

Coupling methods in stochastic deposition models  
(outline of PhD thesis)

Márton Balázs

(Institute of Mathematics, Technical University Budapest)

Supervisor: Prof. Bálint Tóth

Budapest,

99.99.2003

# 1 Introduction

In the first part of this outline we define the class of models we consider, present the steady states for them and show how to couple a pair of these models. In the second part we state the results of the thesis.

## 1.1 The model

The class of models described here is a generalization of the so-called misanthrope process. For  $-\infty \leq \omega^{\min} \leq 0$  and  $1 \leq \omega^{\max} \leq \infty$  (possibly infinite valued) integers, we define

$$I := \{z \in \mathbb{Z} : \omega^{\min} - 1 < z < \omega^{\max} + 1\}$$

and the phase space

$$\Omega = \{\underline{\omega} = (\omega_i)_{i \in \mathbb{Z}} : \omega_i \in I\} = I^{\mathbb{Z}}.$$

For each pair of neighboring sites  $i$  and  $i + 1$  of  $\mathbb{Z}$ , we can imagine a column built of bricks, above the edge  $(i, i + 1)$ . The height of this column is denoted by  $h_i$ . If  $\underline{\omega}(t) \in \Omega$  for a fixed time  $t \in \mathbb{R}$  then  $\omega_i(t) = h_{i-1}(t) - h_i(t) \in I$  is the negative discrete gradient of the height of the “wall”. The growth of a column is described by jump processes. A brick can be added:

$$(\omega_i, \omega_{i+1}) \longrightarrow (\omega_i - 1, \omega_{i+1} + 1) \quad \text{with rate } r(\omega_i, \omega_{i+1}).$$

Conditionally on  $\underline{\omega}(t)$ , these moves are independent. See fig. 1 for some possible instantaneous changes. For small  $\varepsilon$ , the conditional expectation of the growth of the column between  $i$  and  $i + 1$  in the time interval  $[t, t + \varepsilon]$  is  $r(\omega_i(t), \omega_{i+1}(t)) \cdot \varepsilon + o(\varepsilon)$ .

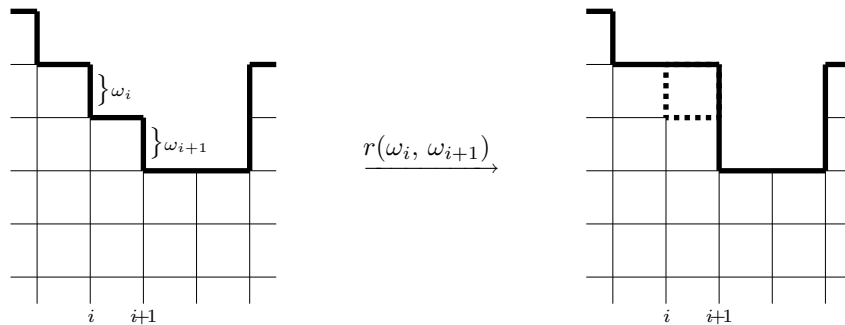


Figure 1: A possible move

The rates must satisfy

$$r(\omega^{\min}, \cdot) \equiv r(\cdot, \omega^{\max}) \equiv 0$$

whenever either  $\omega^{\min}$  or  $\omega^{\max}$  is finite. We assume  $r$  to be non-zero in all other cases. We want the dynamics to smoothen our interface, that is why we assume monotonicity in the following way:

$$(1) \quad r(z + 1, y) \geq r(z, y), \quad r(y, z + 1) \leq r(y, z)$$

for  $y, z, z + 1 \in I$ . This means that the higher neighbors a column has, the faster it grows. Our model is hence *attractive*.

We are going to use product property of the model's stationary measure. For this reason, similarly to Rezakhanlou [18], we assume that for any  $x, y, z \in I$

$$(2) \quad r(x, y) + r(y, z) + r(z, x) = r(x, z) + r(z, y) + r(y, x),$$

and for  $\omega^{\min} < x, y, z < \omega^{\max} + 1$

$$(3) \quad r(x, y - 1) \cdot r(y, z - 1) \cdot r(z, x - 1) = r(x, z - 1) \cdot r(z, y - 1) \cdot r(y, x - 1).$$

These two conditions imply product structure of the stationary measure, see section 1.3. Equation (3) is equivalent to the condition  $r(y, z) = s(y, z+1) \cdot f(y)$  for some function  $f$  and a symmetric function  $s$ .

At time  $t$ , the interface mentioned above is described by  $\underline{\omega}(t)$ . Let  $\varphi : \Omega \rightarrow \mathbb{R}$  be a finite cylinder function i.e.  $\varphi$  depends on a finite number of values of  $\omega_i$ . The growth of this interface is a Markov process, with the formal infinitesimal generator  $L$ :

$$(L\varphi)(\underline{\omega}) = \sum_{i \in \mathbb{Z}} r(\omega_i, \omega_{i+1}) \cdot [\varphi(\dots, \omega_i - 1, \omega_{i+1} + 1, \dots) - \varphi(\underline{\omega})].$$

When constructing the process rigorously, problems may arise due to the unbounded growth rates. The system being one-component and attractive, we assume that existence of dynamics on a set of tempered configurations  $\tilde{\Omega}$  (i.e. configurations obeying some restrictive growth conditions) can be established by applying methods initiated by Liggett and Andjel [11] [1]. Technically we assume that  $\tilde{\Omega}$  is of full measure w.r.t. the canonical Gibbs measures defined in section 1.3. In fact this has been proved for some kinds of these models, see below, and the results stated in section 2.3 concerning regularity of some of our processes may imply existence of the dynamics, this is a work in progress.

## 1.2 Examples

There are three essentially different cases of these models, all of them are of nearest neighbor type.

1. **Generalized exclusion processes** are described by our models in case both  $\omega^{\min}$  and  $\omega^{\max}$  are finite.

- **The totally asymmetric simple exclusion process (SE)** introduced by F. Spitzer [20] is described this way by  $\omega^{\min} = 0$ ,  $\omega^{\max} = 1$ ,

$$r(\omega_i, \omega_{i+1}) = \omega_i \cdot (1 - \omega_{i+1}).$$

Here  $\omega_i$  is the occupation number for the site  $i$ , and  $r(\omega_i, \omega_{i+1})$  is the rate for a particle to jump from site  $i$  to  $i + 1$ . Conditions (1), (2) and (3) for these rates are satisfied.

- **A particle-antiparticle exclusion process** is also shown to demonstrate the generality of the frame described above. Let  $\omega^{\min} =$

$-1$ ,  $\omega^{\max} = 1$ . Fix  $c$  (*creation*),  $a$  (*annihilation*) positive rates with  $c \leq a/2$ . Put

$$r(0, 0) = c, \quad r(0, -1) = \frac{a}{2}, \quad r(1, 0) = \frac{a}{2}, \quad r(1, -1) = a,$$

and all other rates are zero. If  $\omega_i$  is the number of particles at site  $i$ , with  $\omega_i = -1$  meaning the presence of an antiparticle, then this model describes a totally asymmetric exclusion process of particles and antiparticles with annihilation and particle-antiparticle pair creation. These rates also satisfy our conditions.

Other generalizations are possible allowing a bounded number of particles (or antiparticles) to jump to the same site. By the bounded jump rates and by nearest-neighbor type of interaction, the construction of dynamics of these processes is well understood, see e.g. Liggett [14].

2. **Generalized misanthrope processes** are obtained by choosing  $\omega^{\min} > -\infty$ ,  $\omega^{\max} = \infty$ .

- **The zero range process (ZR)** is included by  $\omega^{\min} = 0$ ,  $\omega^{\max} = \infty$ ,

$$r(z, y) = f(z)$$

with an arbitrary  $f : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$  nondecreasing function and  $f(0) = 0$ . Here  $\omega_i$  represents the number of particles at site  $i$ . These rates trivially satisfy conditions (1), (2), (3). The dynamics of this process is constructed by Andjel [1] under the condition that the rate function  $f$  obeys the growth condition  $|f(z+1) - f(z)| \leq K$  for some  $K > 0$  and all  $z \geq 0$ . See section 2.3 for stochastic bounds on this process in the general case.

3. **General deposition processes** are the type of these models where  $\omega^{\min} = -\infty$  and  $\omega^{\max} = \infty$ . In this case, the height difference between columns next to each other can be arbitrary in  $\mathbb{Z}$ . Hence the presence of antiparticles can not be avoided when trying to give a particle representation of the process.

- **Bricklayers' models (BL)**. Let

$$r(z, y) := f(z) + f(-y)$$

with the property

$$f(z) \cdot f(-z+1) = 1$$

for the nondecreasing function  $f$  and for any  $z \in \mathbb{Z}$ . This process can be represented by bricklayers standing at each site  $i$ , laying a brick on the column on their left with rate  $f(-\omega_i)$  and laying a brick to their right with rate  $f(\omega_i)$ . This interpretation gives reason to call these models bricklayers' model. Conditions (1), (2) and (3) hold for  $r$ . Similarly to the ZR process, this model is constructed by Booth [4] and Quant [17] only in case  $|f(z+1) - f(z)|$  is bounded in  $\mathbb{Z}$ . See section 2.3 for stochastic bounds on this process in the general case. Especially, **the exponential bricklayers' model (EBL)** has

$$(4) \quad f(z) = e^{-\frac{\beta}{2}} e^{\beta z}$$

with a positive real parameter  $\beta$ .

### 1.3 Translation invariant stationary product measures

We are interested in translation invariant stationary measures for these processes, i.e. canonical Gibbs-measures. We construct such measures similarly to Rezakhanlou [18] of the following form. Fix  $f(1) > 0$  and define

$$(5) \quad f(z) := \frac{r(z, 0)}{r(1, z-1)} \cdot f(1)$$

for  $\omega^{\min} < z < \omega^{\max} + 1$ . Then  $f$  is a nondecreasing strictly positive function. For  $I \ni z > 0$  we define

$$f(z)! := \prod_{y=1}^z f(y),$$

while for  $I \ni z < 0$  let

$$f(z)! := \frac{1}{\prod_{y=z+1}^0 f(y)},$$

finally  $f(0)! := 1$ . Then we have

$$f(z)! \cdot f(z+1) = f(z+1)!$$

for all  $z \in I$ . Let

$$\bar{\theta} := \begin{cases} \log \left( \liminf_{z \rightarrow \infty} (f(z)!)^{1/z} \right) = \lim_{z \rightarrow \infty} \log(f(z)) & , \text{ if } \omega^{\max} = \infty \\ \infty & , \text{ else} \end{cases}$$

and

$$\underline{\theta} := \begin{cases} \log \left( \limsup_{z \rightarrow \infty} (f(-z)!)^{1/z} \right) = \lim_{z \rightarrow \infty} \log(f(-z)) & , \text{ if } \omega^{\min} = -\infty \\ -\infty & , \text{ else.} \end{cases}$$

By monotonicity of  $f$ , we have  $\bar{\theta} \geq \underline{\theta}$ . We assume  $\bar{\theta} > \underline{\theta}$ . With a generic real parameter  $\theta \in (\underline{\theta}, \bar{\theta})$ , we define

$$Z(\theta) := \sum_{z \in I} \frac{e^{\theta z}}{f(z)!}.$$

Let the product-measure  $\underline{\mu}_\theta$  have marginals

$$(6) \quad \mu_\theta(z) = \underline{\mu}_\theta \{ \underline{\omega} : \omega_i = z \} := \frac{1}{Z(\theta)} \cdot \frac{e^{\theta z}}{f(z)!},$$

this is stationary for the process.

As can be verified, the expectation value  $\varrho(\theta) := \mathbf{E}_\theta(\omega_i)$  is a strictly increasing function of  $\theta$ . We introduce its inverse  $\theta(\varrho)$  and the function

$$(7) \quad \mathcal{H}(\varrho) := \mathbf{E}_{\theta(\varrho)} \{ r(\omega_i, \omega_{i+1}) \},$$

playing an important role in hydrodynamical considerations.

For the SE model, the construction leads to the well-known Bernoulli product-measure with marginals

$$\begin{aligned}\mu(1) &= \underline{\mu}\{\underline{\omega} : \omega_i = 1\} := \varrho, \\ \mu(0) &= \underline{\mu}\{\underline{\omega} : \omega_i = 0\} := 1 - \varrho\end{aligned}$$

with a real number  $\varrho$  between zero and one (the density of the particles). In our notations,  $-\varrho$  describes the average slope of the interface.

For the particle-antiparticle exclusion process, the relative probability of having a particle or an antiparticle as a function of the rates goes as  $\sqrt{c/a}$ , independently for the sites. The density of particles relative to antiparticles can be set by an arbitrary parameter.

Both for the ZR process and for BL models, it turns out that  $f$  defined in (5) and  $f$  in the definition of the rates agree.

## 1.4 The basic coupling

We consider two realizations of a model, namely,  $\underline{\zeta}$  and  $\underline{\eta}$ . We show the basic coupling preserving

$$(8) \quad \zeta_i(t) \geq \eta_i(t),$$

if this property holds initially for  $t = 0$ . We say that  $n = \zeta_i(t) - \eta_i(t) \geq 0$  is the number of *second class particles* present at site  $i$  at time  $t$ . During the evolution of the processes, the total number of these particles is preserved, and each of them performs a nearest neighbor random walk.

The height of the column of  $\underline{\zeta}$  (or  $\underline{\eta}$ ) between sites  $i$  and  $i + 1$  is denoted by  $g_i$  (or  $h_i$ , respectively). (These quantities are just used for easier understanding, they are not essential for the processes.) Let  $g_i \uparrow$  (or  $h_i \uparrow$ ) mean that the column of  $\underline{\zeta}$  (or the column of  $\underline{\eta}$ , respectively) between the sites  $i$  and  $i + 1$  has grown by one brick. Then the coupling rules are shown in table 1. Each line of this table represents a possible move, with rate written in the first column. In the last column,  $\curvearrowright$  (or  $\curvearrowleft$ ) means that a second class particle has jumped from  $i$  to  $i + 1$  (or from  $i + 1$  to  $i$ , respectively). This coupling for the SE model is described (with particle notations) in Liggett [13], [14] and [15]. The rates of these steps are non-negative due to (8) and monotonicity (1) of  $r$ . These rules clearly preserve property (8), since the rate of any move which could destroy this condition becomes zero. Summing the rates corresponding to either  $g_i \uparrow$  or to  $h_i \uparrow$  shows that each  $\underline{\zeta}$  and  $\underline{\eta}$  evolves according to its own rates.

with rate	$g_i \uparrow$	$h_i \uparrow$	a second class particle
$r(\zeta_i, \zeta_{i+1}) - r(\eta_i, \zeta_{i+1})$	•		$\curvearrowright$
$r(\eta_i, \eta_{i+1}) - r(\eta_i, \zeta_{i+1})$		•	$\curvearrowleft$
$r(\eta_i, \zeta_{i+1})$	•	•	

Table 1: Growth coupling rules

When there is only one second class particle in the coupled system of two processes, then we shall call it *the defect tracer*, and we denote its position at time  $t$  by  $Q(t)$ .

## 2 Results

### 2.1 Microscopic shape of shocks in some of our models

The hydrodynamical limit of the nearest neighbor asymmetric simple exclusion model leads to the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} = 0$$

which is a special case of the one-component hyperbolic conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial J(u)}{\partial x} = 0$$

where  $u \mapsto J(u)$  is a smooth, typically convex function. (By changing  $x$  to  $-x$ , concave  $J$ -s can be transformed to convex ones.) This equation has a shock (weak) solution starting with initial data

$$u(0, x) = \begin{cases} u_{\text{left}} & , x < 0 \\ u_{\text{right}} & , x \geq 0 \end{cases}$$

with  $u_{\text{left}} > u_{\text{right}}$ . The stable weak solution is of the form

$$u(t, x) = \begin{cases} u_{\text{left}} & , x < st \\ u_{\text{right}} & , x \geq st \end{cases}$$

where the speed  $s$  of the traveling shock is determined by the Rankine-Hugoniot formula

$$s = \frac{J(u_{\text{right}}) - J(u_{\text{left}})}{u_{\text{right}} - u_{\text{left}}},$$

see e.g. [19]. This is what we see on a macroscopic scale. The microscopic structure (i.e. on the level of particles) of the shock is of great interest. It has been considered in the context of the asymmetric simple exclusion process, and rather complicated microscopic structures have been found [5] [6] [7] [9] [10]. In the more general context of attractive particle systems the microscopic structure of the shock was investigated by [18].

In this part, we show a shock-like stationary distribution for some of our processes. We consider the bricklayers' process  $\omega$ , and we put a defect tracer initially to the origin:  $Q(0) = 0$ . We introduce the process  $\sigma$ , for which

$$\sigma_i(t) = \omega_{i+Q(t)}(t);$$

this is the process  $\omega$  as seen from the random position  $Q(t)$  of the defect tracer. Let

$$\underline{\theta} := \{\theta_i : i \in \mathbb{Z}\}$$

be a sequence of parameters, and define the product measure  $\mu^{(\underline{\theta})}$  with marginals

$$\mu_i(z) = \mu^{(\underline{\theta})} \{ \underline{\sigma} : \sigma_i = z \} := \mu^{(\theta_i)}(z) = \frac{1}{Z(\theta_i)} \cdot \frac{e^{\theta_i z}}{f(z)!}.$$

This measure only differs from the canonical  $\mu^{(\theta)}$  (6) in that the parameter of its one-dimensional marginals depends on the position.

**Theorem 2.1.** *For a bricklayers' model, if  $f$  is not the constant function, then the measure  $\mu^{(\underline{\theta})}$  described above is stationary for  $\underline{\sigma}$  if and only if  $f$  is the rate of an EBL model (4) with any parameter  $\beta > 0$ , and for the  $\underline{\theta}$  parameters of  $\mu^{(\underline{\theta})}$*

$$\theta_i = \begin{cases} \theta_{left} & \text{if } i \leq -1, \\ \theta_{right} := \theta_{left} - \beta & \text{if } i \geq 0 \end{cases}$$

is satisfied with an arbitrary real number  $\theta_{left}$ .

The form of the measure described in this theorem shows after a bit of calculations that the distribution of  $\omega_i$ ,  $i \leq -1$  is shifted by +1 compared to the distribution of  $\omega_j$ ,  $j \geq 0$ . This gives us the picture of a (random) valley with the (randomly) moving defect tracer in its center. Since the position of the defect tracer is not deterministic, we do not see the sharp change between the distribution of the two sides of this valley, if looking the model from outside.

## 2.2 Growth fluctuations in the models

Stochastic deposition models can be used to obtain microscopic description of domain growths, e.g. a colony of cells or an infected area of plants. The fluctuation of the growth is itself of great interest. Moreover, these models are in close connection to interacting particle systems, where the particle diffusion corresponds to rescaled surface fluctuation. It has been known [8] for the simple exclusion process, that the current fluctuation is in close connection to the motion of the called second class particle, and, divided by time, its variance vanishes for an observer moving with the speed of this particle. In this latter case, Prähofer and Spohn [16] suggest this quantity to be in the order of  $t^{2/3}$ .

Here we consider the whole class of the models introduced. In this general frame, we compute the growth fluctuations in order  $\mathcal{O}(t)$ , hence generalize the result of Ferrari and Fontes [8]. We need law of large numbers and a second moment condition for the position of the defect tracer. These have been established for simple exclusion [7], but, as far as we know, only  $L^1$ -convergence is known for most kinds of zero range processes [18]. We prove  $L^n$ -convergence with any  $n$  for the defect tracer of the totally asymmetric zero range process and for our new bricklayers' models via various coupling techniques.

We start our model in a canonical Gibbs-distribution, with parameter  $\theta$ . For a fixed speed value  $V > 0$  we define

$$J^{(V)}(t) := h_{\lfloor Vt \rfloor}(t) - h_0(0),$$

the height of column at site  $\lfloor Vt \rfloor$  at time  $t$ , relative to the initial height of the column at the origin. For  $V < 0$ , we introduce

$$J^{(V)}(t) := h_{\lceil Vt \rceil}(t) - h_0(0),$$



which is the mirror-symmetric form of  $J^{(V)}$  defined above for positive  $V$ 's. For  $V = 0$  we write

$$J(t) = J^{(0)}(t) := h_0(t) - h_0(0).$$

In particle notations of the models,  $J^{(V)}(t)$  is the current, i.e. the algebraic number of particles jumping through the moving window positioned at  $Vt$ , in the time interval  $[0, t]$ . We prove law of large numbers for this quantity:

$$\lim_{t \rightarrow \infty} \frac{J^{(V)}}{t} = \mathbf{E}(r) - V \mathbf{E}(\omega) \quad \text{a.s.}$$

We need law of large numbers and a second-moment condition for the position  $Q(t)$  of the defect tracer if one of the coupled models is started from its canonical Gibbs-measure:

**Condition 2.2.** *With initial distribution  $\underline{\mu}_\theta$  of  $\underline{\omega}$ , weak law of large numbers*

$$(9) \quad \lim_{t \rightarrow \infty} \mathbf{P}_\theta \left( \left| \frac{Q(t)}{t} - C(\theta) \right| > \delta \right) = 0$$

for a speed value  $C(\theta)$  and for any  $\delta > 0$  holds, and the bound

$$(10) \quad \mathbf{E}_\theta \left( \frac{Q(t)^2}{t^2} \right) < K < \infty$$

is satisfied for all large  $t$  for the position  $Q(t)$  of the defect tracer.

Inequality (10) is obvious in case of bounded rates, since in this situation, the process  $|Q(t)|$  is bounded by some Poisson-process.

**Theorem 2.3.** *Assume condition 2.2. Then*

$$(11) \quad \lim_{t \rightarrow \infty} \frac{\mathbf{Var}_\theta(J^{(V)}(t))}{t} = |V - C(\theta)| \cdot \mathbf{Var}_\theta(\omega_0) =: D_J(\theta)$$

for any  $V \in \mathbb{R}$ , where  $\mathbf{Var}_\theta$  stands for the variance w.r.t.  $\mu_\theta$ .

**Theorem 2.4** (Central limit theorem). *Assuming condition 2.2,*

$$\lim_{t \rightarrow \infty} \mathbf{P}_\theta \left( \frac{\tilde{J}^{(V)}(t)}{\sqrt{D_J(\theta)} \cdot \sqrt{t}} \leq x \right) = \Phi(x) = \int_{-\infty}^x \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy,$$

i.e.  $\tilde{J}^{(V)}(t)/\sqrt{t}$  converges in distribution to  $N(0, D_J(\theta))$ , a centered normal random variable with variance  $D_J(\theta)$  of (11). Tilde means here that the mean value of  $J^{(V)}(t)$  is subtracted.

For the SE model, (9) is proven in [7]. It is shown there that

$$\lim_{t \rightarrow \infty} \frac{Q(t)}{t} = 1 - 2\varrho \quad \text{a.s.}$$

Condition 2.2 is satisfied by this law, hence theorem 2.3 gives

$$\lim_{t \rightarrow \infty} \frac{\mathbf{Var}(J^{(V)}(t))}{t} = \varrho(1 - \varrho) |(1 - 2\varrho) - V|,$$

and the central limit theorem 2.4 also holds. These results have been known for SE by Ferrari and Fontes [8].

For the ZR and BL models, we need a condition on the growth rates:

**Condition 2.5.** For ZR and BL processes defined above, the rate function  $f$  is convex.

For the ZR process, under this condition and assuming either strict convexity or concavity of  $\mathcal{H}(\varrho)$  defined in (7), more than (9), namely,  $L^1$ -convergence is established by Rezakhanlou [18] with speed

$$(12) \quad C(\theta) = \frac{e^\theta}{\mathbf{Var}_\theta(\omega)}.$$

As far as we know, the second-moment condition (10) has not yet been proven for this model.

**Theorem 2.6.** For ZR and BL models satisfying condition 2.5 with initial distribution  $\underline{\mu}_\theta$  of  $\underline{\omega}$ , and for any  $n \in \mathbb{Z}^+$ ,

$$\frac{Q(t)}{t} \rightarrow C(\theta) \quad \text{in } L^n,$$

where  $C(\theta)$  is defined in (12) for the ZR process, and

$$(13) \quad C(\theta) := \frac{2 \sinh(\theta)}{\mathbf{Var}_\theta(\omega)}$$

for the BL model.

Hence under condition 2.5, condition 2.2 and thus theorem 2.3 and 2.4 hold for both ZR and BL models with  $C(\theta)$  defined in (12) and (13), respectively. As we expect by mirror symmetric properties of the BL model, the speed  $C(\theta)$  of the defect tracer is zero in case  $\theta = 0$  in this model.

Our methods do not rely on hydrodynamic limits. It follows that  $C(\theta)$  is a nondecreasing function for the totally asymmetric ZR process and BL model under condition 2.5. This shows (non strict) convexity of the function  $\mathcal{H}(\varrho)$  of (7) for these models, since

$$C(\theta(\varrho)) = \frac{d\mathcal{H}(\varrho)}{d\varrho}$$

after some computations, and  $\theta(\varrho)$  is also a monotone function.

**Proposition 2.7.** Under condition 2.5, the function  $\mathcal{H}(\varrho)$  is strictly convex for the BL model. For the ZR process satisfying 2.5, linearity of  $\mathcal{H}(\varrho)$  is equivalent to linearity of the rate function  $f$  on  $\mathbb{Z}$ , which is the case of independent random walk of the particles. If this is not the case, then  $\mathcal{H}(\varrho)$  is strictly convex.

This is an important observation for [2], since this property is only proved for small  $\theta$  values there. It is also remarkable for [18], where strict convexity is just assumed.

We remark that rates for removal of the bricks can also be introduced to obtain a model with both growth and decrease of columns. In particle notations this represents possible left jumps of particles (or right jump of antiparticles, respectively). Therefore, not only the totally asymmetric case, but the general asymmetric case of particle processes (SE or ZR, for example) can also be included in the description. The extension of the proof of theorems 2.3 and 2.4 to

this case is straightforward. However, the coupling arguments used to establish condition 2.2 for ZR and BL models in later sections are not applicable in case of brick-removal.

We see that  $\lim_{t \rightarrow \infty} \frac{\mathbf{Var}(J^{(V)}(t))}{t}$  vanishes if we observe this quantity from the moving position  $Vt = C(\theta)t$ , having the characteristic speed of the hydrodynamical equation. This has been known for the SE model with strongly restricted values of  $\omega_i$ , and now it is proven for the class of more general models with possibly  $\omega_i \in \text{all } \mathbb{Z}$  also. The interesting question, of which the answer is strongly suggested for some models [16], is the correct exponent of  $t$  leading to nontrivial limit of  $\mathbf{Var}(J^{(C)}(t))/t^{2\alpha}$  as  $t \rightarrow \infty$ .  $\alpha$  is believed to be  $1/3$ , in close connection to  $t^{2/3}$  order fluctuations of the position  $Q(t)$  of the defect tracer.

### 2.3 Stochastic bounds on the growth

The shock-like product (time-) stationary measure we found in section 2.1 works for exponential dependence of the growth rates on the relative heights. As the model is not rigorously constructed, the natural question of existence of dynamics arises here.

In the area of interacting particle systems, there are two main situations where construction methods are available. One of them applies when the rate with which the configuration changes at a site is bounded. As described in Liggett [14], the construction can be carried out in this case via functional analysis properties of the infinitesimal generator and via the Hille-Yosida theorem. This is the way how existence of dynamics is usually proved for stochastic Ising models, the voter model, contact processes, simple- and K-exclusion processes.

The other situation is when the growth rates are unbounded, but satisfy a *sublinear* growth condition. This means that the growth rates are bounded from above by a linear function of the local state space. The famous example is the zero range process, where there is a nonnegative number  $\omega_i$  of particles at each site  $i$ , and with rate  $f(\omega_i)$  depending on the number of these particles, one of them jumps to another site. Slightly more than the sublinear condition mentioned above is formulated here by  $|f(k+1) - f(k)| \leq K$  for any  $k \geq 0$  and some  $K > 0$ . Under this condition, it is possible to compare the model to the so-called *multi-type branching process*, or to consider some differential equation arguments, and hence give stochastic bounds on the states realized by the process. This is the way Andjel [1] constructs the process, generalizing the earlier work of Liggett [12]. The method can be extended to more complicated systems, but sublinearity is still an essential condition in the proof of existence.

None of these methods fit to the bricklayers' process with exponential rates in section 2.1. Although a sublinear growth condition would make it possible to use the arguments mentioned above (see Booth [4] or Quant [17]), e.g. [2] sets up a claim to a proof of existence for models with *superlinear* growth rates. Not superlinearity, but convexity considerations also play an important role in hydrodynamical and second class particle-related arguments, see e.g. Balázs [3] or Rezakhanlou [18], and convexity of the growth rates in some cases may imply superlinearity of them.

We consider here the bricklayers' process, where the jump rates are unbounded, and we *do not* require sublinear growth conditions. We use attractiveness of the system instead to show stochastic bounds on it. This is carried out

via coupling considerations, and makes use of auxiliary systems. All our arguments are also valid for the zero range process, hence the bounds also apply to this model for any monotone increasing rate function. Formally, to obtain the statements for zero range, the reader should simply neglect all terms  $f(-\omega_i)$ ,  $e^{-\theta}$ ,  $e^{-\theta_1}$ ,  $e^{-\theta_2}$ .

First we show a process on a finite number of sites. Fix  $n \in \mathbb{N}$ , and define the infinitesimal generator  $L^{(n)}$  acting on functions of  $\underline{\omega}$ :

$$\left(L^{(n)}\varphi\right)(\underline{\omega}) = \sum_{i=-n}^{n-1} [f(\omega_i) + f(-\omega_{i+1})] \cdot \left[\varphi(\underline{\omega}^{(i,i+1)}) - \varphi(\underline{\omega})\right].$$

This is well defined for any  $\underline{\omega} \in \Omega$ . For this finite site-process, which we call the *n-monotone process*, the jump  $\underline{\omega} \rightarrow \underline{\omega}^{(i,i+1)}$  happens with rate  $f(\omega_i) + f(-\omega_{i+1})$ , independently for different sites  $i$ , but only for  $-n \leq i \leq n-1$ . For columns not in this interval, nothing happens.

We say that a measure  $\underline{\pi}$  on  $\Omega$  is a *good measure with parameters  $\theta_1$  and  $\theta_2$* , if there exist  $-\bar{\theta} < \theta_1 < \theta_2 < \bar{\theta}$  such that the measure  $\underline{\mu}^{(\theta_2)}$  (see (6)) dominates  $\underline{\pi}$  and  $\underline{\pi}$  dominates  $\underline{\mu}^{(\theta_1)}$  stochastically. This is equivalent to saying that  $\underline{\eta}$  distributed according to the product measure  $\underline{\mu}^{(\theta_1)}$ ,  $\underline{\zeta}$  distributed according to  $\underline{\pi}$  and  $\underline{\xi}$  distributed according to  $\underline{\mu}^{(\theta_2)}$  can be coupled in such a way that

$$\eta_i \leq \zeta_i \leq \xi_i$$

holds for all  $i \in \mathbb{Z}$ . Note that if  $\underline{\pi}$  is a product of marginals  $\pi_i$  on  $\mathbb{Z}$ , then this is equivalent to the corresponding stochastic dominations for the marginals at each site  $i$ .

**Theorem 2.8.** *Let  $\underline{\omega}(0)$  be distributed according to the good distribution  $\underline{\pi}$  with parameter  $\theta_1, \theta_2$ , and let it evolve according to the n-monotone evolution. The height of column  $i$  at time  $t$  is denoted by  $h_i^{(n)}(t)$ , with the convention  $h_0^{(n)}(0) = 0$ . Then for all  $i \in \mathbb{Z}$ ,  $t > 0$ , the limit*

$$h_i(t) := \lim_{n \rightarrow \infty} h_i^{(n)}(t)$$

exists a.s. for each  $\underline{\omega}(0) \in \Omega$ ,

$$\mathbf{E}[h_i(t) - h_i(0)] \leq t \cdot (e^{\theta_2} + e^{-\theta_1})$$

for all  $t$ , and

$$\mathbf{E}\left([h_i(t) - h_i(0)]^2\right) \leq \text{const.} \cdot t^2$$

for all  $t$  large enough.

Now we refine the results in order to show the bounds when the process starts from a deterministically given initial state  $\underline{\omega}(0) \in \Omega = \mathbb{Z}^{\mathbb{Z}}$ . Of course, we shall not allow all elements of this set. For  $\underline{\omega}$  fixed, we define the set

$$A(\underline{\omega}) := \{(\underline{\zeta}, g_0(0)) \in \Omega \times \mathbb{N} : g_i(0) \geq h_i(0) \text{ for all } i \in \mathbb{Z}\}$$

with columns

$$h_i := \begin{cases} h_0 - \sum_{j=1}^i \omega_j & \text{for } i > 0, \\ h_0 + \sum_{j=i+1}^0 \omega_j & \text{for } i < 0 \end{cases}$$

of  $\underline{\omega}$  and

$$g_i := \begin{cases} g_0 - \sum_{j=1}^i \zeta_j & \text{for } i > 0, \\ g_0 + \sum_{j=i+1}^0 \zeta_j & \text{for } i < 0 \end{cases}$$

of another configuration  $\underline{\zeta}$ . We assume  $h_0(0) = 0$  initially. Imagining the wall of bricks, a typical  $\underline{\zeta}$  has larger negative gradient on the left-hand side than on the right-hand side of the origin.

For a measure  $\underline{\Pi}$  on  $\Omega \times \mathbb{N}$  we call the first marginal  $\underline{\pi}$  on  $\Omega$ , while the second marginal  $\nu$  on  $\mathbb{N}$ . We define

$$\tilde{\Omega} := \{ \underline{\omega} \in \Omega : \text{there exists } \underline{\Pi} \text{ for which } \underline{\pi} \text{ is a good measure,} \\ \nu \text{ has finite second moment and } \underline{\Pi}\{A(\underline{\omega})\} > 0 \}.$$

Note that one has many choice for  $\underline{\Pi}$ , it is not unique.  $\tilde{\Omega}$  is going to be the set of initial configurations for which we shall bound the growth of the process.

**Theorem 2.9.** *Let us fix  $\underline{\omega}(0) \in \tilde{\Omega}$  with  $\underline{\Pi}$ , of which the first marginal  $\underline{\pi}$  is a good measure having parameters  $\theta_1$  and  $\theta_2$ . Let  $\underline{\omega}$  evolve according to the  $n$ -monotone evolution; the height of column  $i$  at time  $t$  is denoted by  $h_i^{(n)}(t)$ , with the convention  $h_0^{(n)}(0) = 0$ . Then for all  $i \in \mathbb{Z}$ ,  $t > 0$ , the limit*

$$h_i(t) := \lim_{n \rightarrow \infty} h_i^{(n)}(t)$$

exists a.s., and

$$\mathbf{E}[h_i(t)] \leq \frac{K \cdot t}{\sqrt{\underline{\Pi}\{A\}}} + \sqrt{\frac{\mathbf{E}([g_i(0)]^2)}{\underline{\Pi}\{A\}}}$$

for all  $t$  large enough and for some  $\infty > K > 0$  depending on  $\theta$ . Here  $g_i(0)$  is the height of column  $i$  for  $\underline{\zeta}(0)$  distributed according to  $\underline{\pi}$ , with  $g_0(0)$  having distribution  $\nu$ .

**Proposition 2.10.** *Fix  $-\bar{\theta} < \theta_1 < \theta_2 < \bar{\theta}$  and  $\mathbf{E}^{(\theta_1)}(z) < K_1 < K_2 < \mathbf{E}^{(\theta_2)}(z)$ . Then*

$$\left\{ \underline{\omega} : K_2 > \limsup_{n \rightarrow -\infty} \frac{1}{|n|} \sum_{i=n+1}^0 \omega_i \quad , \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \omega_i > K_1 \right\} \subset \tilde{\Omega}.$$

*Epecially, for any  $\underline{\theta} < \theta < \bar{\theta}$ , one can find parameters  $\theta_1, \theta_2$  such that the set above has  $\underline{\mu}^{(\theta)}$  measure one, hence  $\underline{\mu}^{(\theta)}\{\tilde{\Omega}\} = 1$  holds.*

## References

- [1] E. D. Andjel. Invariant measures for the zero range process. *Ann. Probab.*, 10(3):525–547, 1982.
- [2] M. Balázs. Microscopic shape of shocks in a domain growth model. *Journal of Stat. Phys.*, 105(3/4):511–524, 2001.

- [3] M. Balázs. Growth fluctuations in a class of deposition models. *Ann. Inst. H. Poincaré Probab. Statist.*, 39:639–685, 2003.
- [4] L. Booth. *Random Spatial Structures and Sums*. PhD thesis, Utrecht University, 2002.
- [5] A. De Masi, C. Kipnis, E. Presutti, and E. Saada. Microscopic structure at the shock in the asymmetric simple exclusion. *Stochastics*, 27:151–165, 1988.
- [6] B. Derrida, J. L. Lebowitz, and E. R. Speer. Shock profiles for the asymmetric simple exclusion process in one dimension. *Journal of Stat. Phys.*, 89(1-2):135–167, 1997.
- [7] P. A. Ferrari. Shock fluctuations in asymmetric simple exclusion. *Probab. Theory Relat. Fields*, 91:81–102, 1992.
- [8] P. A. Ferrari and L. R. G. Fontes. Current fluctuations for the asymmetric simple exclusion process. *Ann. Probab.*, 22:820–832, 1994.
- [9] P. A. Ferrari, L. R. G. Fontes, and Y. Kohayakawa. Invariant measures for a two-species asymmetric process. *Journal of Stat. Phys.*, 76:1153–1177, 1994.
- [10] J. Gärtner and E. Presutti. Shock fluctuations in a particle system. *Ann. Inst. Henri Poincaré*, 53:1–14, 1990.
- [11] T. M. Liggett. Existence theorems for infinite particle systems. *Trans. Am. Math. Soc.*, 165:471–481, 1972.
- [12] T. M. Liggett. An infinite particle system with zero range interactions. *Ann. Probab.*, 1(2):240–253, 1973.
- [13] T. M. Liggett. Coupling the simple exclusion process. *Ann. Probab.*, 4:339–356, 1976.
- [14] T. M. Liggett. *Interacting particle systems*. Springer-Verlag, 1985.
- [15] T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*. Springer-Verlag, 1999.
- [16] M. Prähofer and H. Spohn. Current fluctuations for the totally asymmetric simple exclusion process. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 185–204. Birkhäuser Boston, Boston, MA, 2002.
- [17] C. Quant. *On the construction and stationary distributions of some spatial queueing and particle systems*. PhD thesis, Utrecht University, 2002.
- [18] F. Rezakhanlou. Microscopic structure of shocks in one conservation laws. *Ann Inst. H Poincaré Anal. Non Linéaire*, 12(2):119–153, 1995.
- [19] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*. Springer-Verlag, 1983.
- [20] F. Spitzer. Interaction of Markov processes. *Adv. in Math.*, 5:246–290, 1970.