $M(b) = 1$ then

$$\log \mathbf{E}\Big(\exp\big\{ \gamma_0 l \big(\langle b\,,\,\xi\rangle_l - \Xi(\langle b\,,\,\boldsymbol{\zeta}\rangle_l) \big)^2 \big\} \,\Big|\, \overline{\boldsymbol{\zeta}}_l = \boldsymbol{x} \Big) \le 1. \tag{3.187}$$

Proof. This is actually a corollary of Lemma 12: The bounds (3.186) and (3.187) follow from (3.182), respectively, (3.183) by exponential Gaussian averaging (as in the proof of Lemma 11). □

Proof of Proposition 4

Now we turn to the proof of Proposition 4.

Proof. (i) In order to prove (3.107) first note that by simple numerical approximation (no probability bounds involved)

$$\left| \int_{\mathbb{T}} \big| \{ \widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n) \}(x) \big|^2 \, dx - \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big| \{ \widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n) \}(\tfrac{j}{n}) \big|^2 \right| \le \frac{C}{l} = o\left(\frac{l^2}{n^{1+3\beta+\delta}} \right).$$

We apply Lemma 7 with

$$\mathcal{V} = \big| \{ \widehat{\xi}^n - \Xi(\widehat{\eta}^n, \widehat{\zeta}^n) \}(0) \big|^2 = \big| \langle a\,,\,\xi\rangle_l - \Xi\big(\langle a\,,\,\eta\rangle_l, \langle a\,,\,\zeta\rangle_l \big) \big|^2$$

We use the bound (3.187) of Lemma 13 with the function $b = a$. Note that $\gamma = \gamma_0 l$ can be chosen in (3.106). This yields the bound (3.107).
(ii) In order to prove (3.108) we start again with numerical approximation:

$$\left| \int_{\mathbb{T}} \big| \partial_x \widehat{\xi}^n(x) \big|^2 \, dx - \frac{1}{n} \sum_{j \in \mathbb{T}^n} \big| \partial_x \widehat{\xi}^n(\tfrac{j}{n}) \big|^2 \right| \le C \frac{n^2}{l^3} = o(n^{1-3\beta-\delta}).$$

We apply Lemma 7 with

$$\mathcal{V} = \left|\partial_x \widehat{\xi}^n(0)\right|^2 = \frac{n^2}{l^2} \left| \langle a' \, , \, \xi \rangle_l \right|^2.$$

We use now the bound (3.186) of Lemma 13 with the function $b = a'$. Now we can choose $\gamma = \gamma_0 l^3/n^2$ and this will yield the bound (3.108).

(iii) Next we prove (3.109). We apply Lemma 7 with

$$\begin{aligned}\mathcal{V} &= \frac{|\,\partial_x \widehat{\xi}^n(0)\,|^2}{\widehat{\eta}^n(0)} = \frac{n^2}{l^3} \frac{\left|\,\sum_k a'(k/l)\xi_k\,\right|^2}{\sum_k a(k/l)\eta_k} \\ &= \frac{n^2}{2l^3} \frac{\left|\,\sum_k a'(k/l)(\xi_k - \xi_{-k})\,\right|^2}{\sum_k a(k/l)(\eta_k + \eta_{-k})},\end{aligned}$$

where in the last equality we use the fact that the weighting function $x \mapsto a(x)$ is *even*. We compute the exponential moment $\mathbf{E}^{2l+1}_{N,Z}\big(\exp\{\gamma\mathcal{V}\}\big)$. Let X be a standard Gaussian random variable, which is independent of all other random variables appearing in this paper and denote by $\langle \dots \rangle$ averaging with respect to it. We have

$$\begin{aligned}&\mathbf{E}^{2l+1}_{N,Z}\Big(\exp\big\{\gamma\mathcal{V}\big\}\Big) \\ &= \mathbf{E}^{2l+1}_{N,Z}\Big(\exp\big\{\gamma\frac{n^2}{2l^3}\frac{\left|\,\sum_k a'(k/l)(\xi_k - \xi_{-k})\,\right|^2}{\sum_k a(k/l)(\eta_k + \eta_{-k})}\big\}\Big) \\ &= \Big\langle \mathbf{E}^{2l+1}_{N,Z}\Big(\exp\big\{X\sqrt{\gamma}\frac{n}{l^{3/2}}\frac{\sum_k a'(k/l)(\xi_k - \xi_{-k})}{\sqrt{\sum_k a(k/l)(\eta_k + \eta_{-k})}}\big\}\Big)\Big\rangle \\ &= \Big\langle \mathbf{E}^{2l+1}_{N,Z}\Big(\mathbf{E}^{2l+1}_{N,Z}\Big(\exp\big\{X\sqrt{\gamma}\frac{n}{l^{3/2}}\frac{\sum_k a'(k/l)(\xi_k - \xi_{-k})}{\sqrt{\sum_k a(k/l)(\eta_k + \eta_{-k})}}\big\}\,\Big|\,\big\{\eta_k + \eta_{-k}\big\}_{k=0}^{l}\Big)\Big)\Big\rangle \\ &\le \Big\langle \mathbf{E}^{2l+1}_{N,Z}\Big(\exp\big\{CX^2\gamma\frac{n^2}{l^3}\frac{\sum_k a'(k/l)^2(\eta_k + \eta_{-k})}{\sum_k a(k/l)(\eta_k + \eta_{-k})}\big\}\Big)\Big\rangle \\ &\le \Big\langle \exp\big\{CX^2\gamma\frac{n^2}{l^3}\big\}\Big\rangle,\end{aligned}$$

where we used the facts that the random variables η_k are non-negative, Ω is finite and $\eta(\omega) = 0$ implies $\xi(\omega) = 0$. In the last step we used the inequality

$$a'(x)^2 \le Ca(x),$$

which follows from the conditions on $a(x)$, see subsection 3.4.3.

From this bound it follows that in Lemma 7 we can choose $\gamma = \gamma_0 l^3/n^2$, with a small but fixed γ_0, and hence the second bound in (3.109) follows.

□

3.11 Appendix: Some details about the PDE (3.1)

Hyperbolicity: One has to analyze Jacobian the matrix

$$D := \begin{pmatrix} (\rho u)_\rho & (\rho u)_u \\ (\rho + \gamma u^2)_\rho & (\rho + \gamma u^2)_u \end{pmatrix} = \begin{pmatrix} u & \rho \\ 1 & 2\gamma u \end{pmatrix}.$$

The eigenvalues with the corresponding right and left eigenvectors are:

$$Dr = \lambda r, \quad Ds = \mu s, \quad l^\dagger D = \lambda l^\dagger, \quad m^\dagger D = \mu m^\dagger,$$

($v^\dagger$ stands for the transpose of the column 2-vector v). The eigenvalues and eigenvectors are

$$\left.\begin{matrix} \lambda \\ \mu \end{matrix}\right\} = \pm\frac{1}{2}\left\{\sqrt{(2\gamma-1)^2u^2+4\rho} \pm (2\gamma+1)u, \right\}$$

and

$$\left.\begin{matrix} r^\dagger \\ s^\dagger \end{matrix}\right\} = \left(\frac{1}{2}\left\{\mp\sqrt{(2\gamma-1)^2u^2+4\rho} - (2\gamma-1)u\right\}, 1\right),$$

$$\left.\begin{matrix} l^\dagger \\ m^\dagger \end{matrix}\right\} = \left(1, -\frac{1}{2}\left\{\pm\sqrt{(2\gamma-1)^2u^2+4\rho} - (2\gamma-1)u\right\}\right).$$

So, we can conclude that the pde (3.1) is (strictly) hyperbolic in the domain

$$\begin{aligned} \gamma \neq 1/2: &\qquad \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} : (\rho, u) \neq (0,0)\}, \\ \gamma = 1/2: &\qquad \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} : \rho \neq 0\}. \end{aligned}$$

Riemann invariants: The Riemann invariants $w = w(\rho, u)$, $z = z(\rho, u)$ of the pde are given by the relations

$$(w_\rho, w_u) \cdot s = 0 = (z_\rho, z_u) \cdot r.$$

That is, the level lines $w =$ const., respectively $z =$ const. are determined by the ordinary differential equations

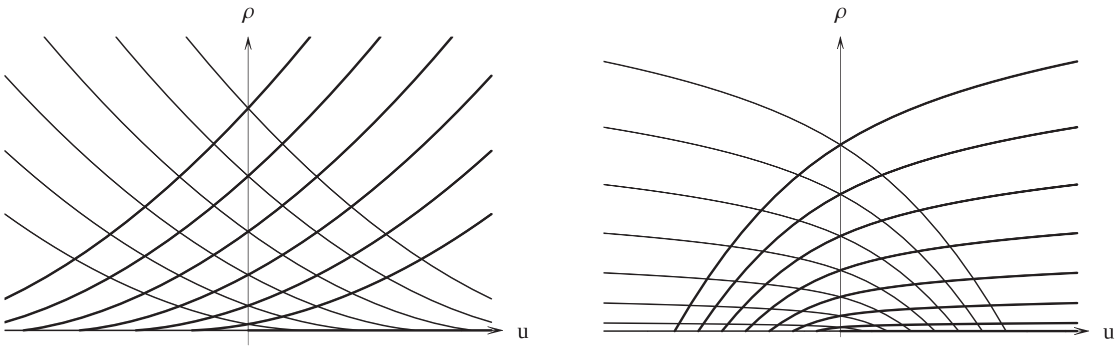
$$\left.\begin{matrix} w = \text{const.} \\ z = \text{const.} \end{matrix}\right\} : \quad \frac{d\rho}{du} = \mp\frac{1}{2}\left\{\sqrt{(2\gamma-1)^2u^2+4\rho} \pm (2\gamma-1)u\right\}$$

Actually only the level lines of the functions $w(\rho, u)$, respectively, $z(\rho, u)$ are determined. In our case the Riemann invariants can be found explicitly. For $\gamma \neq 3/4$ we get

$$\left.\begin{matrix} w(\rho, u) \\ z(\rho, u) \end{matrix}\right\} = F\left\{\left(\sqrt{(2\gamma-1)^2u^2+4\rho} \pm (2\gamma-1)u\right)^{\frac{2\gamma-1}{2\gamma-2}}\left(\sqrt{(2\gamma-1)^2u^2+4\rho} \mp (2\gamma-2)u\right)\right\}$$

Where $F : \mathbb{R} \to \mathbb{R}$ is an appropriately chosen bijection (mind, that only the level sets of the Riemann invariants are determined).

Note that due to the changes of sign of $2\gamma - 1$ and $2\gamma - 2$, the above expression gives rise to *qualitatively different* behavior of the Riemann invariants. The picture changes qualitatively at the critical values $\gamma = 1/2$, $\gamma = 3/4$ and $\gamma = 1$. In Figure 3.3 we present the qualitative picture of the level lines of $w(\rho, u)$ and $z(\rho, u)$ for $3/4 < \gamma < 1$, and $\gamma > 1$, respectively. (For economy reasons we omit the graphical representation of the other cases, but encourage the reader to sketch it.) In all cases the Riemann invariants satisfy the convexity conditions



(a) $3/4 < \gamma < 1$ (b) $1 < \gamma$

Figure 3.3: Level lines of Riemann-invariants

$$\begin{aligned} w_{\rho\rho} w_u^2 - 2 w_{\rho u} w_\rho w_u + w_{uu} w_\rho^2 \geq 0, \\ z_{\rho\rho} z_u^2 - 2 z_{\rho u} z_\rho z_u + z_{uu} z_\rho^2 \geq 0, \end{aligned} \tag{3.188}$$

in $\mathbb{R}_+ \times \mathbb{R}$ for all γ. (The sign of the function $F(\cdot)$ is so chosen, that these expressions be non-negative.) The inequalities are strict in the interior of $\mathbb{R}_+ \times \mathbb{R}$, except for the $\gamma = 1$ case, when these expressions identically vanish. These conditions are equivalent to saying that the level sets $\{(\rho, u) \in [0, \infty) \times (-\infty, \infty) : w(\rho, u) < c\}$ and $\{(\rho, u) \in [0, \infty) \times (-\infty, \infty) : z(\rho, u) < c\}$ be convex. See [17], [18] or [29] for the importance of these convexity conditions.

It is of crucial importance for our problem that the level curves $w(\rho, u) = w =$ const. expressed as $u \mapsto \rho(u, w)$ are convex for $\gamma < 1$, linear for $\gamma = 1$ and concave for $\gamma > 1$.

Genuine nonlinearity: Genuine nonlinearity holds if and only if

$$(\lambda_\rho, \lambda_u) \cdot r \neq 0 \neq (\mu_\rho, \mu_u) \cdot s.$$

in the interior of the domain $\mathbb{R}_+ \times \mathbb{R}$. Elementary computations show that

$$\left. \begin{array}{l} (\lambda_\rho, \lambda_u) \cdot r = 0 \\ (\mu_\rho, \mu_u) \cdot s = 0 \end{array} \right\} \quad \Leftrightarrow \quad \rho = -\frac{4\gamma(2\gamma - 1)^2}{(\gamma + 1)^2} u^2 \text{ and } \left\{ \begin{array}{l} u \leq 0 \\ u \geq 0 \end{array} \right. . \tag{3.189}$$

Thus, for $\gamma \geq 0$, $\gamma \neq 0, 1/2$ the system is genuinely nonlinear on the closed domain $\mathbb{R}_+ \times \mathbb{R}$; for $\gamma = 0, 1/2$ it is genuinely nonlinear in the interior of $\mathbb{R}_+ \times \mathbb{R}$ (with genuine nonlinearity marginally lost on the boundary, $\rho = 0$). For $\gamma < 0$ genuine nonlinearity is lost in the interior of $\mathbb{R}_+ \times \mathbb{R}$.

Lax entropies and entropy solutions: Lax entropies of the pde (3.1) are solutions of the linear hyperbolic partial differential equation

$$\rho S_{\rho\rho} + (2\gamma - 1)uS_{\rho u} - S_{uu} = 0.$$

It turns out that the system is sufficiently rich in Lax entropies. In particular a Lax entropy globally convex in $\mathbb{R}_+ \times \mathbb{R}$ is

$$S(\rho, u) = \rho \log \rho + \frac{u^2}{2}. \tag{3.190}$$

Construction of other Lax entropies with particular features (e.g. possessing scale similarity, or polynomial in $\sqrt{\rho}$ and u, etc.) is a very instructive exercise.

The Maximum Principle and positively invariant domains: For $\gamma \geq 0$ our systems satisfy the conditions of the Lax's Maximum Principle proved in [17]. Namely: (i) they do posses a globally strictly convex Lax entropy bounded from below, see (3.190); (ii) the Riemann invariants $w(\rho, u)$ and $z(\rho, u)$ satisfy the convexity condition (3.188); (iii) they are genuinely nonlinear in the interior of $\mathcal{D}$, see (3.189).

Hence it follows that *convex domains bounded by level curves of $w(\rho, u)$ and $z(\rho, u)$ are positively invariant for entropy solutions.*

First we conclude, that $\mathcal{D}$ itself is positively invariant domain, as it should be.

Second: a very essential difference between the cases $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$ follows, which is of crucial importance for the main result of the present paper. In the case $\gamma < 1$ all convex domains bounded by level curves of the Riemann invariants are *unbounded (non-compact)* and thus there is no a priori bound on the solutions. Even starting with smooth initial data with compact support nothing prevents the entropy solutions to blow up indefinitely after appearance of the shocks. On the other hand, if $\gamma \geq 1$ any compact subset of $\mathcal{D}$ is contained in a compact convex domain bounded by level sets of the Riemann invariants, which fact yields a priori bounds on the entropy solutions, given bounded initial data. A microscopic consequence of this fact is that the proof of our main theorem is valid only for $\gamma > 0$.

Acknowledgement: It is our pleasure to thank József Fritz for the many discussions on the content of this paper, his permanent interest and encouragement. We also thank Peter Lax for a very inspirative consultation on hyperbolic conservation laws.
The kind hospitality of Institut Henri Poincaré (Paris) and that of the Isaac Newton Institute (Cambridge), where parts of this work were completed, is gratefully acknowledged.
The research work of the authors is partially supported by the Hungarian Scientific Research Fund (OTKA) grant no. T037685.

Chapter 4

Hydrodynamic limit for perturbation of a hyperbolic equilibrium point in two-component systems*

We consider one-dimensional, locally finite interacting particle systems with two conservation laws. The models have a family of stationary measures with product structure and we assume the existence of a uniform bound on the inverse of the spectral gap which is quadratic in the size of the system. Under Eulerian scaling the hydrodynamic limit for the macroscopic density profiles leads to a two-component system of conservation laws. The resulting pde is hyperbolic inside the physical domain of the macroscopic densities, with possible loss of hyperbolicity at the boundary.

We investigate the propagation of small perturbations around a *hyperbolic* equilibrium point. We prove that the perturbations essentially evolve according to two *decoupled* Burgers equations. The scaling is not Eulerian: if the lattice constant is n^{-1}, the perturbations are of order $n^{-\beta}$ then time is speeded up by $n^{1+\beta}$. Our derivation holds for $0 < \beta < \frac{1}{5}$. The proof relies on Yau's relative entropy method, thus it applies only in the regime of smooth solutions.

This result is an extension of [28] and [34] where the analogue result was proved for systems with one conservation law. It also complements [36] where it was shown that perturbations around a non-hyperbolic boundary equilibrium point are driven by a universal two-by-two system of conservation laws.

4.1 Introduction

There are several results dealing with the perturbation analysis of hydrodynamic limits for interacting particle systems. In the landmark paper [4] the authors prove that for the asymmetric simple exclusion, in dimensions higher than 2, perturbations of order n^{-1} of a constant profile

*This chapter contains the submitted paper [40].

evolve according to a certain parabolic equation under diffusive scaling (time rescaled by n^2, space by n). It is well-known, that under Eulerian scaling (time rescaled by n, space by n) the hydrodynamic limit leads to a hyperbolic conservation law (the Burgers equation), the perturbation limit gives the same equation with the Navier-Stokes correction. (For a survey on the microscopic interpretations of the Navier-Stokes equations see the end of Chapter 7 of [13].)

Motivated by [4] T. Seppäläinen investigated a similar problem in one dimension for the so-called totally asymmetric stick process. In [28] he proves that an $\mathcal{O}(n^{-\beta})$ perturbation of the constant profile is governed by the Burgers equation (even after the appearance of shocks) if time is rescaled by $n^{1+\beta}$ and space by n, where $\beta \in (0, \frac{1}{2})$ is a fixed constant. Independently, in [34] the authors partially extend this result by proving that one gets *universally* the Burgers equation in the hydrodynamic limit for similar perturbations of equilibrium for a wide class of one-dimensional interacting particle systems with one conservation law. The models are not reversible and not necessarily attractive. The proof relies on H. T. Yau's relative entropy method, it only applies in the smooth regime of solutions and it only works for $\beta \in (0, \frac{1}{5})$. It is conjectured that the result should hold for all $\beta \in (0, \frac{1}{2})$ even without the smoothness condition as in the result of [28].

This universal result may be explained by the following arguments. Under Eulerian scaling these systems admit in the hydrodynamic limit a hyperbolic conservation law of the form

$$\partial_t u + \partial_x J(u) = 0. \tag{4.1}$$

Taking a point u_0 with $J''(u_0) \neq 0$ simple (although formal) calculations yield that solutions of (4.1), with initial conditions which are small perturbations u_0, are governed by the Burgers equation. See [34] for the 'more precise' formulation.

In the present paper we give an extension of the results of [28, 34] for systems with 2 conserved quantities. In [35] a general one-dimensional family of lattice-models was introduced. The models are locally finite interacting particle systems with two conservation laws which possess a family of stationary measures with product structure. In that paper it is shown (in the regime of smooth solutions) that in Eulerian scaling we get a hydrodynamic limit of the form

$$\begin{cases} \partial_t u + \partial_x \Phi(u, v) & = & 0, \\ \partial_t v + \partial_x \Psi(u, v) & = & 0, \end{cases} \tag{4.2}$$

where $(u, v) \in \mathcal{D}$ and $\mathcal{D}$ is a convex compact polygon, the the physical domain (see (4.6) for the definition). We note, that [21] gives the first major result about the Eulerian hydrodynamic limit for multi-component hyperbolic systems, namely for Hamiltonian systems perturbed by a weak noise. In [35] it was also shown that an Onsager-type symmetry relation holds for the macroscopic flux functions Φ, Ψ (see Lemma 14). One of the consequences of this relation is that inside the physical domain $\mathcal{D}$ the pde (4.2) is (weakly) hyperbolic, i.e. the Jacobian can be diagonalized in the real sense. Experience shows, that the limiting pde is strongly hyperbolic (the Jacobian has two distinct real eigenvalues) in the whole physical domain except some special points on the boundary $\partial\mathcal{D}$.

We consider perturbations of order $n^{-\beta}$ around a constant equilibrium point $(u_0, v_0) \in \mathcal{D}$, which is *strictly* hyperbolic. We prove that rescaling time by $n^{1+\beta}$ and space by n the evolution of the perturbations are governed by two *decoupled* equations. (These are 'usually' Burgers equations, see the remark at the end of subsection 4.3.1.) This result agrees with the formal perturbation of the pde (4.2) e.g. with the method of weakly nonlinear geometric optics (see [3, 10]).

The reason for the decoupling of the resulting pde system is the strict hyperbolicity, basically, the two different eigenvalues (sound speeds) cause the equations to separate. In the paper [36] perturbation around a special *non-hyperbolic* point was considered in a similar setting, it was proved that in that case in the limit the evolution obeys a two-by-two system of conservation laws which cannot be decoupled. The treatment of that problem needs more complex tools than our proofs, sophisticated pde methods are used besides Yau's method.

Our proof follows the relative entropy method using similar steps as [34] (thus it only applies in the regime of smooth solutions), but it also heavily relies on the Onsager-type symmetry relation proved in [35]. We assume the existence of a uniform bound on the inverse of the spectral gap, quadratic in system size, to be able to prove the so-called one block estimate. We do not deal with the proof of the spectral gap bound, but we remark that with the techniques of [16] one can get the desired gap estimates for a large class of systems. Our result holds for $\beta \in (0, \frac{1}{5})$. Assuming the stronger (but harder to prove) logarithmic-Sobolev bound we could get the result for $\beta \in (0, \frac{1}{3})$.

4.2 Microscopic models

We consider the family of microscopic models investigated in [35]. We go over the definitions and the important properties, for the details we refer the reader to the original paper. There are several concrete examples introduced in [35], we do not list them here.

4.2.1 State space, conserved quantities, generator

Throughout this paper we denote by $\mathbb{T}^n$ the discrete tori $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{N}$, and by $\mathbb{T}$ the continuous torus $\mathbb{R}/\mathbb{Z}$. We will denote the local spin state by Ω, we only consider the case when Ω is finite. The state space of the interacting particle system is

$$\Omega^n := \Omega^{\mathbb{T}^n}.$$

Configurations will be denoted

$$\underline{\omega} := (\omega_j)_{j\in\mathbb{T}^n} \in \Omega^n,$$

The two conserved quantities are denoted by

$$\zeta : \Omega \to \mathbb{Z}, \quad \eta : \Omega \to \mathbb{Z},$$

we also use the notations $\zeta_j = \zeta(\omega_j)$, $\eta_j = \eta(\omega_j)$. We assume that the conserved quantities are different and non-trivial, i.e. the functions ζ, η and the constant function 1 on Ω are linearly independent.

We consider the *rate function* $r : \Omega \times \Omega \times \Omega \times \Omega \to \mathbb{R}_+$. The dynamics of the system consists of elementary jumps effecting nearest neighbor spins, $(\omega_j, \omega_{j+1}) \longrightarrow (\omega'_j, \omega'_{j+1})$, performed with rate $r(\omega_j, \omega_{j+1}; \omega'_j, \omega'_{j+1})$.

We require that the rate function r satisfy the following conditions:

(A) If $r(\omega_1, \omega_2; \omega'_1, \omega'_2) > 0$ then

$$\begin{array}{rcl} \zeta(\omega_1) + \zeta(\omega_2) & = & \zeta(\omega'_1) + \zeta(\omega'_2), \\ \eta(\omega_1) + \eta(\omega_2) & = & \eta(\omega'_1) + \eta(\omega'_2). \end{array} \tag{4.3}$$

This means that ζ and η are indeed conserved quantities.

(B) For every $Z \in [n \min \zeta, n \max \zeta] \cap \mathbb{Z}$, $N \in [n \min \eta, n \max \eta] \cap \mathbb{Z}$ the set

$$\Omega^n_{Z,N} := \left\{ \underline{\omega} \in \Omega^n : \sum_{j \in \mathbb{T}^n} \zeta_j = Z, \sum_{j \in \mathbb{T}^n} \eta_j = N \right\}$$

is an irreducible component of Ω^n, i.e. if $\underline{\omega}, \underline{\omega}' \in \Omega^n_{Z,N}$ then there exists a series of elementary jumps with positive rates transforming $\underline{\omega}$ into $\underline{\omega}'$. This ensures that there are no hidden conservation laws.

(C) There exists a probability measure π on Ω which puts positive mass on each element of Ω and for any ω_1, ω_2, ω'_1, $\omega'_2 \in \Omega$

$$Q(\omega_1, \omega_2) + Q(\omega_2, \omega_3) + Q(\omega_3, \omega_1) = 0,$$

where

$$Q(\omega_1, \omega_2) := \sum_{\omega'_1, \omega'_2 \in \Omega} \left\{ \frac{\pi(\omega'_1)\pi(\omega'_2)}{\pi(\omega_1)\pi(\omega_2)} r(\omega'_1, \omega'_2; \omega_1, \omega_2) - r(\omega_1, \omega_2; \omega'_1, \omega'_2) \right\}.$$

This condition will imply that the measure $\prod_{j \in \mathbb{T}^n} \pi$ is stationary for our process on Ω^n.

For a precise formulation of the infinitesimal generator on Ω^n we first define the map $\Theta_j^{\omega',\omega''} : \Omega^n \to \Omega^n$ for every $\omega', \omega'' \in \Omega$, $j \in \mathbb{T}^n$:

$$\left(\Theta_j^{\omega',\omega''} \underline{\omega}\right)_i = \begin{cases} \omega' & \text{if} \quad i = j \\ \omega'' & \text{if} \quad i = j+1 \\ \omega_i & \text{if} \quad i \neq j, j+1. \end{cases}$$

The infinitesimal generator of the process defined on Ω^n is

$$L^n f(\underline{\omega}) = \sum_{j \in \mathbb{T}^n} \sum_{\omega', \omega'' \in \Omega} r(\omega_j, \omega_{j+1}; \omega', \omega'')(f(\Theta_j^{\omega',\omega''} \underline{\omega}) - f(\underline{\omega})).$$

We denote by $\mathcal{X}^n_t$ the Markov process on the state space Ω^n with infinitesimal generator L^n.

4.2.2 Stationary measures

For every $\theta, \tau \in \mathbb{R}$ let $G(\theta,\tau)$ be the moment generating function defined below:

$$G(\theta,\tau) := \log \sum_{\omega\in\Omega} e^{\theta\zeta(\omega)+\tau\eta(\omega)}\pi(\omega).$$

We define the probability measures

$$\pi_{\theta,\tau}(\omega) := \pi(\omega)\exp(\theta\zeta(\omega)+\tau\eta(\omega)-G(\theta,\tau)) \tag{4.4}$$

on Ω. Using condition (C), by very similar considerations as in [1], [2], [27] or [34] one can show that for any $\theta,\tau\in\mathbb{R}$ the product measure

$$\pi^n_{\theta,\tau} := \prod_{j\in\mathbb{T}^n} \pi_{\theta,\tau}$$

is stationary for the Markov process $\mathcal{X}^n_t$ on Ω^n with infinitesimal generator L^n. We will refer to these measures as the *canonical* measures. Since $\sum_j \zeta_j$ and $\sum_j \eta_j$ are conserved, the canonical measures on Ω^n are not ergodic. The conditioned measures defined on $\Omega^n_{Z,N}$ by:

$$\pi^n_{Z,N}(\underline{\omega}) := \pi^n_{\theta,\tau}\left(\underline{\omega}\,\middle|\,\sum_j \zeta_j = Z,\, \sum_j \eta_j = N\right) = \frac{\pi^n_{\theta,\tau}(\underline{\omega})1\!\!1\{\underline{\omega}\in\Omega^n_{Z,N}\}}{\pi^n_{\theta,\tau}(\Omega^n_{Z,N})}$$

are also stationary and due to condition (B) satisfied by the rate functions they are also ergodic. We shall call these measures the *microcanonical measures* of our system. It is easy to see that the measure $\pi^n_{Z,N}$ does not depend on the values of θ,τ.

4.2.3 Expectations, fluxes

Expectation, variance, covariance with respect to the measures $\pi^n_{\theta,\tau}$ will be denoted by $\mathbf{E}_{\theta,\tau}(.)$, $\mathbf{Var}_{\theta,\tau}(.)$, $\mathbf{Cov}_{\theta,\tau}(.)$.

We compute the expectations of the conserved quantities with respect to the canonical measures, as functions of the parameters θ and τ:

$$\begin{aligned} u(\theta,\tau) &:= \mathbf{E}_{\theta,\tau}(\zeta) = \sum_{\omega\in\Omega}\zeta(\omega)\pi_{\theta,\tau}(\omega) = \partial_\theta G(\theta,\tau) = G_\theta, \\ v(\theta,\tau) &:= \mathbf{E}_{\theta,\tau}(\eta) = \sum_{\omega\in\Omega}\eta(\omega)\pi_{\theta,\tau}(\omega) = \partial_\tau G(\theta,\tau) = G_\tau. \end{aligned}$$

We will usually note partial derivatives by using the respective subscripts, as long as it does not cause confusion. Elementary calculations show, that the matrix-valued function

$$\begin{pmatrix} u_\theta & u_\tau \\ v_\theta & v_\tau \end{pmatrix} = \begin{pmatrix} G_{\theta\theta} & G_{\theta\tau} \\ G_{\theta\tau} & G_{\tau\tau} \end{pmatrix} =: G''(\theta,\tau)$$

is equal to the covariance matrix $\mathbf{Cov}_{\theta,\tau}(\zeta,\eta)$ and as a consequence the function $(\theta,\tau)\mapsto(u(\theta,\tau),v(\theta,\tau))$ is invertible. We denote the inverse function by $(u,v)\mapsto(\theta(u,v),\tau(u,v))$.

Denote by $(u,v) \mapsto S(u,v)$ the convex conjugate (Legendre transform) of the strictly convex function $(\theta,\tau) \mapsto G(\theta,\tau)$:

$$S(u,v) := \sup_{\theta,\tau} \big(u\theta + v\tau - G(\theta,\tau) \big), \tag{4.5}$$

and

$$\begin{aligned} \mathcal{D} &:= \{(u,v) \in \mathbb{R}_+ \times \mathbb{R} : S(u,v) < \infty\} \\ &= \text{co}\{(\zeta(\omega), \eta(\omega)) : \omega \in \Omega\}, \end{aligned} \tag{4.6}$$

where co stands for convex hull. In probabilistic terms: $S(u,v)$ is the rate function for joint large deviations of $(\sum_j \zeta_j, \sum_j \eta_j)$. If (u,v) is inside $\mathcal{D}$ then we have

$$\theta(u,v) = S_u(u,v), \qquad \tau(u,v) = S_v(u,v).$$

With slight abuse of notation we shall denote: $\pi_{\theta(u,v),\tau(u,v)} =: \pi_{u,v}$, $\pi^n_{\theta(u,v),\tau(u,v)} =: \pi^n_{u,v}$, $\mathbf{E}_{\theta(u,v),\tau(u,v)} =: \mathbf{E}_{u,v}$, etc. Clearly, $\pi_{u,v}$ can be defined naturally on the boundary of $\mathcal{D}$, in that case $\pi_{u,v}$ does not puts zero weight on some of the elements of Ω.

We introduce the flux of the conserved quantities. The infinitesimal generator L^n acts on the conserved quantities as follows:

$$\begin{aligned} L^n\zeta_i &= -\phi(\omega_i,\omega_{i+1}) + \phi(\omega_{i-1},\omega_i) =: -\phi_i + \phi_{i-1}, \\ L^n\eta_i &= -\psi(\omega_i,\omega_{i+1}) + \psi(\omega_{i-1},\omega_i) =: -\psi_i + \psi_{i-1}, \end{aligned}$$

where

$$\begin{aligned} \phi(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2' \in \Omega} r(\omega_1,\omega_2;\omega_1',\omega_2')(\zeta(\omega_2') - \zeta(\omega_2)) + C_1 \\ \psi(\omega_1,\omega_2) &:= \sum_{\omega_1',\omega_2' \in \Omega} r(\omega_1,\omega_2;\omega_1',\omega_2')(\eta(\omega_2') - \eta(\omega_2)) + C_2 \end{aligned} \tag{4.7}$$

(The constants C_1, C_2 may be chosen arbitrarily, we will fix them later.) We shall denote the expectations of these functions with respect to the canonical measure $\pi^2_{u,v}$ by

$$\Phi(u,v) := \mathbf{E}_{u,v}(\phi), \qquad \Psi(u,v) := \mathbf{E}_{u,v}(\psi). \tag{4.8}$$

The following lemma was proved in [35].

Lemma 14. *Suppose we have a particle system with two conserved quantities and rates satisfying conditions (A) and (C). Then*

$$\partial_\theta \Psi\left(u(\theta,\tau), v(\theta,\tau)\right) = \partial_\tau \Phi\left(u(\theta,\tau), v(\theta,\tau)\right).$$

The first derivative matrix of the fluxes Φ and Ψ (with resp. to u, v) will be denoted by

$$D = D(u,v) := \begin{pmatrix} \Phi_u & \Phi_v \\ \Psi_u & \Psi_v \end{pmatrix}. \tag{4.9}$$

From Lemma 14 it follows that $D(u,v)$ is (weakly) hyperbolic, it can be diagonalized in a real sense (see [35]). We denote the two eigenvalues of D by λ and μ, and the corresponding right and left eigenvectors by $\mathbf{r} = (r_1, r_2)^\dagger, \mathbf{s} = (s_1, s_2)^\dagger$ and $\mathbf{l} = (l_1, l_2), \mathbf{m} = (m_1, m_2)$:

$$D\mathbf{r} = \lambda\mathbf{r}, \quad \mathbf{l}D = \lambda\mathbf{l},$$
$$D\mathbf{s} = \mu\mathbf{s}, \quad \mathbf{m}D = \mu\mathbf{m}.$$

Although we do not denote it explicitly, all of these are functions of (u, v). We can assume

$$|\mathbf{r}| = |\mathbf{s}| = 1, \quad \mathbf{l}\cdot\mathbf{r} = 1, \quad \mathbf{m}\cdot\mathbf{s} = 1.$$

The second derivatives of the macroscopic fluxes are denoted by Φ'', Ψ'', these are symmetric two-by-two matrices depending on (u, v).

4.2.4 The spectral gap condition

Let l be a positive integer and (Z, N) integers with $Z \in [l \min \zeta, l \max \zeta]$, $N \in [l \min \eta, l \max \eta]$. Expectation with respect to the measure $\pi^l_{Z,N}$ is denoted by $\mathbf{E}^l_{Z,N}(\,\cdot\,)$. For $f : \Omega^l_{Z,N} \to \mathbb{R}$ let

$$L^l_{Z,N} f(\underline{\omega}) := \sum_{j=1}^{l-1} \sum_{\omega',\omega''} r(\omega_j, \omega_{j+1}; \omega', \omega'')\big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big),$$

$$D^l_{Z,N}(f) := \frac{1}{2}\sum_{j=1}^{l-1} \mathbf{E}^l_{Z,N}\left(\sum_{\omega',\omega''} r(\omega_j, \omega_{j+1}; \omega', \omega'')\big(f(\Theta^{\omega',\omega''}_{j,j+1}\underline{\omega}) - f(\underline{\omega})\big)^2\right).$$

$L^l_{Z,N}$ is the infinitesimal generator restricted to the hyperplane $\Omega^l_{Z,N}$, and $D^l_{Z,N}$ is the Dirichlet form associated to $L^l_{Z,N}$ (or to its symmetric part). Note, that $L^l_{Z,N}$ is defined with *free boundary conditions.*

We will assume the following additional condition on our models:

(D) There exists a positive constant W independent of l, Z, N such that for any $f : \Omega^l_{Z,N} \to \mathbb{R}$ with $\mathbf{E}^l_{Z,N} f = 0$

$$\mathbf{E}^l_{N,Z} f^2 \leq W\, l^2\, D^l_{Z,N}(f).$$

Remark. Presumably (D) is true for all (or a large class of) the models satisfying conditions (A)-(C). The techniques of [16] should be suitable to get the desired gap estimates, but we do not know about any published results covering our case.

4.3 Perturbation of the Eulerian hdl

In [35] it was proved by the application of Yau's relative entropy method, that under Eulerian scaling the local density profiles of the conserved quantities evolve according to the following system of partial differential equations:

$$\begin{cases} \partial_t u + \partial_x \Phi(u,v) &= 0 \\ \partial_t v + \partial_x \Psi(u,v) &= 0. \end{cases} \tag{4.10}$$

This pde is usually a strictly hyperbolic conservation law (i.e. $D(u,v)$ has two distinct real eigenvalues), weak hyperbolicity follows from Lemma 14 (see [35]). Since the relative entropy method needs smoothness conditions for the solution of the limiting equation, the previous result holds only up to a finite time, till the appearance of the first shock. We note, that [21] gives the first major result about the Eulerian hydrodynamic limit for multi-component hyperbolic systems, also with the application of Yau's method.

4.3.1 Formal perturbation

We will investigate the hydrodynamic behavior of small perturbations of an equilibrium point. For that we need to understand the asymptotics of small perturbations of a constant solution of (4.10). One of the perturbation techniques is the so-called method of weakly nonlinear geometric optics (see e.g. [3, 10]) which gives the following *formal* result.

Fix a point (u_0, v_0) in $\mathcal{D}$ and suppose that this point is strictly hyperbolic, i.e.

$$\lambda \neq \mu, \tag{4.11}$$

at (u_0, v_0). Suppose $(u_\varepsilon(t,x), v_\varepsilon(t,x))$ is the solution of the pde (4.10) with initial conditions

$$\begin{aligned} u_\varepsilon(0,x) &= u_0 + \varepsilon u^*(x), \\ v_\varepsilon(0,x) &= v_0 + \varepsilon v^*(x), \end{aligned}$$

where $u^*(x), v^*(x)$ are fixed $\mathbb{T} \mapsto \mathbb{R}$ smooth functions. Denote

$$\begin{aligned} \sigma_0(x) &:= \mathbf{l} \cdot (u^*(x), v^*(x))^\dagger, \qquad c_\sigma := \textstyle\int_{\mathbb{T}} \sigma_0(y) dy, \\ \delta_0(x) &:= \mathbf{m} \cdot (u^*(x), v^*(x))^\dagger, \qquad c_\delta := \textstyle\int_{\mathbb{T}} \delta_0(y) dy, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} a_1 &:= \mathbf{l} \cdot (\mathbf{r}^\dagger \, \Phi'' \, \mathbf{r}, \mathbf{r}^\dagger \, \Psi'' \, \mathbf{r})^\dagger, \qquad a_2 := \mathbf{l} \cdot (\mathbf{r}^\dagger \, \Phi'' \, \mathbf{s}, \mathbf{r}^\dagger \, \Psi'' \, \mathbf{s})^\dagger, \\ b_1 &:= \mathbf{m} \cdot (\mathbf{s}^\dagger \, \Phi'' \, \mathbf{s}, \mathbf{s}^\dagger \, \Psi'' \mathbf{s})^\dagger, \qquad b_2 := \mathbf{m} \cdot (\mathbf{r}^\dagger \, \Phi'' \, \mathbf{s}, \mathbf{r}^\dagger \, \Psi'' \mathbf{s})^\dagger, \end{aligned} \tag{4.13}$$

where $\mathbf{l}, \mathbf{m}, \mathbf{r}, \mathbf{s}$ and Φ'', Ψ'' are the respective vector- and matrix-valued functions taken at (u_0, v_0).

Then, according to the formal computations of the geometric optics method,

$$\begin{pmatrix} u_\varepsilon(t,x) \\ v_\varepsilon(t,x) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \varepsilon \sigma(\varepsilon t, x - \lambda t) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \varepsilon \delta(\varepsilon t, x - \mu t) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} + \mathcal{O}(\varepsilon^2), \tag{4.14}$$

as $\varepsilon \to 0$, where σ and δ are the solutions of the following Cauchy problems:

$$\left\{ \begin{array}{rcl} \partial_t\sigma(t,x) + \partial_x \left(a_1 \cdot \frac{1}{2}\sigma(t,x)^2 + c_\delta\, a_2\, \sigma(t,x)\right) & = & 0, \\ \sigma(0,x) & = & \sigma_0(x), \end{array} \right. \tag{4.15}$$

and

$$\left\{ \begin{array}{rcl} \partial_t\delta(t,x) + \partial_x \left(b_1 \cdot \frac{1}{2}\delta(t,x)^2 + c_\sigma\, b_2\, \delta(t,x)\right) & = & 0, \\ \delta(0,x) & = & \delta_0(x). \end{array} \right. \tag{4.16}$$

Remarks

1. This result means that a small perturbation of a constant solution of (4.10) is governed by the solutions of two decoupled equations (at least, by formal computations). If a_1 and b_1 are nonzero, then these equations are linear transforms of the Burgers equation. Otherwise the respective equations become linear transport equations. It is easy to check, that $a_1 \neq 0$, $b_1 \neq 0$ hold exactly when the point (u_0, v_0) is *genuinely nonlinear*, i.e.

$$\nabla\lambda \cdot \mathbf{r} \neq 0, \quad \nabla\mu \cdot \mathbf{s} \neq 0$$

at (u_0, v_0).

2. The geometric optics method is based on series expansion, thus it needs smoothness as a condition which could only be true up to a finite time in our case. Surprisingly, this formal method gives good approximation of the solutions even after the shocks. In [3] the authors prove that the equation (4.14) is valid, in the sense that for any $t > 0$ the L_1-norm of the difference of the two sides is bounded by $C\,t\,\varepsilon^2$. In fact, this result is valid for the case if we consider the pde (4.10) on $\mathbb{T}$ (as we do), on $\mathbb{R}$ they have even stronger bounds.

4.3.2 The main result

Our main theorem is a similar result on the microscopic level. We will apply Yau's method, thus our results will hold in the regime of smooth solutions, only up to a finite time before the first appearance of shocks.

Suppose, that (u_0, v_0) is a point in the physical domain which is strictly hyperbolic, see (4.11). Let $u^*(x), v^*(x)$ be smooth real functions on $\mathbb{T}$. Define $\sigma(t,x), \delta(t,x)$ according to (4.12), (4.13), (4.15) and (4.16), and suppose that they are smooth in $\mathbb{T} \times [0,T]$. Fix a small positive parameter β, and suppose that a particle system on Ω^n satisfying conditions (A)-(D) has initial distribution for which the density profiles of the two conserved quantities are 'close' to the functions $u_0 + n^{-\beta}u^*(\cdot)$, $v_0 + n^{-\beta}v^*(\cdot)$. I.e. the profiles are a small perturbation of the constant (u_0, v_0) profile. We also assume, that $(u_0 + n^{-\beta}u^*(x),\ v_0 + n^{-\beta}v^*(x)) \in \mathcal{D}$ holds for every $x \in \mathbb{T}$, at least for $n > n_0$. Then, uniformly for $0 \leq t \leq T$, at time $n^{1+\beta}t$ the respective density profiles will be 'close' to the functions $u_0 + n^{-\beta}u^{(n)}(t,\cdot), v_0 + n^{-\beta}v^{(n)}(t,\cdot)$, where

$$\begin{pmatrix} u^{(n)}(t,x) \\ v^{(n)}(t,x) \end{pmatrix} := \sigma(t, x - \lambda n^\beta t) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \delta(t, x - \mu n^\beta t) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \tag{4.17}$$

For the precise formulation of the result we need to introduce some additional notations. We will denote by μ_t^n the true distribution of the system at microscopic time $n^{1+\beta}t$:

$$\mu_t^n := \mu_0^n \exp\{n^{1+\beta}tL^n\} \tag{4.18}$$

We define the time-dependent reference measure ν_t^n as

$$\nu_t^n := \prod_{j\in\mathbb{T}^n} \pi_{u_0+n^{-\beta}u^{(n)}(t,\frac{j}{n}),v_0+n^{-\beta}v^{(n)}(t,\frac{j}{n})}, \tag{4.19}$$

with $u^{(n)}, v^{(n)}$ defined in (4.17). This measure mimics on a microscopic level the macroscopic profiles $u_0 + n^{-\beta}u^{(n)}(t,\cdot), v_0 + n^{-\beta}v^{(n)}(t,\cdot)$. We also choose an absolute reference measure

$$\pi^n := \prod_{j\in\mathbb{T}^n} \pi_{u_0^n,v_0^n}, \tag{4.20}$$

which is a stationary measure of our Markov process on Ω^n. The point (u_0^n, v_0^n) is chosen in a way that it lies *inside* the domain $\mathcal{D}$ and

$$|u_0 - u_0^n| + |v_0 - v_0^n| < n^{-\beta}. \tag{4.21}$$

If (u_0, v_0) is inside $\mathcal{D}$, then we may choose $(u_0^n, v_0^n) = (u_0, v_0)$. By choosing (u_0^n, v_0^n) inside $\mathcal{D}$ we get that any probability measure on Ω^n is absolutely continuous with respect to π^n. Condition (4.21) ensures that π^n is 'close enough' to μ_t^n in entropy sense, uniformly in t.

Theorem. *Let $\beta \in (0, \frac{1}{5})$ be fixed. Under the stated conditions, if*

$$H(\mu_0^n|\nu_0^n) = o(n^{1-2\beta}), \tag{4.22}$$

then

$$H(\mu_t^n|\nu_t^n) = o(n^{1-2\beta}), \tag{4.23}$$

uniformly for $0 \le t \le T$.

The following corollary is a simple consequence of the Theorem and the entropy inequality.

Corollary. *Assume the conditions of Theorem 4.3.2. Let $g : \mathbb{T} \to \mathbb{R}$ be a test function. Then for any $t \in [0, T]$*

$$\left| n^{-1+\beta} \sum_{j\in\mathbb{T}^n} g(\frac{j}{n}) \Big(\zeta_j(n^{1+\beta}t) - u_0\Big) - \int_{\mathbb{T}} g(x) \Big(\sigma(t, x - \lambda n^\beta t)r_1 + \delta(t, x - \mu n^\beta t)s_1\Big)\, dx \right| \xrightarrow{\mathbf{P}} 0,$$

$$\left| n^{-1+\beta} \sum_{j\in\mathbb{T}^n} g(\frac{j}{n}) \Big(\eta_j(n^{1+\beta}t) - v_0\Big) - \int_{\mathbb{T}} g(x) \Big(\sigma(t, x - \lambda n^\beta t)r_2 + \delta(t, x - \mu n^\beta t)s_2\Big)\, dx \right| \xrightarrow{\mathbf{P}} 0.$$

Remarks.

1. The Theorem states that if the initial distribution of the system is 'close' to ν_0^n in relative entropy sense then at time $n^{1+\beta}t$ it will be close to ν_t^n. The fact, that 'close' should mean $o(n^{1-2\beta})$ can be easily justified, see e.g. [34] or [35].
2. If instead of condition (D) we assume a similar uniform bound on the logarithmic-Sobolev constant then our Theorem is valid for $\beta \in (0, \frac{1}{3})$.

4.4 Proof

We will assume, that

$$(u_0, v_0) = (0, 0), \qquad \Phi_v(0,0) = \Psi_u(0,0) = 0. \tag{4.24}$$

It is easy to see, that we can always reduce the general case to get (4.24), via some suitable linear transformations on (ζ, η). Also, with the proper choice of the constants in the definition (4.7) we can set

$$\Phi(0,0) = \Psi(0,0) = 0. \tag{4.25}$$

Assumptions (4.24) imply, that

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad \mathbf{l} = \mathbf{r}^\dagger = (1, 0), \qquad \mathbf{m} = \mathbf{s}^\dagger = (0, 1), \tag{4.26}$$

and

$$u^{(n)}(t, x) = \sigma(t, x - \lambda n^\beta t), \qquad v^{(n)}(t, x) = \delta(t, x - \mu n^\beta t). \tag{4.27}$$

We introduce the notations

$$\Phi'' = \begin{pmatrix} \Phi_{uu} & \Phi_{uv} \\ \Phi_{vu} & \Phi_{vv} \end{pmatrix} =: \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}, \qquad \Psi'' = \begin{pmatrix} \Psi_{uu} & \Psi_{uv} \\ \Psi_{vu} & \Psi_{vv} \end{pmatrix} =: \begin{pmatrix} b_3 & b_2 \\ b_2 & b_1 \end{pmatrix}. \tag{4.28}$$

Clearly, these definitions agree with the definition (4.13).

We define the functions $\overline{\sigma}(t, x_1, x_2), \overline{\delta}(t, x_1, x_2)$ as

$$\begin{aligned} \overline{\sigma}(t, x_1, x_2) &:= \frac{1}{\lambda - \mu} \left(a_2\, \sigma(t, x_1)\delta(t, x_2) + a_2\, \sigma_x(t, x_1) \int_0^{x_2} (\delta(t, z) - c_\delta) dz + \frac{a_3}{2}\, \delta(t, x_2)^2 \right) \\ \overline{\delta}(t, x_1, x_2) &:= \frac{1}{\mu - \lambda} \left(b_2\, \sigma(t, x_1)\delta(t, x_2) + b_2\, \delta_x(t, x_2) \int_0^{x_1} (\sigma(t, z) - c_\sigma) dz + \frac{b_3}{2}\, \sigma(t, x_1)^2 \right) \end{aligned} \tag{4.29}$$

The defining partial differential equations (4.15), (4.16) of the functions σ, δ are conservation laws, thus for any $0 \le t \le T$:

$$\int_{\mathbb{T}} \sigma(z, t) dz = c_\sigma, \qquad \int_{\mathbb{T}} \delta(z, t) dz = c_\delta.$$

From that it follows that $\overline{\sigma}, \overline{\delta}$ are well-defined smooth functions on $[0, T] \times \mathbb{T} \times \mathbb{T}$ (i.e. periodic in x_1 and x_2) with bounded derivatives.

4.4.1 Changing the time-dependent reference measure

The usual way to prove a result like Theorem 4.3.2 is to get a Grönwall-type estimate on $H(\mu_t^n | \nu_t^n)$:

$$H(\mu_t^n | \nu_t^n) - H(\mu_0^n | \nu_0^n) \le C \int_0^t H(\mu_s^n | \nu_s^n) + o(n^{1-2\beta}),$$

via bounding the derivative $\partial_t H(\mu_t^n|\nu_t^n)$. We will use a slightly different approach, by proving a similar estimate for $H(\mu_t^n|\widetilde{\nu}_t^n)$:

$$H(\mu_t^n|\widetilde{\nu}_t^n) - H(\mu_0^n|\widetilde{\nu}_0^n) \le C \int_0^t H(\mu_s^n|\widetilde{\nu}_s^n) + o(n^{1-2\beta}). \tag{4.30}$$

Here

$$\widetilde{\nu}_t^n := \prod_{j\in\mathbb{T}^n} \pi_{n^{-\beta}\widetilde{u}^{(n)}(t,\frac{j}{n}),n^{-\beta}\widetilde{v}^{(n)}(t,\frac{j}{n})} \tag{4.31}$$

and $\widetilde{u}^{(n)}, \widetilde{v}^{(n)}$ are smooth functions defined as

$$\begin{aligned} \widetilde{u}^{(n)}(x,t) &:= u^{(n)}(x,t) + n^{-\beta}\,\overline{\sigma}(t, x-\lambda n^\beta t, x-\mu n^\beta t) \\ &= \sigma(t, x-\lambda n^\beta t) + n^{-\beta}\,\overline{\sigma}(t, x-\lambda n^\beta t, x-\mu n^\beta t), \\ \widetilde{v}^{(n)}(x,t) &:= v^{(n)}(x,t) + n^{-\beta}\,\overline{\delta}(t, x-\lambda n^\beta t, x-\mu n^\beta t), \\ &= \delta(t, x-\mu n^\beta t) + n^{-\beta}\,\overline{\delta}(t, x-\lambda n^\beta t, x-\mu n^\beta t). \end{aligned} \tag{4.32}$$

Because of Lemma 15 and condition (4.22) we have $H(\mu_0^n|\widetilde{\nu}_0^n) = o(n^{1-2\beta})$ and therefore from (4.30)

$$H(\mu_t^n|\widetilde{\nu}_t^n) = o(n^{1-2\beta})$$

will follow uniformly for $0 \le t \le T$. Using Lemma 15 again we get Theorem 4.3.2.

Lemma 15. *Let $\mu_t^n, \nu_t^n, \widetilde{\nu}_t^n$ be the measures defined as before, with $t \in [0,T]$. Then*

$$H(\mu_t^n|\nu_t^n) = o(n^{1-2\beta}) \Longleftrightarrow H(\mu_t^n|\widetilde{\nu}_t^n) = o(n^{1-2\beta}).$$

Proof. We start with

$$H(\mu_t^n|\nu_t^n) - H(\mu_t^n|\widetilde{\nu}_t^n) = -\int_{\Omega^n} \log \frac{d\nu_t^n}{d\widetilde{\nu}_t^n} d\mu_t^n. \tag{4.33}$$

By subsections 4.2.2 and 4.2.3 we can calculate that

$$\begin{aligned} \log \frac{d\nu_t^n}{d\widetilde{\nu}_t^n}(\underline{\omega}) = \sum_{j\in\mathbb{T}^n} \Big\{ & \left(\theta(n^{-\beta}u^{(n)}, n^{-\beta}v^{(n)}) - \theta(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right)\zeta_j \\ & + \left(\tau(n^{-\beta}u^{(n)}, n^{-\beta}v^{(n)}) - \tau(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right)\eta_j \\ & - G\left(\theta(n^{-\beta}u^{(n)}, n^{-\beta}v^{(n)}), \tau(n^{-\beta}u^{(n)}, n^{-\beta}v^{(n)})\right) \\ & + G\left(\theta(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)}), \tau(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right)\Big\}, \end{aligned}$$

where, for typographical reasons, we omitted the arguments $(t, \frac{j}{n})$ from the functions $u^{(n)}, v^{(n)}$, $\widetilde{u}^{(n)}, \widetilde{v}^{(n)}$.

From the previous expression via power-series expansion:

$$\begin{aligned}\log \frac{d\nu_t^n}{d\widetilde{\nu}_t^n}(\underline{\omega}) &\leq \mathcal{O}(n^{1-3\beta}) + Cn^{-2\beta} \sum_{j\in\mathbb{T}^n} \left(\left|\zeta_j - u^{(n)}(t, \frac{j}{n})\right| + \left|\eta_j - v^{(n)}(t, \frac{j}{n})\right| \right), \\ &= \mathcal{O}(n^{1-3\beta}) + Cn^{-2\beta} \sum_{j\in\mathbb{T}^n} \left(\left|\zeta_j - \widetilde{u}^{(n)}(t, \frac{j}{n})\right| + \left|\eta_j - \widetilde{v}^{(n)}(t, \frac{j}{n})\right| \right),\end{aligned}$$

with uniform error terms. Using this with (4.33) and the entropy inequality with respect to ν_t^n and $\widetilde{\nu}_t^n$ the lemma follows. □

We also note, that applying the same arguments as in the proof of Lemma 15 we get that from the conditions (4.21) and (4.22)

$$H(\mu_0^n|\pi^n) = \mathcal{O}(n^{1-2\beta})$$

follows. Since π^n is a stationary measure,

$$H(\mu_t^n|\pi^n) \leq H(\mu_0^n|\pi^n) = \mathcal{O}(n^{1-2\beta}) \tag{4.34}$$

for all $t \geq 0$.

The proof of the following lemma is a simple application of the entropy inequality with the entropy bound (4.34). Mind that because of (4.24) and (4.25) we have

$$\int_{\Omega^n} \zeta_i d\pi^n = \int_{\Omega^n} \eta_i d\pi^n = \int_{\Omega^n} \phi_i d\pi^n = \int_{\Omega^n} \psi_i d\pi^n = 0.$$

Lemma 16. *Suppose $b_1, b_2, \ldots$ are real numbers with $|b_j| \leq 1$ and ξ_j stands for either of η_j, ζ_j, ψ_j or ϕ_j. Then*

$$\int_{\Omega^n} \frac{1}{n} \sum_{j\in\mathbb{T}^n} b_j \xi_j \, d\mu_t^n \leq Cn^{-\beta}.$$

with an absolute constant C.

In the rest of the paper we prove inequality (4.30).

4.4.2 Preparatory computations

We define

$$\begin{aligned}\widetilde{\theta}^{(n)}(t,x) &:= n^\beta \theta(n^{-\beta}\widetilde{u}^{(n)}(t,x), n^{-\beta}\widetilde{v}^{(n)}(t,x)), \\ \widetilde{\tau}^{(n)}(t,x) &:= n^\beta \tau(n^{-\beta}\widetilde{u}^{(n)}(t,x), n^{-\beta}\widetilde{v}^{(n)}(t,x)), \\ \theta_0^n &:= n^\beta \theta(n^{-\beta}u_0^n, n^{-\beta}v_0^n), \\ \tau_0^n &:= n^\beta \tau(n^{-\beta}u_0^n, n^{-\beta}v_0^n).\end{aligned} \tag{4.35}$$

It is easy to check, that the partial derivatives $\partial_x\widetilde{\theta}^{(n)}(t,x), \partial_x\widetilde{\tau}^{(n)}(t,x), \partial_x^2\widetilde{\theta}^{(n)}(t,x), \partial_x^2\widetilde{\tau}^{(n)}(t,x)$ are uniformly bounded in $[0,T]\times\mathbb{T}$. From subsection 4.2.2 we have

$$\begin{aligned}\widetilde{f}_t^n &:= \frac{d\widetilde{\nu}_t^n}{d\pi^n}\\ &= \exp\sum_{j\in\mathbb{T}^n}\Big\{n^{-\beta}(\widetilde{\theta}^{(n)}(t,\frac{j}{n})-\theta_0^n)\zeta_j + n^{-\beta}(\widetilde{\tau}^{(n)}(t,\frac{j}{n})-\tau_0^n)\eta_j \\ &\qquad -G\left(n^{-\beta}\widetilde{\theta}^{(n)}(t,\frac{j}{n}), n^{-\beta}\widetilde{\tau}^{(n)}(t,\frac{j}{n})\right) + G\left(n^{-\beta}\widetilde{\theta}_0^{(n)}, n^{-\beta}\widetilde{\tau}_0^{(n)}\right)\Big\}.\end{aligned} \tag{4.36}$$

Differentiating the identity

$$H(\mu_t^n|\widetilde{\nu}_t^n) - H(\mu_t^n|\pi^n) = -\int_{\Omega^n}\log\widetilde{f}_t^n\mu_t^n$$

and noting that $\partial_t H(\mu_t^n|\pi^n)\le 0$ we get the following bound on $\partial_t H(\mu_t^n|\widetilde{\nu}_t^n)$:

$$n^{2\beta-1}\partial_t H(\mu_t^n|\widetilde{\nu}_t^n) \le -\int_{\Omega^n}\left(n^{3\beta}L^n\log\widetilde{f}_t^n + n^{-1+2\beta}\partial_t\log\widetilde{f}_t^n\right)d\mu_t^n.$$

Integrating with respect to the time:

$$n^{2\beta-1}(H(\mu_t^n|\widetilde{\nu}_t^n) - H(\mu_0^n|\widetilde{\nu}_0^n)) \le -\int_0^t\int_{\Omega^n}\left(n^{3\beta}L^n\log\widetilde{f}_s^n + n^{-1+2\beta}\partial_t\log\widetilde{f}_s^n\right)d\mu_s^n dt. \tag{4.37}$$

We estimate the two terms on the right-hand side separately in the next two subsections.

4.4.3 Estimating the first term of (4.37)

From the definitions

$$\begin{aligned}n^{3\beta}L^n\log\widetilde{f}_t^n(\underline{\omega}) &= n^{-1+2\beta}\sum_{j\in\mathbb{T}^n}\left(\phi_j - \Phi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\right)\partial_x\widetilde{\theta}^{(n)}(t,\frac{j}{n})\\ &\qquad + \left(\psi_j - \Psi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\right)\partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n})\\ &\quad + \mathrm{Err}_1^n(t,\underline{\omega}) + \mathrm{Err}_2^n(t),\end{aligned} \tag{4.38}$$

where

$$\begin{aligned}\mathrm{Err}_1^n(t,\underline{\omega}) &:= n^{-1+2\beta}\sum_{j\in\mathbb{T}^n}\Big\{\phi_j\left(\nabla^n\widetilde{\theta}^{(n)}(t,\frac{j}{n}) - \partial_x\widetilde{\theta}^{(n)}(t,\frac{j}{n})\right)\\ &\qquad + \psi_j\left(\nabla^n\widetilde{\tau}^{(n)}(t,\frac{j}{n}) - \partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n})\right)\Big\},\end{aligned} \tag{4.39}$$

$$\begin{aligned}\mathrm{Err}_2^n(t) &:= n^{-1+2\beta}\sum_{j\in\mathbb{T}^n}\Big\{\Phi\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\theta}^{(n)}(t,\frac{j}{n})\\ &\qquad + \Psi\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n})\Big\}.\end{aligned} \tag{4.40}$$

We used the (slightly abused) shorthanded notations

$$\widetilde{u}_j^{(n)} = \widetilde{u}^{(n)}(t,\frac{j}{n}), \qquad \widetilde{v}_j^{(n)} = \widetilde{v}^{(n)}(t,\frac{j}{n}),$$

and ∇^n denotes the discrete gradient:

$$\nabla^n g(x) := n\left(g(x+\frac{1}{n}) - g(x)\right).$$

Using the smoothness of $\widetilde{\theta}^{(n)}$ and $\widetilde{\tau}^{(n)}$ and Lemma 16 the expectation of the first error term can be easily estimated uniformly in $t \in [0,T]$:

$$\int_{\Omega^n} |\mathrm{Err}_1^n(t,\underline{\omega})|\, d\mu_t^n = \mathcal{O}(n^{-1+\beta}). \tag{4.41}$$

Using Lemma 14 it is easy to see that there exists a smooth function $U(u,v)$ such that

$$\partial_\theta U\left(u(\theta,\tau), v(\theta,\tau)\right) = \Phi\left(u(\theta,\tau), v(\theta,\tau)\right), \qquad \partial_\tau U\left(u(\theta,\tau), v(\theta,\tau)\right) = \Psi\left(u(\theta,\tau), v(\theta,\tau)\right).$$

Thus $\mathrm{Err}_2^n(t)$ takes the form:

$$\mathrm{Err}_2^n(t) = n^{3\beta-1}\sum_{j\in\mathbb{T}^n} \partial_x U\left(n^{-\beta}\widetilde{u}^{(n)}(t,\frac{j}{n}), n^{-\beta}\widetilde{v}^{(n)}(t,\frac{j}{n})\right),$$

from which

$$\mathrm{Err}_2(t) = \mathcal{O}(n^{-1+2\beta}), \tag{4.42}$$

uniformly for $t \in [0,T]$. From previous bounds we have

$$\begin{aligned}\int_0^t \int_{\Omega^n} n^{3\beta} L^n \log \widetilde{f}_s^n(\underline{\omega}) d\mu_s^n ds = \\ n^{-1+2\beta}\int_0^t\int_{\Omega^n}\sum_{j\in\mathbb{T}^n}\left(\phi_j - \Phi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\right)\partial_x\widetilde{\theta}^{(n)}(s,\frac{j}{n}) \\ + \left(\psi_j - \Psi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\right)\partial_x\widetilde{\tau}^{(n)}(s,\frac{j}{n}) d\mu_t^n ds \\ + \mathcal{O}(n^{-1+2\beta})\end{aligned} \tag{4.43}$$

In the next step we introduce the block averages. We will denote the block size with $l = l(n)$, it will be large microscopically, but small on the macroscopic scale. In the first computations we only assume $l \gg n^{2\beta}$, the exact order of l will only be determined at the end of the proof, after collecting all the error terms. For a local function κ_j ($j \in \mathbb{T}^n$) we define its block average with

$$\kappa_j^l := \frac{1}{l}\sum_{i=0}^{l-1}\kappa_{j+i}.$$

By partial summation for a smooth function $\rho(x) : \mathbb{T} \mapsto \mathbb{R}$ we have

$$\left|\sum_{j\in\mathbb{T}^n}\kappa_j\,\rho(\frac{j}{n}) - \sum_{j\in\mathbb{T}^n}\kappa_j^l\,\rho(\frac{j}{n})\right| \le \|\partial_x\rho\|_\infty \left|\sum_{j\in\mathbb{T}^n}\kappa_j\right|\frac{l}{n}$$

Using this with Lemma 16 we can replace ϕ_j, ψ_j in (4.43) with the the respective block averages:

$$\int_0^t \int_{\Omega^n} n^{3\beta} L^n \log \widetilde{f}_s^n(\underline{\omega}) d\mu_s^n ds = \\ n^{-1+2\beta} \int_0^t \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \Big\{ \Big(\phi_j^l - \Phi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\Big) \partial_x \widetilde{\theta}^{(n)}(s, \frac{j}{n}) \\ + \Big(\psi_j^l - \Psi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\Big) \partial_x \widetilde{\tau}^{(n)}(s, \frac{j}{n}) \Big\} d\mu_t^n ds \quad (4.44) \\ + \mathcal{O}(n^{\beta-1} l).$$

Finally, using Lemma 17 (the one-block estimate), we replace the block averages ϕ_j^l, ψ_j^l by their 'local equilibrium value': $\Phi(\zeta_j^l, \eta_j^l)$ and $\Psi(\zeta_j^l, \eta_j^l)$, respectively:

$$\int_0^t \int_{\Omega^n} n^{3\beta} L^n \log \widetilde{f}_s^n(\underline{\omega}) d\mu_s^n ds = \\ n^{-1+2\beta} \int_0^t \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \Big(\Phi(\zeta_j^l, \eta_j^l) - \Phi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\Big) \partial_x \widetilde{\theta}^{(n)}(s, \frac{j}{n}) \\ + \Big(\Psi(\zeta_j^l, \eta_j^l) - \Psi(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)})\Big) \partial_x \widetilde{\tau}^{(n)}(s, \frac{j}{n}) d\mu_t^n ds \quad (4.45) \\ + \mathcal{O}(n^{\beta-1} l \vee n^{-1-\beta} l^3 \vee l^{-1})$$

Lemma 17 (One block estimate).

$$\frac{1}{n}\int_0^t \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \Big|\phi_j^l - \Phi(\zeta_j^l, \eta_j^l)\Big| d\mu_s^n dt \leq C(n^{-1-3\beta} l^3 + l^{-1}),$$
$$\frac{1}{n}\int_0^t \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \Big|\psi_j^l - \Psi(\zeta_j^l, \eta_j^l)\Big| d\mu_s^n dt \leq C(n^{-1-3\beta} l^3 + l^{-1}).$$

The proof relies on the spectral gap condition (D). It uses the Feynman-Kac formula, the Raleigh-Schrödinger perturbation technique and the 'equivalence of ensembles' (see the Appendix of [13] for all three). A detailed proof can be found in [34] for the one component case which can be easily adapted for our purposes.

Remark. If instead of the condition (D) we have a similar uniform bound on the logarithmic-Sobolev constant, then the previous lemma may be strengthened: it holds with the bound $C(n^{-1-3\beta} l^2 + l^{-1})$.

4.4.4 Estimating the second term of (4.37)

Performing the time-derivation we obtain:

$$n^{-1+2\beta} \partial_t \log \widetilde{f}_t^n = \frac{1}{n} \sum_{j\in\mathbb{T}^n} \Big\{ \Big(n^\beta \zeta_j - \widetilde{u}_j^{(n)}\Big) \partial_t \widetilde{\theta}^{(n)}(t, \frac{j}{n}) + \Big(n^\beta \eta_j - \widetilde{v}_j^{(n)}\Big) \partial_t \widetilde{\tau}^{(n)}(t, \frac{j}{n}) \Big\} \quad (4.46)$$

By the definitions of $\widetilde{u}^{(n)}, \widetilde{v}^{(n)}$ and Taylor-expansion we readily get that

$$\begin{aligned}\partial_t \widetilde{u}^{(n)}(t,x) &= \sigma_t(t, x-\lambda n^\beta t) - \lambda n^\beta \sigma_x(t, x-\lambda n^\beta t) \\ &\quad -\lambda \overline{\sigma}_{x_1}(t, x-\lambda n^\beta t, x-\mu n^\beta t) - \mu \overline{\sigma}_{x_2}(t, x-\lambda n^\beta t, x-\mu n^\beta t) \\ &\quad +\mathcal{O}(n^{-\beta}),\end{aligned}$$

and

$$\begin{aligned} & n^{2\beta}\partial_x \Phi\left(n^{-\beta}\widetilde{u}^{(n)}(x,t), n^{-\beta}\widetilde{v}^{(n)}(x,t)\right) \\ &\quad = \lambda n^\beta \sigma_x(t, x-\lambda n^\beta t) \\ &\qquad +\lambda \overline{\sigma}_{x_1}(t, x-\lambda n^\beta t, x-\mu n^\beta t) + \lambda \overline{\sigma}_{x_2}(t, x-\lambda n^\beta t, x-\mu n^\beta t) \\ &\qquad +\partial_x\left(\frac{1}{2}a_1\sigma(t, x-\lambda n^\beta t)^2 + c_\delta a_2 \sigma(t, x-\lambda n^\beta t)\right) \\ &\qquad +\partial_x\left(a_2\sigma(t, x-\lambda n^\beta t)(\delta(t, x-\mu n^\beta t) - c_\delta) + \frac{1}{2}a_3\delta(t, x-\mu n^\beta t)^2\right) \\ &\qquad +\mathcal{O}(n^{-\beta}),\end{aligned}$$

with uniform error terms. ($\overline{\sigma}_{x_1}$ and $\overline{\sigma}_{x_2}$ are the partial derivatives of $\overline{\sigma}(t, x_1, x_2)$ with respect to the second and third variable.) Adding up these equations and checking the definitions for $\sigma, \delta, \overline{\sigma}$ we see that all the significant terms on the right hand side cancel to give:

$$\partial_t \widetilde{u}^{(n)} + \partial_x\left(n^{2\beta}\Phi(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right) = \mathcal{O}(n^{-\beta}). \tag{4.47}$$

Similarly,

$$\partial_t \widetilde{v}^{(n)} + \partial_x\left(n^{2\beta}\Psi(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right) = \mathcal{O}(n^{-\beta}). \tag{4.48}$$

From (4.47) and (4.48):

$$\begin{aligned}\partial_t \widetilde{\theta}^{(n)} &= \theta_u \partial_t \widetilde{u}^{(n)} + \theta_v \partial_t \widetilde{v}^{(n)} \\ &= -n^{2\beta}\theta_u \partial_x\left(\Phi(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right) - n^{2\beta}\theta_v \partial_x\left(\Psi(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})\right) + \mathcal{O}(n^{-\beta}) \\ &= -n^\beta(\theta_u\Phi_u + \theta_v\Psi_u)\partial_x\widetilde{u}^{(n)} - n^\beta(\theta_u\Phi_v + \theta_v\Psi_v)\partial_x\widetilde{v}^{(n)} + \mathcal{O}(n^{-\beta}) \\ &= -n^\beta(\theta_u\Phi_u + \tau_u\Psi_u)\partial_x\widetilde{u}^{(n)} - n^\beta(\theta_v\Phi_u + \tau_v\Psi_u)\partial_x\widetilde{v}^{(n)} + \mathcal{O}(n^{-\beta}) \\ &= -n^\beta\Phi_u\partial_x\widetilde{\theta}^{(n)} - n^\beta\Psi_u\partial_x\widetilde{\tau}^{(n)} + \mathcal{O}(n^{-\beta}) \end{aligned} \tag{4.49}$$

In the fourth line we used $\tau_u = \theta_v$ and Lemma 14. To simplify notations, we omitted the arguments (t,x) from the functions $\widetilde{\theta}^{(n)}, \widetilde{\tau}^{(n)}, \widetilde{u}^{(n)}, \widetilde{v}^{(n)}$, and the arguments $(n^{-\beta}\widetilde{u}^{(n)}, n^{-\beta}\widetilde{v}^{(n)})$ from all the partial derivatives of θ, τ, Φ, Ψ with respect to u, v. Similarly,

$$\partial_t \widetilde{\tau}^{(n)} = -n^\beta\Phi_v\partial_x\widetilde{\theta}^{(n)} - n^\beta\Psi_v\partial_x\widetilde{\tau}^{(n)} + \mathcal{O}(n^{-\beta}). \tag{4.50}$$

Hence from (4.46):

$$
\begin{aligned}
&n^{-1+2\beta}\partial_t \log \widetilde{f}_t^n \\
&\quad= -n^{-1+2\beta}\sum_{j\in\mathbb{T}^n}\Big\{\left(\zeta_j - n^{-\beta}\widetilde{u}_j^{(n)}\right)\Phi_u\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\theta}^{(n)}(t,\frac{j}{n}) \\
&\qquad\qquad + \left(\zeta_j - n^{-\beta}\widetilde{u}_j^{(n)}\right)\Psi_u\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n}) \\
&\qquad\qquad + \left(\eta_j - n^{-\beta}\widetilde{v}_j^{(n)}\right)\Phi_v\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\theta}^{(n)}(t,\frac{j}{n}) \\
&\qquad\qquad + \left(\eta_j - n^{-\beta}\widetilde{v}_j^{(n)}\right)\Psi_v\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\tau}^{(n)}(t,\frac{j}{n})\Big\} \\
&\qquad +\mathrm{Err}_3(t,\underline{\omega}),
\end{aligned}
\tag{4.51}
$$

where

$$
\mathrm{Err}_3(t,\underline{\omega}) = \frac{1}{n}\sum_{j\in\mathbb{T}^n}(\zeta_j - n^{-\beta}\widetilde{u}_j^{(n)})\, b_j(t) + (\eta_j - n^{-\beta}\widetilde{v}_j^{(n)})\, c_j(t) \tag{4.52}
$$

and $b_j(t)$ and $c_j(t)$ are uniformly bounded constants. Using Lemma 16 we get that

$$
\int_{\Omega^n} |\mathrm{Err}_3(t,\underline{\omega})|\, d\mu_t^n = \mathcal{O}(n^{-\beta}). \tag{4.53}
$$

We can exchange ζ_j, η_j with their block-averages ζ_j^l, η_j^l (as in the previous subsection) which (after the time-integration) gives the following estimate:

$$
\begin{aligned}
&\int_0^t \int_{\Omega^n} n^{-1+2\beta}\partial_t \log \widetilde{f}_s^n d\mu_s^n ds \\
&\quad= -\int_0^t \int_{\Omega^n} n^{-1+2\beta}\sum_{j\in\mathbb{T}^n}\Big\{\left(\zeta_j^l - n^{-\beta}\widetilde{u}_j^{(n)}\right)\Phi_u\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\theta}^{(n)}(s,\frac{j}{n}) \\
&\qquad\qquad + \left(\zeta_j^l - n^{-\beta}\widetilde{u}_j^{(n)}\right)\Psi_u\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\tau}^{(n)}(s,\frac{j}{n}) \\
&\qquad\qquad + \left(\eta_j^l - n^{-\beta}\widetilde{v}_j^{(n)}\right)\Phi_v\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\theta}^{(n)}(s,\frac{j}{n}) \\
&\qquad\qquad + \left(\eta_j^l - n^{-\beta}\widetilde{v}_j^{(n)}\right)\Psi_v\left(n^{-\beta}\widetilde{u}_j^{(n)}, n^{-\beta}\widetilde{v}_j^{(n)}\right)\partial_x\widetilde{\tau}^{(n)}(s,\frac{j}{n})\Big\}\, d\mu_s^n ds \\
&\qquad +\mathcal{O}(n^{-\beta} \vee n^{\beta-1}l),
\end{aligned}
\tag{4.54}
$$

4.4.5 Block replacement

For a function $\Upsilon(u,v)$ we denote

$$
\mathrm{R}_\Upsilon(u_1,v_1;u_2,v_2) := \Upsilon(u_1,v_1) - \Upsilon(u_2,v_2) - \Upsilon_u(u_2,v_2)(u_1-u_2) - \Upsilon_v(u_2,v_2)(v_1-v_2).
$$

Collecting the estimates of the previous subsections we have

$$\begin{aligned}
&H(\mu_t^n|\widetilde{\nu}_t^n) - H(\mu_t^n|\widetilde{\nu}_0^n) \\
&\quad \le C\, n^{2\beta-1} \int_0^t \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \left(\left| \mathrm{R}_\Phi \left(\zeta_j^l, \eta_j^l; n^{-\beta}\widetilde{u}^{(n)}(s,\frac{j}{n}), n^{-\beta}\widetilde{v}^{(n)}(s,\frac{j}{n}) \right) \right| \right. \\
&\qquad\qquad \left. + \left| \mathrm{R}_\Psi \left(\zeta_j^l, \eta_j^l; n^{-\beta}\widetilde{u}^{(n)}(s,\frac{j}{n}), n^{-\beta}\widetilde{v}^{(n)}(s,\frac{j}{n}) \right) \right| \right) d\mu_s^n ds \\
&\quad +\mathcal{O}(n^{-\beta} \vee n^{\beta-1} l \vee n^{-1-\beta} l^3)
\end{aligned} \tag{4.55}$$

The second derivatives of Φ and Ψ are bounded thus

$$\begin{aligned}
&\left| \mathrm{R}_\Phi \left(\zeta_j^l, \eta_j^l; n^{-\beta}\widetilde{v}^{(n)}(s,\frac{j}{n}), n^{-\beta}\widetilde{u}^{(n)}(s,\frac{j}{n}) \right) \right| + \left| \mathrm{R}_\Psi \left(\zeta_j^l, \eta_j^l; n^{-\beta}\widetilde{v}^{(n)}(s,\frac{j}{n}), n^{-\beta}\widetilde{u}^{(n)}(s,\frac{j}{n}) \right) \right| \\
&\qquad \le C \left((\zeta_j^l - n^{-\beta}\widetilde{u}^{(n)}(s,\frac{j}{n}))^2 + (\eta_j^l - n^{-\beta}\widetilde{v}^{(n)}(s,\frac{j}{n}))^2 \right),
\end{aligned}$$

which means that it is sufficient to estimate

$$n^{2\beta-1} \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \left(\zeta_j^l - n^{-\beta}\widetilde{u}^{(n)}(t,\frac{j}{n}) \right)^2 d\mu_t^n \ \text{ and } \ n^{2\beta-1} \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \left(\eta_j^l - n^{-\beta}\widetilde{v}^{(n)}(t,\frac{j}{n}) \right)^2 d\mu_t^n$$

uniformly in t. We estimate the first expression, the other will follow the same way. We denote

$$\widehat{\zeta}_j^l := \widehat{\zeta}_j^l(t,\underline{\omega}) = \frac{1}{l} \sum_{i=0}^{l-1} \left(\zeta_{j+i} - n^{-\beta}\widetilde{u}^{(n)}(t,\frac{j+i}{n}) \right).$$

Since $\partial_x \widetilde{u}^{(n)}(t,x)$ is uniformly bounded for $(t,x) \in [0,T]\times\mathbb{T}$, we have

$$(\widehat{\zeta}_j^l)^2 - (\zeta_j^l - n^{-\beta}\widetilde{u}_j^{(n)})^2 = \mathcal{O}(n^{-\beta-1} l)$$

uniformly in $j \in \mathbb{T}^n$, $t \in [0,T]$ and it is enough to estimate

$$n^{2\beta-1} \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \left(\widehat{\zeta}_j^l \right)^2 d\mu_t^n.$$

Applying the entropy inequality with respect to the time-dependent reference measure $\widetilde{\nu}_t^n$ and using Hölder's inequality:

$$n^{2\beta-1} \int_{\Omega^n} \sum_{j\in\mathbb{T}^n} \left(\widehat{\zeta}_j^l \right)^2 d\mu_t^n \le \frac{1}{\gamma} n^{2\beta-1} H(\mu_t^n|\widetilde{\nu}_t^n) + \frac{1}{\gamma} l^{-1} n^{2\beta-1} \sum_{j\in\mathbb{T}^n} \log \mathbf{E}_{\widetilde{\nu}_t^n} \exp\left(\gamma l (\widehat{\zeta}_j^l)^2 \right) \tag{4.56}$$

for any $\gamma > 0$. $\mathcal{D}$ is compact, ζ is bounded thus there exists a positive constant C such that

$$\log \mathbf{E}_{u,v} \exp\left((\zeta - u) y \right) \le C y^2$$

for all $(u,v) \in \mathcal{D}$ and $y \in \mathbb{R}$. Thus as a consequence of Lemma 18, there exists a small $\gamma > 0$ for which

$$\frac{1}{n} \sum_{j\in\mathbb{T}^n} \log \mathbf{E}_{\widetilde{\nu}_t^n} \exp\left(\gamma l (\widehat{\zeta}_j^l)^2 \right) < 1.$$

Substituting into (4.56):

$$n^{2\beta-1}\int_{\Omega^n}\sum_{j\in\mathbb{T}^n}\left(\zeta_j^l - n^{-\beta}\widetilde{u}^{(n)}(t,\frac{j}{n})\right)^2 d\mu_t^n < Cn^{2\beta-1}H(\mu_t^n|\widetilde{\nu}_t^n) + \mathcal{O}(n^{2\beta}l^{-1}).$$

Collecting all the estimates, from (4.55) we get

$$n^{2\beta-1}\left(H(\mu_t^n|\widetilde{\nu}_t^n) - H(\mu_0^n|\widetilde{\nu}_0^n)\right) \le Cn^{2\beta-1}\int_0^t H(\mu_s^n|\widetilde{\nu}_s^n)ds + \mathcal{O}(n^{-\beta}\vee n^{\beta-1}l \vee n^{-1-\beta}l^3 \vee n^{2\beta}l^{-1}).$$

Choosing l with

$$n^{2\beta} \ll l \ll n^{\frac{1+\beta}{3}}$$

the error term becomes $o(1)$ and the Theorem follows via the Grönwall inequality. (If we have the logarithmic-Sobolev condition, and thus a stronger version of Lemma 17, then l can be chosen with $n^{2\beta} \ll l \ll n^{\frac{1+\beta}{2}}$ to make all the error terms $o(1)$.)
The proof of Lemma 18 can be found in [34] or [36].

Lemma 18. *Suppose $\xi_1, \xi_2, \ldots$ are independent random variables with $\mathbf{E}\xi_i = 0$ for which*

$$\log \mathbf{E}\exp(y\xi_i) \le Cy^2$$

with a positive constant C independent of i and y. Then there exists a small positive constants γ depending only on C such that

$$\log \mathbf{E}\exp\left(\gamma l(\xi_i^l)^2\right) < 1.$$

Acknowledgements: The author thanks Bálint Tóth for introducing him into the theory of hydrodynamic limits, and particularly to the problem of the present paper. He also thanks Fraydoun Rezakhanlou for drawing his attention to the paper [3] and the method of geometric optics.
This work was partially supported by the Hungarian Scientific Research Fund (OTKA) grant no. T037685.

Bibliography

[1] M. Balázs: Growth fluctuations in interface models. *Annales de l'Institut Henri Poincaré — Probabilitées et Statistiques* **39**: 639-685 (2003)

[2] C. Cocozza: Processus des misanthropes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **70**: 509-523 (1985)

[3] R. J. DiPerna, A. Majda: The Validity of Nonlinear Geometric Optics for Weak Solutions of Conservation Laws. *Commun. Math. Phys.* **98**: 313-347 (1985)

[4] R. Esposito, R. Marra, H.T. Yau: Diffusive limit of asymmetric simple exclusion. *Reviews of Mathematical Physics* **6**: 1233-1267 (1994)

[5] L.C. Evans: *Partial Differential Equations.* Graduate Studies in Mathematics **19**, AMS, Providence RI, 1998

[6] J. Fritz: *An Introduction to the Theory of Hydrodynamic Limits.* Lectures in Mathematical Sciences **18**. Graduate School of Mathematics, Univ. Tokyo, 2001.

[7] J. Fritz: Entropy pairs and compensated compactness for weakly asymmetric systems. *Advanced Studies in Pure Mathematics* (2003) (to appear), www.math.bme.hu/ jofri.

[8] J. Fritz, B. Tóth: Derivation of the Leroux system as the hydrodynamic limit of a two-component lattice gas. (2003) `http://arxiv.org/abs/math.PR/0304481`

[9] P. R. Garabedian: *Partial Differential Equations.* AMS Chelsea, Providence RI, 1998

[10] J. K. Hunter, J. B. Keller: Weakly nonlinear high frequency waves. Commun Pure Appl. Math. **36**, 547-569 (1983)

[11] F. John: *Partial Differential Equations.* Applied Mathematical Sciences, vol. 1, Springer, New York-Heidelberg-Berlin, 1971.

[12] M. Kardar, G. Parisi, Y.-C. Zhang: Dynamic scaling of growing interfaces. *Physical Reviews Letters* **56**: 889-892 (1986)

[13] C. Kipnis, C. Landim: *Scaling Limits of Interacting Particle Systems.* Springer, 1999.

[14] E. Kosygina: The behaviour of the specific entropy in the hydrodynamic scaling limit, *Ann. Probab.* **29**: 1086-1110 (2001)

[15] L. D. Landau, E. M. Lifshitz: *Mécanique des fluides*, Éditions Mir, Moscou, 1971

[16] C. Landim, S. Sethuraman, S.R.S. Varadhan: Spectral gap for zero range dynamics. *Ann. Probab.* **24**: 1871-1902 (1986)

[17] P. Lax: Shock waves and entropy. In: *Contributions to Nonlinear Functional Analysis*, ed.: E.A. Zarantonello. Academic Press, 1971, pp. 603-634

[18] P. Lax: *Systems of Conservation Laws and the Mathematical Theory of Shock Waves.* SIAM, CBMS-NSF 11, 1973.

[19] H. A. Levine, B. D. Sleeman: A system of reaction diffusion equations arising in the theory of reinforced random walks. *SIAM Journal of Applied Mathematics* **57** 683-730 (1997)

[20] R. J. Leveque: *Numerical Methods in Conservation Laws.* Lectures In Mathematics, ETH Zürich, Birkhäuser Verlag Basel, 1990

[21] S. Olla, S. R. S. Varadhan, and H. T. Yau: Hydrodynamical limit for Hamiltonian system with weak noise, *Commun. Math. Phys.* **155**: 523-560 (1993)

[22] H. G. Othmer, A. Stevens: Aggregation, blowup, and collapse: the abc's of taxis in reinforced random walks. *SIAM Journal of Applied Mathematics* **57**: 1044-1081 (1997)

[23] V. Popkov, G.M. Schütz: Shocks and excitation dynamics in driven diffusive two channel systems. *Journal of Statistical Physics* **112**: 523-540 (2003)

[24] M. Rascle: On some "viscous" perurbations of quasi-linear first order hyperbolic systems arising in biology. *Contemporary Mathematics* **17**: 133-142 (1983)

[25] L. E. Reichl: *A Modern Course in Statistical Physics*, Second Edition, John Wiley and Sons, 1998

[26] F. Rezakhanlou: Hydrodynamic limits for attractive particle systems on $\mathbb{Z}^d$. *Commun. Math. Phys.* **140**: 417-448 (1991)

[27] F. Rezakhanlou: Microscopic structure of shocks in one conservation laws. *Annales de l'Institut Henri Poincaré — Analyse Non Lineaire* **12**: 119-153 (1995)

[28] T. Seppäläinen: Perturbation of the equilibrium for a totally asymmetric stick process in one dimension. *Annals of Probability* **29**: 176-204 (2001)

[29] D. Serre: *Systems of Conservation Laws.* Vol 1-2. Cambridge University Press, 2000

[30] J. Smoller: *Shock Waves and Reaction Diffusion Equations*, Second Edition, Springer, 1994.

[31] A. Sommerfeld: *Mechanics of deformable bodies, Lectures on Theoretical Physics, Vol. II.*, Academic Press, NY, 1950

[32] H. Spohn: *Large Scale Dynamics of Interacting Particles.* Springer-Verlag, Berlin-Heidelberg-New York 1991.

[33] B. Tóth: The 'true' self-avoiding walk with bond repulsion on $\mathbb{Z}$: limit theorems. *Ann. Prob.* **23:** 1523-1556 (1995)

[34] B. Tóth, B. Valkó: Between equilibrium fluctuations and Eulerian scaling. Perturbation of equilibrium for a class of deposition models. *Journal of Statistical Physics* **109**: 177-205 (2002)

[35] B. Tóth, B. Valkó: Onsager relations and Eulerian hydrodynamic limit for systems with several conservation laws. *Journal of Statistical Physics* **112**: 497-521 (2003)

[36] B. Tóth, B. Valkó:Perturbation of singular equilibria of hyperbolic two-component systems: a universal hydrodynamic limit, submitted preprint, **arxiv.org/abs/math.PR/0312256**

[37] B. Tóth, W. Werner: The true self-repelling motion. *Probability Theory and Relative Fields* **111**: 375-452 (1998)

[38] B. Tóth, W. Werner: Hydrodynamic equation for a deposition model. In: *In and out of equilibrium. Probability with a physics flavor,* V. Sidoravicius Ed., Progress in Probability **51**, Birkhäuser, 227-248 (2002)

[39] S. R. S. Varadhan: Nonlinear diffusion limit for a system with nearest neighbor interactions II. In: *Asymptotic Problems in Probability Theory, Sanda/Kyoto 1990* 75–128. Longman, Harlow 1993.

[40] B. Valkó: Hydrodynamic limit for perturbation of a hyperbolic equilibrium point in two-component systems, submitted preprint, **arxiv.org/abs/math.PR/0402017**

[41] H. T. Yau: Relative entropy and hydrodynamics of Ginzburg-Landau models. *Letters in Mathematical Physics* **22**: 63-80 (1991)

[42] H. T. Yau: Logarithmic Sobolev inequality for generalized simple exclusion processes. *Probability Theory and Related Fields* **109**: 507-538 (1997)

[43] H. T. Yau: Scaling limit of particle systems, incompressible Navier-Stokes equations and Boltzmann equation. In: *Proceedings of the International Congress of Mathematics, Berlin 1998*, vol 3, pp 193-205, Birkhäuser (1999)

Acknowledgements

I would like to thank Prof. Bálint Tóth for being an excellent supervisor. He introduced me to an exciting field of mathematics with interesting and challenging problems. It has been a great experience to learn from him and to work with him.

I am grateful to Prof. József Fritz for the many discussions on hydrodynamics and hyperbolic conservation laws. His observations and remarks always provided substantial help for our proofs.

Last, but not least, I thank my wife for the constant support and encouragement.