

Stabilization of sampled-data nonlinear systems by receding horizon
control via discrete-time approximations

Thesis of the Ph. D. Dissertation

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Introduction

During the last decades, considerable attention has been paid to the stabilization problem of nonlinear systems. Among the solution methods for this problem, receding horizon control (RHC) strategies, also known as model predictive control (MPC), have become quite popular (see e.g. [2], [3], [5], [6], [7], [9], [10], [11]). Owing to the use of computers in the implementation of the controllers, the investigation of sampled-data control systems has become an important area of control science. An overview and analysis of existing approaches for the stabilization of sampled-data systems can be found in ([12], [13]).

One way to design a digital controller is to design a continuous-time controller based on the continuous-time plant model and then discretize it using fast sampling for digital implementation. However, some difficulty may arise during the application of this method: 1) because of hardware limitations, it may be impossible to reduce the sampling period to a sufficiently small value that ensures the desired performance of the system; 2) the exact solution of the nonlinear continuous-time model is typically unknown, therefore an approximation procedure is unavoidable; 3) it may be difficult to implement an arbitrarily time-varying control function.

The second way to design a digital controller is to discretize the continuous-time plant model and design a controller on the basis of the discrete-time model. While for linear systems we can in principle compute the exact discrete-time model of the plant, this is not the case for nonlinear systems. As a result, the controller design can be carried out by means of an approximate discrete-time model. In [12] and [13] a systematic investigation of the connection between the exact and approximate discrete-time models is carried out. Moreover, results in [12] and [13] present a set of sufficient conditions which guarantee that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant. Results in ([12], [13]) provide a framework for controller design via approximate discrete-time models, but they did not explain how the actual controller design can be carried out within this framework.

There are several ways to design controllers satisfying the conditions given in ([12], [13]). In [4], optimization-based methods are studied; the design is carried out either via an infinite horizon optimization problem or via an optimization problem over a finite horizon with varying length. To relax the computational burden needed in the case of infinite horizon optimization and in the case of optimization over a varying time interval, the application of the RHC method offers good vistas. The RHC method obtains the feedback control by solving a finite horizon optimal control problem at each time instant using the current state of the plant as the initial state for the optimization and applying "the first part" of the optimal control. The present work studies the conditions under which the stabilizing RHC computed for the approximate discrete-time model also stabilizes the exact discrete-time system both in the cases when the sampling period T is fixed, but the integration parameter h used in obtaining the approximate model can be chosen to be arbitrarily small and when these two parameters coincide, but can be adjusted.

All of this investigations deal with the case when the sampling rates of the control function and the state measurement coincide i.e. a single-rate approach is presented. Moreover, the measurement result and the corresponding controller are assumed to be available instantaneously. The latter assumption is of course unrealistic and may be considered as one of the reasons why different rates of control and measurement samplings have to be taken into account. Besides the measurement and computational delay, the nature of the problem may involve different measurement and control sampling rates. The notion of multirate sampled-data feedback (which was introduced to the best of our knowledge by [14]) is used in the this thesis in this sense. Polushin and Marquez address the design of multirate controllers based on the knowledge of a continuous-time stabilizing feedback for the exact model as well as on that of a discrete-time stabilizing feedback for the approximate model under the assumption of "low measurement rate" and in the presence of measurement delay. In this thesis we drive a multirate version of the RHC algorithm based on discrete-time approximate models of the plant, and establish sufficient conditions which guarantee that the proposed control stabilizes the original exact model in the presence of measurement and computational delays.

One of the very important applications of the optimal control theory is to determine optimal treatment schedules of Acquired Immune Deficiency Syndrome (AIDS). One way to design optimal treatment is to design an open-loop optimal controller by using Pontryagin's Maximum Principle (see [1] and [8]). However, some drawbacks may arise during the application of this method: 1) the optimization is performed over a finite time horizon, and no care is taken over the evolution of the process behind this time horizon; 2) the optimal controller is obtained as a continuous-time controller, in spite of the fact that continuous variation of the drug dose seems hard to apply in the real treatment of patients; 3) the optimal

controller is given in an open-loop form and it does not deal with the changes that may happen in the system during the treatment. To overcome these drawbacks, the application of the RHC methods based on the discrete-time model seems to be obvious. We applied our theoretical results to two HIV/AIDS models.

Problem statement

Consider the nonlinear control system described by

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$, with $f(0, 0) = 0$, U is closed and $0 \in U$. Let $\Gamma \subset \mathbb{R}^n$ be a given compact set containing the origin and consisting of all initial states to be taken into account. The system is to be controlled digitally using piecewise constant control functions

$$u(t) = u(kT) =: u_k, \quad \text{if } t \in [kT, (k+1)T), \quad k \in \mathbb{N},$$

where $T > 0$ is the sampling period. Let $t \in [0, T] \mapsto \phi(t, x_0, \bar{u})$ denote the solution of (1) with $u(t) = \bar{u}$ and $\phi(0, x_0, \bar{u}) = x_0$.

The exact discrete-time model of system (1) can be defined as

$$x_{k+1} = F_T^E(x_k, u_k), \quad (2)$$

where $F_T^E(x, u) := \phi^E(T, x, u)$. We emphasize that F_T^E in (2) is not known in most cases, therefore the controller design can be carried out by means of an approximate discrete-time model

$$x_{k+1} = F_{T,h}^A(x_k, u_k), \quad (3)$$

where parameter h is a modelling parameter, which is typically the step size of the underlying numerical method.

The problem is to define a state-feedback controller

$$v_{T,h}^A : \tilde{\Gamma} \rightarrow U \quad (4)$$

using the approximate model (3) which stabilizes the origin for the exact model (2) in an appropriate sense, where $\tilde{\Gamma} \supset \Gamma$.

One may think that this is a simple question if we have a convergent numerical method. To show that this is not the case we have presented three examples for which a family of receding horizon control law is designed to stabilize the family of approximate models, but the exact discrete-time model is destabilized by the same family of controllers.

New Scientific Results

Results on stabilizing receding horizon control of sampled-data nonlinear systems via their approximate discrete-time models have been presented. We have investigated both situations when the sampling period T is fixed and the integration parameter h can be chosen to be arbitrarily small, and when these two parameters coincide but can be adjusted arbitrary. Both single rate and multirate versions of the receding horizon algorithm for the stabilization of sampled-data nonlinear systems has been investigated. In the later case "low measurement rate" is assumed, and the presence of measurement and computational delays are taken into account. Sufficient conditions have been established which guarantee that the controller that renders the origin to be asymptotically stable for the approximate model also stabilizes the exact discrete-time model for sufficiently small integration and/or sampling parameters. These conditions concern directly the data of the problem and the design parameters of the method, but not the results of the design procedure.

Our Theoretical results have been applied to recently developed models of the interaction of the HIV virus and the immune system of the human body.

Thesis 1: The necessary and sufficient condition for the existence of a stabilizing state-feedback controller has been presented.

Since we want to find a state-feedback controller, it seems to be reasonable to investigate when it does exist. We have formulated a necessary and sufficient condition for the existence of such a controller in

the following theorem. Let $\Gamma \subset \mathbb{R}^n$ be a given compact set, containing a neighborhood of the origin and let $\Delta > 0$ be such that $\Gamma \subset \mathcal{B}_\Delta$.

Theorem 1 System (2) is practically asymptotically stabilizable (PAS) with a parameterized family of feedbacks in Γ about the origin if and only if it is practically asymptotically controllable (PAC) with a parameterized family of control functions from Γ to the origin.

Thesis 2: A suitable version of the receding horizon control method has been chosen and several properties of the method has been established which are important to establish the closed-loop stability of the exact discrete-time model.

We have described a version of the RHC method suitable in connection with the stabilization problem of the exact discrete-time systems via their approximate discrete-time model. In order to define a receding horizon feedback controller, let (3) be subject to the cost function

$$J_{T,h}(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} T l_h(x_k^A, u_k) + g(x_N^A), \quad (5)$$

where $0 < N \in \mathbb{N}$ and $\mathbf{u} = \{u_0, u_1, \dots, u_{N-1}\}$ and $x_k^A = \phi_k^A(x, \mathbf{u})$, $k = 0, 1, \dots, N$, denote the solution of (3), and l_h and g are given functions satisfying some assumptions (see Chapter 2).

Consider the optimization problem

$$P_{T,h}^A(N, x) : \min \{J_{T,h}(N, x, \mathbf{u}) : u_k \in U_{\Delta_1}\}. \quad (6)$$

Let $\mathbf{u}^*(x) = \{u_0^*(x), u_1^*(x), \dots, u_{N-1}^*(x)\}$ denote the solution of (6) then its first element, i.e. $u_0^*(x)$ is applied at the state x . Since the optimal solution of $P_{T,h}^A(N, x)$ naturally depends on x , in this way a feedback has been defined on the basis of the approximate discrete-time model i.e.

$$v_{T,h}^A(x) := u_0^*(x).$$

For any $x \in \mathbb{R}^n$, let

$$V_N(x) = \inf \{J_{T,h}(N, x, \mathbf{u}) : u_k \in U_{\Delta_1}\},$$

The optimal value function $V_N(\cdot)$ has been used as a Lyapunov function to establish the stability of the approximate discrete-time model with the receding horizon control.

We have formulated some properties of the RHC method which depend on the consistency between the exact and the approximate model: as an auxiliary result we showed a control sequence that steers the trajectory of the approximate system to the terminal set, while this trajectory remain uniformly bounded. As a consequence, we obtained that the approximate model is asymptotically controllable from Γ to the origin. Moreover, we derived a uniform upper bound for V_N and a criterion for the horizon length (in terms of the real continuous-time) ensuring that the optimal trajectory ends in the terminal set.

Thesis 3: Local practical asymptotic stability of the closed-loop exact discrete-time model has been proven in the case of fixed sampling parameter T .

We have discussed the case when the sampling parameter T is fixed and the discretization parameter h can be assigned arbitrarily and independently of T .

We have shown that the value function has the following properties

- 1) $V_N(\cdot)$ is continuous in \mathcal{B}_Δ uniformly in small h .
- 2) $\exists \sigma_1, \sigma_2 \in \mathcal{K}_\infty$ and a positive definite function σ_3 such that

$$\sigma_1(\|x\|) \leq V_N(x) \leq \sigma_2(\|x\|), \quad (7)$$

$$V_N(F_{T,h}^A(x, v_{T,h}^A(x))) - V_N(x) \leq -T\sigma_3(\|x\|), \quad (8)$$

for all $x \in \mathcal{B}_\Delta$. On the basis of the consistency between the exact and approximate models we have proven that the family $(F_T^E, v_{T,h}^A)$ is locally practically asymptotically stable about the origin. The effectiveness of the method has been illustrated by simulation example.

Thesis 4: Semiglobal practical asymptotic stability of the closed-loop exact discrete-time model has been proven in the case of varying sampling rate $T = h$.

In the case of $T = h$ we have stated conditions that guarantee that the receding horizon controller defined for the approximate discrete-time model with $T = h$ results in a sampled-data system which is

practically asymptotically stable about the origin with the prescribed basin of attraction Γ . The main difficulty in this case is to derive a class- \mathcal{K}_∞ lower bound for V_N which is independent of T . In fact, we can only determine a uniform lower bound outside an arbitrary small ball around the origin. The effectiveness of the method has been illustrated by simulation example.

Thesis 5: An algorithm for multirate sampled-data systems with delays has been proposed and the stability of the closed-loop system has been proven.

We have addressed the problem of state feedback stabilization of (2) under "low measurement rate" in the presence of measurement and computational delays. Here we considered the case of $T \neq h$.

We assume that state measurements can be performed at the time instants jT^m , $j = 0, 1, \dots$:

$$y_j := x(jT^m), \quad j = 0, 1, \dots,$$

where T^m is the measurement sampling period. In this case, different measurement and control sampling rates are used.

The result of the measurement y_j becomes available for the computation of the controller at $jT^m + \tau_1$ ($> jT^m$), while the computation requires $\tau_2 > 0$ length of time i.e. the (re)computed controller is available at $T_j^* := jT^m + \tau_1 + \tau_2$, $j = 0, 1, \dots$. We assume that $\tau_1 = \ell_1 T$, $\tau_2 = \ell_2 T$ and $T^m = \ell T$ for some integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ and $\ell \geq \ell_1 + \ell_2 =: \bar{\ell}$.

Because of the measurement and computational delays, on the time interval $[0, \tau_1 + \tau_2)$ a precomputed control function \mathbf{u}^c can only be used. It is reasonable to assume that the initial states can be kept within the PAC domain of the exact system with such a precomputed controller.

Furthermore, a "new" controller computed according to the measurement $y_j = x(jT^m)$ will only be available at $jT^m + \bar{\ell}T$, so in the time interval $[jT^m, jT^m + \bar{\ell}T)$, the "old" controller has to be applied. Since the corresponding exact trajectory is unknown, an approximation ζ_j^A to the exact state $x(jT^m + \bar{\ell}T)$ can only be used which can be defined as follows. Assume that a control sequence $\{u_0(\zeta_{j-1}^A), \dots, u_{\ell-1}(\zeta_{j-1}^A)\}$ has been defined for $j \geq 1$. Let

$$\mathbf{v}^p(\zeta_{j-1}^A) = \{u_{\ell-\bar{\ell}}(\zeta_{j-1}^A), \dots, u_{\ell-1}(\zeta_{j-1}^A)\},$$

and define ζ_j^A by

$$\zeta_j^A = \mathcal{F}_{\bar{\ell}}^A(y_j, \mathbf{v}^p(\zeta_{j-1}^A)), \quad \zeta_0^A = \phi_{\bar{\ell}}^A(x, \mathbf{u}^c),$$

where $\mathcal{F}_{\bar{\ell}}^A(y, \{u_0, \dots, u_{\bar{\ell}-1}\}) = F_{T, h}^A(\dots F_{T, h}^A(F_{T, h}^A(y, u_0), u_1), \dots, u_{\bar{\ell}-1})$.

The ℓ -step exact discrete-time model is given by:

$$\xi_{j+1}^E = \mathcal{F}_{\ell}^E(\xi_j^E, \mathbf{v}^{(j)}), \quad \xi_0^E = \phi_{\ell}^E(x, \mathbf{u}^c),$$

where $\mathcal{F}_{\ell}^E(\xi_j^E, \mathbf{v}) = \phi_{\ell}^E(\xi_j^E, \mathbf{v})$, and $\mathbf{v}^{(j)} = \{u_0^{(j)}, \dots, u_{\ell-1}^{(j)}\}$.

Our aim is to solve the following problem: for given T , T^m , τ_1 and τ_2 find a control strategy

$$\mathbf{v}_{\ell, h}(x) = \{u_0(x), \dots, u_{\ell-1}(x)\},$$

using the approximate model (3) which stabilizes the origin for the exact system (2) in an appropriate sense.

We have established sufficient conditions which guarantee that the proposed control stabilizes the original exact model in the presence of measurement and computational delays. We have illustrated that, if the occurring delays are not taken into account, then instability of the closed-loop may occur.

Thesis 6: The theoretical results have successfully been applied to two kinds of HIV/AIDS models.

In the second part of the work, we have applied the theoretical results for two kinds HIV/AIDS models. The first model takes into account the latently infected cells (such cells contain the virus but are not producing it) and the actively infected cells (such cells are producing the virus). The second model considers both infectious virions and non-infectious virions.

The system has two steady states which we call uninfected E^0 and infected E^+ . In the real situation and when there is no treatment, E^0 is usually unstable for the AIDS patients. The drug dose is considered as control input and the goal is to determine a control strategy which stabilizes this unstable equilibrium.

In order to apply our theoretical results, we have verified all the required conditions. We have used both the one-step and the ℓ -step versions of the receding horizon control method to determine the treatment schedules.

The Authors Publications

- 1- Gyurkovics, É., **A. M. Elaiw**, "Stabilization of sampled-data nonlinear systems by receding horizon control via discrete-time approximations", *Automatica*, **40(12)**, pp. 2017-2028, 2004.
- 2- **Elaiw, A. M.**, "Stabilization of sampled-data nonlinear systems by an ℓ -step receding horizon control based on their approximate discrete-time models", *Pure Mathematics and Applications*, **14(3)**, pp. 181-197, 2003.
- 3- **Elaiw, A. M.**, "Receding horizon control method applied to antiviral treatment of AIDS", *Miskolc Mathematical Notes*, **5(2)**, pp. 173-186, 2004.
- 4- **Elaiw, A. M.**, K. Kiss, M. A. L. Caetano, "Stabilization of HIV/AIDS model by receding horizon control", *Journal of Applied Mathematics and Computing*, 18(1-2) 2005.
- 5- **Elaiw, A. M.**, É. Gyurkovics, "Stabilizing receding horizon control of sampled-data nonlinear systems via their approximate discrete-time models", in: *Bars, R., E. Gyurkovics (eds), Proceedings of the 12th IFAC Workshop on Control Applications of Optimization, Visegrád, Hungary*, pp. 65-61, 2003.
- 6- Gyurkovics, É., **A. M. Elaiw**, "Stabilization of sampled-data nonlinear systems by receding horizon control via discrete-time approximations", *Proceedings of the 2nd IFAC Conference on Control Systems Design*, Bratislava, Slovak Republic, 2003.
- 7- **Elaiw, A. M.**, "Receding horizon control applied to antiviral treatment of AIDS", *Seventh Colloquium on the Qualitative Theory of Differential Equations*, Hungary 2003 (oral presentation).
- 8- **Elaiw, A. M.**, É. Gyurkovics, "Stabilization of HIV/AIDS model using an ℓ -step receding horizon control based treatment scheduling", in: *Allgöwer F., (ed), Proceedings of the 6th IFAC Symposium on Nonlinear Control Systems*, Stuttgart, Germany, pp. 801-806, 2004.
- 9- **Elaiw, A. M.**, É. Gyurkovics, "Multirate sampling and delays in receding horizon control stabilization of nonlinear systems", accepted by *16th IFAC World Congress, Prague, Czech Republic*, July 4-8 2005.

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Selected Publication on Topics Related to Dissertation

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