

Multidegrees of Singularities and Nonreductive Quotients

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A thesis submitted for the degree
PhD in Mathematics
November 2007

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Acknowledgments

I would like to thank my supervisor, András Szenes for all his support and directions. I am especially grateful for his patience and humanity; without his unselfish helpfulness and care, and also paternal rigour I would not have survived my PhD studies.

I would also like to thank Richárd Rimányi for introducing me to singularity theory, and explaining the problem to me. Also thanks to András Szucs for the beautiful lectures in Algebraic Topology. I am greatly indebted to and Michéle Vergne, whose ideas profoundly influenced this thesis and helped my work. I would like to express my gratitude to Frances Kirwan for her generous support and help during my stay in Oxford, and for introducing me to some new ideas of GIT and symplectic geometry.

My years as a PhD student would have been poorer without the friendship and help of other graduate students I met in Budapest and all around Europe. I would particularly thank to Zsolt Szilagyi, Balazs Komuves, Elisabeth Baldwin and Vladimir Dotsenko.

Last but not least I would like to acknowledge the patience and support of my family.

Chapter 1

Introduction

This thesis gives a compact iterated residue formula for the multidegrees of A_n singularities. Multidegrees are generalisations of ordinary degree of algebraic geometry in the equivariant world; these are equivariant cohomology classes, and they coincide with equivariant Poincaré duals in topology.

More precisely, if we are given a complex representation V of a complex semisimple Lie group G and a G -invariant subvariety $\eta \subset V$, one can associate a multidegree $\text{mdeg}[\eta, V] \in H_G^*(V) = H^*(BG)$ in the equivariant cohomology ring of V . This is a homogeneous polynomial in the variables associated to the generators of the Lie algebra of the maximal torus in G , whose degree equals to the codimension of (the maximal dimensional component of) η in V . As expected, one can derive from this the ordinary degree of η . However, the G -action is an extra structure on η , and it provides more information stored in the multidegree.

The topological definition of equivariant Poincaré dual is the following. If $\eta \subset V$ is nice enough, the cycle $B\eta = \eta \times_G EG$ in $BV = V \times_G EG$ defines a homology class in $H_*(BV, \mathbb{Z})$, whose Poincaré dual is the equivariant dual of η , which lives in $H^*(BV, \mathbb{Z}) = H_G^*(V)$. However, this definition is as simple as impractical, and in concrete examples almost useless. In the algebraic case, when $\eta \subset V$ is a subvariety, the characteristic properties of the equivariant dual can serve as axioms, and we are going to use the axiomatic viewpoint as the definition of multidegree. They are also applicable for subschemes of V , extending the definition in the non-reduced case. The first purely algebraic definition (and the term multidegree) was introduced by Joseph.

This thesis focuses on the rather special case when $V = \{(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)\}$ is the vector space of map germs fixing the origo. Different parametrizations of \mathbb{C}^n and \mathbb{C}^k do not change the map, so we are concerned with $\text{Diff}_d(k) \times \text{Diff}_d(n)$ -invariant subsets of V under the left-right symmetry action $(A, B) \cdot f = AfB^{-1}$ for $(A, B) \in \text{Diff}_d(k) \times \text{Diff}_d(n)$, $f \in V$.

Left-right symmetry orbits in a wider sense are called *singularities*. They are invariant under the action of the Lie-subgroup $\text{GL}_k \times \text{GL}_n \subset \text{Diff}_d(k) \times \text{Diff}_d(n)$, and computing the multidegrees of singularities is a fundamental problem of global

singularity theory. They are also called *Thom polynomials* since the pioneering work of Thom in the 1950's.

The main difficulty of computation of Thom polynomials is the highly nontrivial symmetry group $\text{Diff}_d(k) \times \text{Diff}_d(n)$. By now there are three known efficient methods for calculating them. The first, classical way is the method of resolutions – this is also effective at computations of other invariants of singular varieties. The second, is based on an idea of R. Rimányi and called the method of restriction equations. A standard reference of this is [42], and a nice summary of this method and its applications is [27].

This thesis translates the problem from singularity theory and topology into (almost) pure algebra. Using an algebraic model of Gaffney [20] and Porteous [38] for A_n singularities allows us to use methods in algebraic geometry and algebraic topology, especially localisation methods for studying group actions and their quotient spaces. The heart of this thesis is to handle the problem of quotienting by the diffeomorphism group. Since this is a non-reductive group, the standard methods of Mumford's Geometric Invariant Theory are not applicable. Instead of that, we give a new compactification method by embedding the quotient into a compact Schubert variety, and then using abelian localisation of Berline and Vergne on the closure. Finally, this can be turned into an iterated residue formula.

Overview

We begin with a quick summary of the notions of global singularity theory and the theory of Thom polynomials. For a more detailed review we refer the reader to [1, 26].

Consider a holomorphic map $f : N \rightarrow K$ between two complex manifolds, of dimensions $n \leq k$. We say that $p \in N$ is a *singular* point of f (or f has a singularity at p) if the rank of the differential $df_p : T_p N \rightarrow T_{f(p)} K$ less than n .

Often, the topology of the situation forces f to have singularities at some points of N . To introduce a finer classification of singular points, choose local coordinates near $p \in N$ and $f(p) \in K$, and consider the resulting map-germ $\check{f}_p : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$, which may be thought of as a sequence of k power series in n variables without constant terms. The group of infinitesimal local coordinate changes $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ acts on the linear space $\mathcal{J}(n, k)$ of all such map-germs, and thus a reasonable notion of a *singularity* is a $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -orbit – or $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -invariant subset – in $\mathcal{J}(n, k)$. Given such orbit an $O \subset \mathcal{J}(n, k)$, for any holomorphic map $f : N \rightarrow K$ we can define the set

$$Z_O[f] = \{p \in N; \check{f}_p \in O\},$$

which is an analytic subvariety of N , independent of any coordinate choices. Assuming N is compact and f is sufficiently generic, this subvariety will have a Poincaré dual class $\alpha_O[f] \in H^*(N, \mathbb{Z})$; one of the fundamental problems of global singularity

theory is the computation of this class. This is indeed useful: for example, if we can prove that $\alpha_O[f]$ does not vanish, then we can guarantee that the singularity O occurs at some point of the map f .

A basic principle, introduced by René Thom (cf. [48]), is that to every singularity O one can associate a bivariant characteristic class τ_O , which, when evaluated on the pair (TN, f^*TK) produces the Poincaré dual class $\alpha_f[O]$, assuming that f is sufficiently generic. Originally, Thom defined this class in the parallel theory of smooth maps between real manifolds [48, 24]. One of the consequences of this principle is that the class $\alpha_O[f]$ depends only on the homotopy class of f .

Let us state this principle in more concrete terms. Denote by $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$ the space of those polynomials in the variables $(\lambda_1, \dots, \lambda_n, \theta_1, \dots, \theta_k)$ which are invariant under the permutations of the λ s and the permutations of the θ s. According to the structure theorem of symmetric polynomials $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$ itself is a polynomial ring in the elementary symmetric polynomials:

$$\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k} = \mathbb{C}[c_1(\boldsymbol{\lambda}), \dots, c_n(\boldsymbol{\lambda}), c_1(\boldsymbol{\theta}), \dots, c_k(\boldsymbol{\theta})].$$

By the Chern-Weil map, any polynomial $b \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$, and every pair of bundles (E, F) over N of ranks n and k , respectively, produces a characteristic class $b(E, F) \in H^*(N, \mathbb{C})$. Now Thom's principle reads:

*For every $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -orbit O of codimension j in $\mathcal{J}(n, k)$, there exists a homogeneous polynomial $\text{Tp}_O \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$ of degree j such that for an arbitrary, sufficiently generic map $f : N \rightarrow K$, the cycle $Z_f[O] \subset N$ is Poincaré dual to the characteristic class $\text{Tp}_O(TN, f^*TK)$.*

The statement may be formulated for more general $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -invariant subsets O as well; the corresponding polynomial Tp_O is called the *Thom polynomial* of O . We will see in Chapter 2 that Tp_O is the multidegree of O in $\mathcal{J}(n, k)$. The computation of these polynomials is a central problem of singularity theory.

The structure of the $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -action on $\mathcal{J}(n, k)$ is rather complicated; even the parametrization of the orbits is difficult. There is, however, a simple invariant on the space of orbits: to each map-germ $\check{f} : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$, we can associate the finite-dimensional nilpotent algebra $A_{\check{f}}$ defined as the quotient of the algebra of power series $\mathbb{C}[[x_1, \dots, x_n]]$ by the pull-back subalgebra $\check{f}^*(\mathbb{C}[[y_1, \dots, y_k]])$. This algebra $A_{\check{f}}$ is trivial if the map-germ \check{f} is non-singular, and it does not change along a $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -orbit, in other words it can be defined without choosing coordinates.

According to Thom's principle, this last observation means that to each finite-dimensional nilpotent algebra A , and pair of integers (n, k) one can associate a doubly symmetric polynomial $\text{Tp}_A^{n \rightarrow k} \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$; in the sense described above, this will serve as a universal Poincaré dual of points with nilpotent algebra A in the source spaces of holomorphic maps

Again, it turns out in Chapter 2 that $\text{Tp}_A^{n \rightarrow k}$ is the multidegree of the variety

$$\Sigma(A) = \{\check{f} \in \mathcal{J}(n, k); A_{\check{f}} \simeq A\}$$

in the complex vector space $\mathcal{J}(n, k)$.

The computation of this family of polynomials for general nilpotent algebras is a difficult problem. A few structural statements are known, however (cf. Chapter 2 for more details).

First, the polynomial $\mathrm{Tp}_A^{n \rightarrow k}$ lies in the subring of $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{S_n \times S_k}$ generated by the relative Chern classes (cf. [10, 40]) defined by the generating series

$$1 + c_1 q + c_2 q^2 + \dots = \frac{\prod_{j=1}^k (1 + \theta_j q)}{\prod_{i=1}^n (1 + \lambda_i q)}.$$

Next, the Thom polynomial, expressed in terms of these relative Chern classes, only depends on the codimension $j = k - n$. More precisely, there is a unique polynomial $\mathrm{TD}_A^c(c_1, c_2, \dots)$ such that

$$\mathrm{Tp}_A^{n \rightarrow k}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \mathrm{TD}_A^{k-n}(c_1(\boldsymbol{\lambda}, \boldsymbol{\theta}), c_2(\boldsymbol{\lambda}, \boldsymbol{\theta}), \dots),,$$

as long as the relevant subset in $\mathcal{J}_d(n, k)$ has the right codimension.

Finally (cf. [16]), performing the substitution $c_i \mapsto c_{i-1}$ in the homogeneous part of TD_A^c of maximal degree produces TD_A^{c-1} .

In this thesis, we will concentrate on the so-called Morin singularities [33], which correspond to the situation when the algebra A is generated by a single element. The list of these algebras is simple: $A_d = t\mathbb{C}[t]/t^{d+1}$, $d = 1, 2, \dots$

The main result of this thesis is a compact iterated residue formula for the the Thom polynomial $\mathrm{Tp}_{A_d}^{n \rightarrow k}$ for arbitrary d, n and k . For simplicity of notation we will denote this polynomial by $\mathrm{Tp}_d^{n \rightarrow k}$, or sometimes simply by Tp_d , omitting the dependence on the parameters n and k .

This problem has a rich history. The case $d = 1$ is the classical formula of Porteous: $\mathrm{Tp}_1 = c_{k-n+1}$. The Thom polynomial in the $d = 2$ case was computed by Ronga in [40]. More recently, in [4], the authors proposed a formula for Tp_3 ; P. Pragacz has given a sketch of a proof for this conjecture [39]. Finally, using his method of restriction equations, Rimányi was able to treat the zero-codimension case [43] (cf. [20] for the case $d = 4$): he could compute $\mathrm{Tp}_d^{n \rightarrow n}$ for $d \leq 8$.

Our approach combines the test-curve model of Porteous [38] and Gaffney [20] with localization techniques in equivariant cohomology [6, 45, 49].

The first observation is that the set $\Sigma(A_d)$ fibres equivariantly over a base space with respect to the action of a sub-torus of the diffeomorphism group. The base of this fibration is a non-compact quotient $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^n, \mathbb{C}^k) // H$ of the vector space of regular $k \times n$ matrices by a linear action of a non-reductive finite-dimensional subgroup H of the diffeomorphism group of \mathbb{C} . The main ingredient is a subtle compactification of this non-reductive quotient, which makes it possible to use a two-step localisation process:

1. First it is observed that $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^n, \mathbb{C}^k) // H$ fibres over the partial flag variety $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}^n, \mathbb{C}^k) // B$, where $B \subset \mathrm{GL}_n$ is a Borel containing H . We transform

the abelian localization on the flag manifold into a an iterated residue formula, allowing attention to be restricted to the fibre over one distinguished fixed flag under the torus action.

2. We employ abelian localization on a subtle compactification of the fibre over the distinguished flag.

The result is a residue formula which is a sum of terms indexed by the fixed points in the compactified distinguished fibre. However something unexpected happens: all but one of these terms contribute zero to the result, so that all the relevant information is stored at one fixed point.

We obtain a formula which reduces the computation of $\mathrm{Tp}_d^{n \rightarrow k}$ to a certain problem of commutative algebra which only depends on d , namely the computation of the multidegree of a Borel-orbit in a vector space. Computation of multidegrees for toric varieties is a well-understood task (see [47]), but much less is known for Borel orbits. This problem is trivial for $d = 1, 2, 3$, hence we instantly recover essentially all known results. We also compute it for $d = 4, 5, 6$ explicitly in Chapter 5, where we also compute some part of these polynomials in general. However, the entire polynomial is not known for $d \geq 7$.

Organization of the thesis

The thesis is structured as follows: we describe the basic setup and notions of singularity theory in §2.1, essentially repeating the above construction using more formal notation. Next, in §2.2 we recall the notion of equivariant Poincaré duals, which is a convenient language in which one can describe Thom polynomials. We also present the localization formulas of Berline-Vergne [6] and Rossmann [45], which are crucial to our computations later. In §3 we develop a calculus localizing equivariant Poincaré duals by combining the localization principles with Vergne’s integral formula for equivariant Poincaré duals. With these preparations, we proceed to describe the test curve model for Morin singularities in §4.1. The heart of our work is §4.2, where we reinterpret the model using a double fibration in a way which allows us to compactify our model space and apply the localization formulas. The following section, §4.3 is a rather straightforward application of the localization techniques of §2.2 to the double fibration constructed in §4.2. The resulting formula (4.3.27), in principle, reduces the computation of our Thom polynomials to a finite problem, but this formula is difficult to use for concrete calculations. Remarkably, however, the formula undergoes through several simplifications, which we explain in §4.4. At the end of §4.3, we summarize our constructions and results in a key diagram, which will hopefully orient the reader.

The simplifications bring us to our main result: Theorem 4.4.16 and formula (4.4.19). This formula is simple, but still contains an unknown quantity: a certain homogeneous polynomial \widehat{Q}_d in d variables, which does not depend on n and k . This

polynomial is the multidegree of the Borel-orbit $\widehat{\mathcal{O}}_d$ described in 4.4.16. The first few values of this polynomial are as follows:

$$\widehat{Q}_1 = \widehat{Q}_2 = \widehat{Q}_3 = 1, \quad \widehat{Q}_4(z_1, z_2, z_3, z_4) = 2z_1 + z_2 - z_4.$$

The computation of \widehat{Q}_d for general d is a finite but difficult problem. At the moment, we do not have an efficient algorithm for solving this problem. We discuss certain partial results in the final Chapter of this thesis; in §5.1 we give the computations for $d = 2, 3, 4, 5, 6$, and in 5.2 we prove for general d , that a certain part of \widehat{Q}_d coincides with the multidegree of the toric part of $\widehat{\mathcal{O}}_{d-1}$.

We end the thesis with an application of our theorem to positivity of Thom series. Rimanyi conjectured in [43] that the Thom polynomials Tp_d expressed in terms of relative Chern classes have positive coefficients. Our formalism suggests a stronger positivity conjecture; we formulate this conjecture in §5.3, and check it for the first few values of d .

Statement of Originality

Most of this thesis is based on common work with my supervisor András Szenes. We intend to present the results as a collaboration, and I have therefore based the structure of this thesis closely to the joint paper [3].

My own contribution is Chapter 5, the concrete formulas for Thom-polynomials for $d = 4, 5, 6$, and Section 5.2 with Theorem 5.2.13. This is entirely my own work.

Chapter 2

Global Singularity Theory and Multidegrees

2.1 Basic notions of singularity theory

2.1.1 The setup

We start with a brief introduction to singularity theory. We suggest [32],[1],[48] as references for the subject.

Let (e_1, \dots, e_n) be the basis of \mathbb{C}^n , and denote the corresponding coordinates by (x_1, \dots, x_n) . Introduce the notation $\mathcal{J}(n) = \{h \in \mathbb{C}[[x_1, \dots, x_n]]; h(0) = 0\}$ for the algebra of power series without a constant term, and let $\mathcal{J}_d(n)$ be the space of d -jets of holomorphic functions on \mathbb{C}^n near the origin, i.e. the quotient of $\mathcal{J}(n)$ by the ideal of those power series whose lowest order term is of degree at least $d + 1$. As a linear space, $\mathcal{J}_d(n)$ may be unidentified with polynomials on \mathbb{C}^n of degree at most d without a constant term.

In this thesis, we will call an algebra *nilpotent* if it is finite-dimensional, and there exists a positive integer N such that the product of any N elements of the algebra vanishes. Then the algebra $\mathcal{J}_d(n)$ is nilpotent, since $\mathcal{J}_d(n)^{d+1} = 0$.

Our basic object is $\mathcal{J}_d(n, k)$, the d -jets of holomorphic maps $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$. This is a finite-dimensional complex vector space, which one can identify $\mathcal{J}_d(n) \otimes \mathbb{C}^k$; hence $\dim \mathcal{J}_d(n, k) = k \binom{n+d}{d} - k$. We will sometimes call the elements of $\mathcal{J}_d(n, k)$ *map-germs of order d* , or simply map-germs. In this thesis we will always assume $n \leq k$.

One can compose d -jets of maps via substitution and elimination of terms of degree greater than d ; this leads to the composition maps

$$\mathcal{J}_d(n, k) \times \mathcal{J}_d(m, n) \rightarrow \mathcal{J}_d(m, k), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1. \quad (2.1.1)$$

The case of $d = 1$ reduces to multiplication of matrices. More generally, by taking the linear parts of jets, we obtain a map

$$\text{Lin} : \mathcal{J}_d(n, k) \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^k)$$

from the d -jets of maps into k -by- n matrices, which is compatible with the compositions (2.1.1).

Consider now the set

$$\text{Diff}_d(n) = \{\Delta \in \mathcal{J}_d(n, n); \text{Lin}(\Delta) \text{ invertible}\}.$$

It is an algebraic group with respect to the composition map (2.1.1) which has a faithful representation on $\mathcal{J}_d(n)$. Using the compositions (2.1.1) again, we obtain the so-called *left-right* action of the group $\text{Diff}_d(k) \times \text{Diff}_d(n)$ on $\mathcal{J}_d(n, k)$:

$$[(\Delta_L, \Delta_R), \Psi] \mapsto \Delta_L \circ \Psi \circ \Delta_R^{-1}$$

Note that the action of $\text{Diff}_d(n)$ is linear, while the action of $\text{Diff}_d(k)$ is not.

Singularity theory, in the sense that we are considering here, studies the left-right-invariant algebraic subsets of $\mathcal{J}_d(n, k)$. A natural way to form such subsets is as follows. Observe that to each element $\Psi = (P_1, \dots, P_k) \in \mathcal{J}_d(n, k)$, where $P_i \in \mathcal{J}_d(n)$ for $i = 1, \dots, k$, we can associate the quotient algebra $A_\Psi = \mathcal{J}_d(n)/I\langle P_1, \dots, P_k \rangle$ of the finite-dimensional algebra $\mathcal{J}_d(n)$ by the ideal generated by the elements of the sequence. This is the *nilpotent algebra*¹ of the map germ Ψ . For $\Psi = 0$ this nilpotent algebra is $\mathcal{J}_d(n)$, while for a generic germ, in fact, as soon as $\text{rank}[\text{Lin}(\Psi)] = n$, we have $A_\Psi = 0$.

Now let A be a nilpotent \mathbb{C} -algebra, as defined above. Consider the subset

$$\Theta_A = \{(P_1, \dots, P_k) \in \mathcal{J}_d(n, k); \mathcal{J}_d(n)/I\langle P_1, \dots, P_n \rangle \cong A\} \quad (2.1.2)$$

of the map germs of order d . Again, the dependence on the parameters d, n and k will be usually omitted. Let us collect some simple properties of Θ_A .

Proposition 2.1.1. (*[1]*) *Let A be a nilpotent algebra. Assume that $A^{d+1} = 0$ and $n \geq \dim(A/A^2)$.*

- *For k sufficiently large, Θ_A is a non-empty algebraic subvariety of $\mathcal{J}_d(n, k)$,*
- *The codimension Θ_A in $\mathcal{J}_d(n, k)$ equals $(k - n + 1) \dim(A)$; in particular, it does not depend on d and depends only on the difference $k - n$.*
- *Θ_A is $\text{Diff}_d(k) \times \text{Diff}_d(n)$ -invariant.*

A key observation is that although two germs with the same nilpotent algebra may be in different $\text{Diff}_d(k) \times \text{Diff}_d(n)$ orbits, there is a group acting on $\mathcal{J}_d(n, k)$ whose orbits are exactly the sets Θ_A for nilpotent algebras A . This group is defined as

$$\mathcal{K}_d(n, k) = \text{GL}_k(\mathbb{C} \oplus \mathcal{J}_d(n)) \rtimes \text{Diff}_d(n), \quad (2.1.3)$$

¹Instead of this algebra, it is customary to use the so-called *local algebra* of Ψ , which is simply the augmentation of A_Ψ by the constants \mathbb{C} .

where the algebra $\mathbb{C} \oplus \mathcal{J}_d(n)$ is the augmentation of $\mathcal{J}_d(n)$ by constants. The vector space $\mathcal{J}_d(n)$ is naturally a module over $\mathbb{C} \oplus \mathcal{J}_d(n)$, and hence $\mathcal{K}_d(n, k)$ acts on $\mathcal{J}_d(n, k)$ via

$$[(M, \Delta), \Psi] \mapsto M \cdot (\Psi \circ \Delta^{-1}),$$

where “ \cdot ” stands for matrix multiplication.

Remark 2.1.2. 1. Two germs in the same \mathcal{K}_d -orbit are called *contact equivalent*, or \mathcal{K} -equivalent (cf. [1]). The term V -equivalence ([29]) is also used.

2. It is not difficult to check that that a left-right orbit of a map germ is always contained in its \mathcal{K}_d -orbit.

3. The group $\mathrm{GL}_k(\mathbb{C} \oplus \mathcal{J}_d(n))$ may be thought of as the group of order- d map germs $\mathbb{C}^n \rightarrow \mathrm{GL}(\mathbb{C}^k)$ at the origin $0 \in \mathbb{C}^n$. One arrives at the same notion of equivalence on $\mathcal{J}_d(n, k)$ if one considers the action of the much larger group of map germs $\mathbb{C}^n \rightarrow \mathrm{Diff}_d(k)$ in the definition (cf. [1]).

Proposition 2.1.3. [31],[32],[1] *Two map germs in $\mathcal{J}_d(n, k)$ have the same nilpotent algebra if and only if they are in the same \mathcal{K}_d -orbit.*

The group \mathcal{K}_d is connected, hence we have the following

Corollary 2.1.4. *For a nilpotent algebra A , the set Θ_A is an irreducible subvariety of $\mathcal{J}_d(n, k)$.*

Finally, we describe the relationship between $\mathcal{K}_d(n, k)$ -orbits and $\mathrm{Diff}(\mathbb{C}^k) \times \mathrm{Diff}(\mathbb{C}^n)$ -orbits in $\mathcal{J}_d(n, k)$. Roughly, the statement is that for fixed A and sufficiently large n , there is a dense left-right orbit in Θ_A .

Let r be a non-negative integer. An *unfolding* of a map germ $\Psi \in \mathcal{J}_d(n, k)$ is a map germ $\widehat{\Psi} \in \mathcal{J}_d(n+r, k+r)$ of the form

$$(x_1, \dots, x_n, y_1, \dots, y_r) \mapsto (F(x_1, \dots, x_n, y_1, \dots, y_r), y_1, \dots, y_r)$$

where $F \in \mathcal{J}_d(n+r, k)$ is an r -dimensional deformation of Ψ , i.e satisfies

$$F(x_1, \dots, x_n, 0, \dots, 0) = \Psi(x_1, \dots, x_n).$$

The *trivial unfolding* is the map germ

$$(x_1, \dots, x_n, y_1, \dots, y_r) \mapsto (\Psi(x_1, \dots, x_n), y_1, \dots, y_r).$$

Definition 2.1.5. [1],[32] A map germ $\Psi \in \mathcal{J}_d(n, k)$ is *stable* if all unfoldings of Ψ are left-right equivalent to the trivial unfolding.

Informally, a germ of a holomorphic map $f : N \rightarrow K$ of complex manifolds at a point $x \in N$ is *stable*, if for any small deformation \tilde{f} there is a point in the vicinity of x at which the germ of \tilde{f} is left-right equivalent to the germ of f at x .

Now we can formulate the relationship between contact and left-right orbits precisely.

Proposition 2.1.6. [1],[32]

1. If $\widehat{\Psi}$ is an unfolding of Ψ , then $A_{\widehat{\Psi}} \cong A_{\Psi}$.
2. Every map germ has a stable unfolding.
3. If a map germ is stable, then its left-right orbit is dense in its contact orbit.

The topology of contact singularities seems to be more tractable (cf. Proposition 2.2.9), hence in this thesis we will focus on them; stability will not play a role.

2.1.2 Morin singularities

In the previous paragraph, we introduced a singularity class, i.e. a family of $\text{Diff}(\mathbb{C}^k) \times \text{Diff}(\mathbb{C}^n)$ -invariant subsets $\Theta_A[n, k] \subset \mathcal{J}_d(n, k)$ for each nilpotent algebra A . The focus of the present thesis is the study of the topological invariants of these classes in the specific case of the algebra

$$A_d = t\mathbb{C}[t]/t^{d+1}.$$

The corresponding singularity class is called the A_d -singularity or *Morin-singularity* [1],[33]. We introduce the simplified notation

$$\Theta_d[n, k] \text{ instead of } \Theta_{A_d}[n, k],$$

and we will omit the parameters n and k this causes no confusion.

Let us specialize to this algebra the setup we introduced the previous paragraph. We have

- $(A_d)^{d+1} = 0$, hence we can work in $\mathcal{J}_d(n, k)$.
- The variety $\Theta_d[n, k]$ is non-empty for any $n \leq k$. For $n = k = 1$ we simply have $\Theta_d[1, 1] = \{0\}$, the constant zero germ in $\mathcal{J}_d 1, 1$. This germ is not stable.
- There are stable map germs in $\mathcal{J}_d(n, k)$ with nilpotent algebra A_d , whenever $n \geq d$. An example in $\mathcal{J}_N(d, d)$ for $N \geq d$ with minimal source dimension $n = d$ is

$$(x_1, \dots, x_d) \mapsto (x_d^{d+1} + x_1 x_d^{d-1} + x_2 x_d^{d-2} + \dots + x_{d-1} x_d, x_1, \dots, x_{d-1}). \quad (2.1.4)$$

Finally, we recall that the A_d -singularities fit into a wider family of the so-called *Thom-Boardman* singularity classes. ([7],[1]). In general, a Thom-Boardman class is determined by a non-increasing sequence of positive integers $i_1 \geq \dots \geq i_d$. For a general sequence, this class is the union of several \mathcal{K}_d -orbits, but the special values $i_1 = \dots = i_d = 1$, the Thom-Boardman class contains exactly those maps with local algebra isomorphic to A_d .

The description of Θ_d as a Thom-Boardman class is rather different from (2.1.2); we provide it below for reference.

Observe that

- eliminating the terms of degree d results in an algebra homomorphism $\pi_{d \rightarrow d-1} I : \mathcal{J}_d(n) \rightarrow \mathcal{J}_{d-1}(n)$, and
- partial differentiation $f \rightarrow \partial f / \partial x_j$ is a well-defined map $\mathcal{J}_d(n) \rightarrow \mathcal{J}_{d-1}(n)$ for $j = 1, \dots, n$.

Define the following operation on ideals of $\mathcal{J}_d(n)$.

Definition 2.1.7. Given a proper ideal I in the algebra $\mathcal{J}_d(n)$, define δI to be the ideal of $\mathcal{J}_{d-1}(n)$ generated by $\pi_{d \rightarrow d-1} I$ together with the truncated determinants of the n -by- n matrices of the form

$$\det \left(\frac{\partial Q_i}{\partial x_j} \right)_{i,j=1}^n \in \mathcal{J}_{d-1}(n),$$

with arbitrary $Q_1, \dots, Q_n \in I$.

Proposition 2.1.8. Denoting by $I\langle P_1, \dots, P_k \rangle$ the ideal in $\mathcal{J}_d(n)$ generated by the elements P_1, \dots, P_k , we have

$$\Theta_d = \{(P_1, \dots, P_k) \in \mathcal{J}_d(n, k); \text{codim}(\delta^{d-1} I\langle P_1, \dots, P_k \rangle \subset \mathcal{J}_1(n)) = 1\}. \quad (2.1.5)$$

Remark 2.1.9. Relaxing the condition for $\text{codim}(\delta^{d-1} I\langle P_1, \dots, P_k \rangle)$ we obtain the Zariski closure

$$\bar{\Theta}_d = \{(P_1, \dots, P_k) \in \mathcal{J}_d(n, k); \delta^{d-1} I\langle P_1, \dots, P_k \rangle \neq \mathcal{J}_1(n)\} \subset \mathcal{J}_d(n, k),$$

which contains all other Boardman classes $\Sigma^{i,j,\dots}$ with $i \geq 2$ (cf. [1],[7]).

2.2 Equivariant Poincaré duals And Thom polynomials

Let T be a complexified torus: $T \cong (\mathbb{C}^*)^r$. The *equivariant Poincaré dual* is an invariant associated to algebraic or analytic T -invariant subvarieties of T -representations; this invariant takes values in homogeneous polynomials on the Lie algebra $\text{Lie}(T)$ of T . The central objects of the present work, Thom polynomials, are special cases of equivariant Poincaré duals (cf. [43],[26]). We review the definitions and properties of equivariant Poincaré duals in some detail here in order to prepare ourselves for the localization formulas of the next section.

The equivariant Poincaré dual has appeared in the literature in several guises: as Joseph polynomial, equivariant multiplicity, multidegree, etc. One of the first definitions was given by Joseph [25], who introduced it as the polynomial governing the asymptotic behavior of the character of the algebra of functions on the subvariety. Rossmann in [45] defined this invariant for analytic subvarieties via an integral-limit

representation, and then used it to write down a very general localization formula for equivariant integrals. This formula will play a central role in our computations.

We start with an axiomatic algebraic definition, following the treatment of [35]; this will provide us with some useful computational tools. After studying an example, we turn to the analytic picture. We first give an overview of Rossmann's localization formula, in which the equivariant Poincaré dual plays a central role, then we describe Vergne's integral representation, which places the equivariant Poincaré dual into the proper context of equivariant cohomology. Finally, in the last two paragraphs, we link the equivariant Poincaré dual to Thom polynomials. This allows us to formulate our problem precisely.

2.2.1 Axiomatic definition

Let W be a complex vector space of dimension N with an action of the complex torus $T = (\mathbb{C}^*)^r$. Denote by $\lambda_1, \dots, \lambda_r$ the standard integral coordinates on the dual of the Lie algebra of T . One can choose coordinates y_1, \dots, y_N on W in such a way that the action in the dual basis be diagonal with weights η_i , $i = 1, \dots, N$; each of these weights is an integral linear combination of the λ s.

We will work with the equivariant cohomology ring $H_T^*(W, \mathbb{C})$ of W ; as W is contractible, $H_T^*(W, \mathbb{C})$ may be identified with the polynomial ring $\mathbb{C}[\lambda_1, \dots, \lambda_r]$.

Let $A \subset \mathbb{Z}^r$ denote the subgroup generated by the weights η_i . We can think of $S = \mathbb{C}[y_1, \dots, y_N]$ as a multigraded polynomial ring, graded by A . Let $\Sigma \subset W$ be a T -invariant closed affine subscheme, $\Sigma = \text{Spec}(M(\Sigma))$, where $M(\Sigma)$ is a finitely generated graded S -module. One can associate to this scheme the *multidegree*

$$\text{mdeg}[(\Sigma, W)] = \text{mdeg}[M(\Sigma)] \in H_T^*(W, \mathbb{C}) \cong \mathbb{C}[\lambda_1, \dots, \lambda_r].$$

The polynomial $\text{mdeg}[\Sigma, W]$ is homogeneous, and has degree equal to the codimension of Σ in W .

We follow [35], sect. 8.5 in giving the characteristic properties of multidegrees, which serve as axioms in our case.

Let $\Sigma_1, \Sigma_2, \dots, \Sigma_c$ denote the maximal dimensional reduced components of Σ . These correspond to the maximal dimensional associated primes in the primary decomposition of the ideal of Σ , see [13], sect. II.3.3. The multiplicity of the component Σ_i is $\text{mult}(\Sigma_i)$, $i = 1, \dots, c$. If $\Sigma = \text{Spec}(M(\Sigma))$, and $\Sigma_i = \text{Spec}(S/\mathfrak{p}_i)$ corresponds to the associated prime \mathfrak{p}_i , then, by definition, $\text{mult}(\Sigma_i)$ equals to $\text{mult}_{\mathfrak{p}_i}(S/\mathfrak{p}_i)$, the length of the largest finite-length submodule in the localization of $M(\Sigma)$ at \mathfrak{p}_i .

We need one more ingredient to give the axioms of the multidegree, namely a special flat deformation of Σ called Groebner degeneration. This corresponds to a fixed monomial order on S . If $M(\Sigma) = S/I(\Sigma)$, one defines the initial ideal $\text{in}(I(\Sigma))$ as the monomial ideal generated by the first terms of the polynomials in $I(\Sigma)$ with respect to the given order. We call $\text{Groeb}(\Sigma) = \text{Spec}(S/\text{in}(I(\Sigma)))$ the Groebner degeneration of Σ with respect to this order.

The axioms are the following. The multidegree $\text{mdeg}[\cdot, W]$ is

additive

$$\text{mdeg}[\Sigma, W] = \sum_{i=1}^c \text{mult}(\Sigma_i) \cdot \text{mdeg}[\Sigma_i, W],$$

degenerative

$$\text{mdeg}[\Sigma, W] = \text{mdeg}[\text{Groeb}(\Sigma), W]$$

with respect to any monomial order on S .

normalized For T -invariant linear subspaces of W the invariant is defined to equal to the product of weights in the normal direction, i.e

$$\text{eP}(\{w \in W; y_i(w) = 0, i \in I\}, W) = \prod_{i \in I} \eta_i. \quad (2.2.1)$$

Remark 2.2.1. Writing the modules instead of the schemes into the brackets we can also write for the S -module $M = S/I$ with maximal associated primes $\mathfrak{p}_1, \dots, \mathfrak{p}_c$

- $\text{mdeg}[M] = \sum_{i=1}^c \text{mult}_{\mathfrak{p}_i}(M) \cdot \text{mdeg}[S/\mathfrak{p}_i]$,
- $\text{mdeg}[M] = \text{mdeg}[\text{in}(M)]$
- $\text{mdeg}[S/\langle y_j : j \in J \rangle] = \prod_{j \in J} \eta_j$.

These axioms determine $\text{eP}[\Sigma, W]$ uniquely (see e.g. [35], sect. 8.5, Theorem 8.44), namely

$$\begin{aligned} \text{mdeg}[\Sigma, W] &= \sum_{i=1}^c \text{mult}(\Sigma_i) \cdot \text{mdeg}[\Sigma_i, W] = \\ &= \sum_{i=1}^c \text{mult}(\Sigma_i) \cdot \text{mdeg}[\text{Groeb}(\Sigma_i), W] \end{aligned} \quad (2.2.2)$$

Since $\text{Groeb}(\Sigma_i) = \text{Spec}(S/\text{in}(\mathfrak{p}_i))$ where $\text{in}(\mathfrak{p}_i)$ is generated by monomials, $\text{Groeb}(\Sigma_i)$ is the union of T -invariant linear subspaces of W ; the normalizing axiom then defines their equivariant Poincar dual.

Conventions and Notations:

1. The natural extension of multidegrees for not necessarily closed varieties is coming from topology, see Definition 2.2.4. If $\Sigma \subset W$ is a subvariety, we define

$$\text{mdeg}[\Sigma, W] = \text{mdeg}[\overline{\Sigma}, W] \quad (2.2.3)$$

2. To simplify our notation, we will omit the vector space W from the notation whenever this does not cause confusion.

3. The simplest example of (2.2.1) is the case of the point: $\Sigma = \{0\}$. We will often use the notation $\text{Euler}^T(W)$ for $\text{mdeg}[\{0\}, W]$, since, indeed, this is the equivariant Euler class of W thought of as a T -vector bundle over a point. We have thus

$$\text{Euler}^T(W) = \prod_{i=1}^N \eta_i. \quad (2.2.4)$$

Lemma 2.2.2. *Assume that, with respect to coordinates (y_1, \dots, y_N) , the complexified torus T acts on W diagonally with weights η_1, \dots, η_N . Let $\Sigma = \text{Spec}(M(\Sigma)) \subset W$ be a closed T -invariant subvariety with $M(\Sigma) = \mathbb{C}[y_1, \dots, y_N]/I(\Sigma)$. Then if for some l , $1 \leq l \leq N$ there is a relation $R \in I(\Sigma)$ such that the coefficient of the monomial y_l in R is nonzero, then $\text{mdeg}[\Sigma]$ is divisible by η_l .*

Proof. One can easily define a monomial order on S such that the initial term of R is y_l : give weights -2 to y_l , and the weight -1 to y_i , $i \neq l$. (see [14], sect. 15.2). Therefore all the associated maximal prime monomial ideals of Σ contain y_l , which means that $\text{Groeb}(\Sigma_i)$ built up from linear subspaces contained in the hyperplane $y_l = 0$. By the additive and normalized properties η_l divides $\text{mdeg}[\text{Groeb}(\Sigma_i, W)]$ for all $i = 1, \dots, c$ in (2.2.2), so Lemma 2.2.2 is proved. \square

Remark 2.2.3. The geometric interpretation of Lemma 2.2.2 is the following. Let $\pi_i : W \rightarrow H_i$ denote the projection onto the hyperplane $H_i = \{y_i = 0\}$. Then, under the conditions of Lemma 2.2.2,

$$\text{mdeg}[\Sigma, W] = \eta_i \cdot \text{eP}[\pi_i(\Sigma), H_i]$$

When Σ has no multiple, embedded or lower dimensional components we can use equivariant integration to compute the multidegree, see section 2.2.3. We introduce the term *equivariant Poincaré dual* for this case.

Definition 2.2.4. If $\Sigma \subset W$ is a reduced scheme of pure dimension we call $\text{mdeg}[\Sigma, W]$ the equivariant Poincaré dual of Σ in W , and use the notation $\text{eP}[\Sigma, W]$.

Remark 2.2.5. The term comes from the topological description of $\text{eP}[\Sigma, W]$, namely, this is the ordinary Poincaré dual of $EG \times_G \Sigma$ in $EG \times_G W$.

2.2.2 The basic example

A simple way to construct T -invariant subvarieties of W is to take the closure of the orbit $T \cdot p$ of a generic point $p \in W$. Because of deformation invariance, $\text{eP}[\overline{T \cdot p}]$ does not depend on the choice of p , hence this polynomial is an invariant of the T -action on W . Nevertheless, to compute this polynomial in general is rather difficult.

Consider the following example: take $W = \mathbb{C}^4$ endowed with a diagonal $T = (\mathbb{C}^*)^3$ -action, whose weights η_1, η_2, η_3 and η_4 span $\text{Lie}(T)^*$, and satisfy $\eta_1 + \eta_3 = \eta_2 + \eta_4$. In other words, the four weights, η_i , $i = 1, \dots, 4$, form the vertices of a

parallelogram in $\text{Lie}(T)^*$ lying in a hyperplane which does not pass through the origin. Choose $p = (1, 1, 1, 1) \in W$; then the closure of the T -orbit of p is given by a single equation:

$$\Sigma = \overline{T \cdot p} = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4; x_1x_3 = x_2x_4\}. \quad (2.2.5)$$

We will compute the equivariant Poincaré dual of this subvariety in a number of ways (cf. [15]).

First method: Groebner degeneration. Use the lexicographic monomial order induced by the order $x_1 > x_2 > x_3 > x_4$ on the coordinates to define the initial ideal of $I(\Sigma) = \langle x_1x_3 - x_2x_4 \rangle \triangleleft \mathbb{C}[x_1, x_2, x_3, x_4]$:

$$\text{in}(I(\Sigma)) = \langle x_1x_3 \rangle \triangleleft \mathbb{C}[x_1, x_2, x_3, x_4]. \quad (2.2.6)$$

Since

$$\text{Groeb}(\Sigma) = \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]/\langle x_1x_3 \rangle) \quad (2.2.7)$$

is the union of two hyperplanes: $\{x_1 = 0\}$ and $\{x_3 = 0\}$, using the additivity and the normalization axioms, we arrive at the result that the equivariant Poincaré dual is $\text{eP}[\Sigma] = \eta_1 + \eta_3 = \eta_2 + \eta_4$. Thus we have

$$\text{eP}[T \cdot p] = \eta_1 + \eta_3. \quad (2.2.8)$$

Second method: complete intersections. The normalization axiom may be reformulated as follows: given a surjective equivariant linear map $\gamma : W \rightarrow E$ from W to another T -module E , we have

$$\text{eP}[\gamma^{-1}(0)] = \text{Euler}^T(E). \quad (2.2.9)$$

In fact, this formula holds even in the case when γ is an equivariant *polynomial* map whose derivative $d\gamma : T_pW \rightarrow E$ at a generic point $p \in \gamma^{-1}(0)$ is surjective, i.e. when the subvariety $\gamma^{-1}(0) \subset W$ is a *complete intersection*. This easily follows from the original definition of Joseph via the explicit resolution of the ideal sheaf of $\gamma^{-1}(0)$ ([25, 45]), or using a deformation argument.

Any hypersurface is a complete intersection; in particular, our orbit is given by a single equation of weight $\eta_1 + \eta_3$: $x_1x_3 - x_2x_4 = 0$, and thus we recover (2.2.8).

2.2.3 Integration and equivariant multiplicities

The notion of equivariant Poincaré dual may be extended to the case of analytic T -invariant varieties defined in a neighborhood of the origin in a T -representations. As observed by Rossmann in [45], this setup may be further generalized to the nonlinear situation in the following sense.

Let Z be a complex manifold with a holomorphic T -action, and let $M \subset Z$ be a T -invariant analytic subvariety with an isolated fixed point $p \in M^T$. Then one can find local analytic coordinates near p , in which the action is linear and diagonal.

Using these coordinates, one can identify a neighborhood of the origin in $T_p Z$ with a neighborhood of p in Z . We denote by $\hat{T}_p M$ the part of $T_p Z$ which corresponds to M under this identification; informally, we will call $\hat{T}_p M$ the T -invariant *tangent cone* of M at p . This tangent cone is not quite canonical: it depends on the choice of coordinates, however, the equivariant Poincaré dual of $\Sigma = \hat{T}_p M$ in $W = T_p Z$ does not. Rossmann named this equivariant Poincaré dual the *equivariant multiplicity of M in Z at p* :

$$\text{emult}_p[M, Z] \stackrel{\text{def}}{=} \text{eP}[\hat{T}_p M, T_p Z]. \quad (2.2.10)$$

In the algebraic framework one might need to pass to the *tangent scheme* of M at p (cf. [17]). This is canonically defined, but we will not need this notion.

An important application of the equivariant multiplicity is Rossmann's localization formula [45]. Let $\mu : \text{Lie}(T) \rightarrow \Omega^\bullet(Z)$ be a holomorphic equivariant map with values in smooth differential forms on Z . (The reader will find the necessary background material about equivariant differential forms and equivariant integration in [22, 5].) Then *Rossmann's localization formula* says that

$$\int_M \mu = \sum_{p \in M^T} \frac{\text{emult}_p[M, Z]}{\text{Euler}^T(T_p Z)} \cdot \mu^{[0]}(p), \quad (2.2.11)$$

where $\mu^{[0]}(p)$ is the differential-form-degree-zero component of μ evaluated at p . Recall that $\text{Euler}^T(T_p Z)$ stands for the product of the weights of the T -action on $T_p Z$.

This formula generalizes the equivariant integration formula of Berline and Vergne [6], which applies when M is smooth. In this case the tangent cone of M at p is a well-defined linear subspace of $T_p Z$, and $\text{emult}_p[M]$ is the ePd of this subspace. Then the fraction in (2.2.11) simplifies, the ambient space Z is eliminated from the picture, and one arrives at (cf. [6]).

$$\int_M \mu = \sum_{p \in M^T} \frac{\mu^{[0]}(p)}{\text{Euler}^T(T_p M)}. \quad (2.2.12)$$

Rossmann proves (2.2.11) by writing down a local integral-limit formula for the equivariant multiplicity, and then applying an adaptation of Stokes theorem, following the method of Bott [8]. As showed by Vergne [49], such a local integration formula for equivariant Poincaré duals may be given in the framework of equivariant cohomology. To describe this formula, we return to our setup of a T -invariant subvariety Σ in a vector space W . The starting point is the Thom isomorphism in equivariant cohomology:

$$H_{T, \text{cpt}}^*(W) = H_T^*(W) \cdot \text{Thom}(W),$$

which presents compactly supported equivariant cohomology as a module over usual equivariant cohomology. The class $\text{Thom}(W) \in H_{T, \text{cpt}}^{\dim W}(W)$ may be represented by

an explicit equivariant differential form with compact support (cf. [30, 12]). Then one simply has (cf. [49])

$$eP[\Sigma] = \int_{\Sigma} \text{Thom}(W). \quad (2.2.13)$$

Remarkably, this formula turns things upside down, and describes $eP[\Sigma]$ as an integral in equivariant cohomology. As we explain in the next section, this allows us to *localize* the equivariant Poincaré dual.

We complete this review by noting that a consequence of (2.2.13) is the following formula. For an equivariantly closed differential form μ with compact support, we have

$$\int_{\Sigma} \mu = \int_W eP[\Sigma] \cdot \mu.$$

This formula serves as the motivation for the term *equivariant Poincaré dual*.

2.2.4 GL_n -actions

In case the torus action extends to the action of the general linear group, we have the following statement.

Lemma 2.2.6. *Let $T = (\mathbb{C}^*)^n$ be the subgroup of diagonal matrices of the complex group GL_n , and denote by $\lambda_1, \dots, \lambda_n$ its basic weights. If Σ is a GL_n -invariant subvariety of the GL_n -module W , then the equivariant Poincaré dual $eP[\Sigma]$ is a symmetric polynomial in $\lambda_1, \dots, \lambda_n$.*

Indeed, this follows from the fact that the Weyl group of GL_n , which is the quotient of the normalizer $N(T)$ by T , acts on the basic weights as the full permutation group.

Now we formulate a topological property of the equivariant Poincaré dual, which is the reason why it is important in singularity theory. Informally, the polynomial $eP[\Sigma]$ measures the *topological likelihood* of a random point in W to land in Σ .

Indeed, let F be a principal GL_n -bundle over a compact complex manifold M . Then, using the Chern-Weil map, any symmetric polynomial $P \in \mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n}$ defines a characteristic class $P(F) \in H^*(M, \mathbb{C})$. Now let Σ be GL_n -invariant closed subvariety of the GL_n -module W , and, denote by W_F the associated vector bundle $F \times_{GL_n} W$ over M , and by Σ_F the subset of W_F corresponding to Σ .

$$\begin{array}{ccc}
 F \times_{GL_n} W = & W_F \longleftarrow \Sigma_F & = F \times_{GL_n} \Sigma \\
 & \uparrow \quad \downarrow & \nearrow \\
 & s \quad \downarrow & \\
 & M &
 \end{array} \quad (2.2.14)$$

Proposition 2.2.7 ([1],[42]). *For a sufficiently generic section $s : M \rightarrow W_F$, the cycle $s^{-1}(\Sigma_F) \subset M$ is Poincaré dual to the characteristic class $eP[\Sigma](F)$ of F corresponding to the symmetric polynomial $eP[\Sigma]$. If this characteristic class is nonzero, then $s^{-1}(\Sigma_F)$ is non-empty for any section s .*

A sufficiently generic section satisfies the following transversality property:

Definition 2.2.8. Let $f : Z \rightarrow X$ be an algebraic map between smooth algebraic varieties, and $Y \subset X$ be a subvariety of codimension d , $Y^0 = Y \setminus \text{Sing}(Y)$ be the complement of the singular locus. The map f is transversal to Y if it is transversal to the smooth manifold Y^0 , $f^{-1}Y$ has also complex dimension d , and $\overline{f^{-1}Y^0} = f^{-1}Y$.

If $f : Z \rightarrow X$ is transversal to $Y \subset X$, then the pull-back of the cohomology class represented by $Y \subset X$ is equal to the cohomology class represented by $f^{-1}(Y) \subset Z$. By a sufficiently generic section in Proposition 2.2.7 we mean a map $s : M \rightarrow W_F$ transversal to $\Sigma_F \subset W_F$.

2.2.5 Thom polynomials and the formulation of the problem

We can specialize the constructions of this section to the setup of global singularity theory detailed in §2.1. Observe that the quotient map $\text{Lin} : \text{Diff}_d(n) \rightarrow \text{GL}_n$ has a canonical section, consisting of linear substitutions. In other words we have a canonical group embedding

$$\text{GL}_n \hookrightarrow \text{Diff}_d(n),$$

and we can restrict the action of the diffeomorphism groups $\text{Diff}_d(k) \times \text{Diff}_d(n)$ on $\mathcal{J}_d(n, k)$ to the group $\text{GL}_k \times \text{GL}_n$.

Then Θ_A is a $\text{GL}_k \times \text{GL}_n$ -invariant subvariety of the space of map germs $\mathcal{J}_d(n, k)$, hence we can associate to $\Theta_A \subset \mathcal{J}_d(n, k)$ a polynomial $eP[\Theta_A, \mathcal{J}_d(n, k)]$ in the two sets of variables

$$\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n), \quad \boldsymbol{\theta} = (\theta_1, \dots, \theta_k), \quad (2.2.15)$$

corresponding the basic weights of the group of diagonal matrices in GL_n and GL_k , respectively. Formulate the statement that the Thom polynomial gives us the Poincaré dual of the appropriate cycle. Need some kind of proof.

In this setup, Proposition 2.2.7 has the following interpretation ([48, 24]). Let A be a nilpotent algebra satisfying $A^{d+1} = 0$, and let $f : N \rightarrow K$ be a sufficiently generic holomorphic map from the compact complex manifold N of dimension n to the complex manifold K of dimension k . This determines a section of the d -jet bundle $J_d(n, k)$ over N . The fiber over $p \in N$ of $J_d(n, k)$ is the vector space $\mathcal{J}_d(n, k)$, with structure group $\text{Diff}_d(k) \times \text{Diff}_d(n)$, and the map f determines a section s_f as $s_f(p) = \check{f}_p$.

The d -jet bundle over N is the pull-back of the universal d -jet bundle associated to the universal principal $\text{Diff}_d(k) \times \text{Diff}_d(n)$ -bundle. Moreover, the quotient map

$\text{Diff} = \text{Diff}_d(k) \times \text{Diff}_d(n) \rightarrow \text{GL} = \text{GL}_k \times \text{GL}_n$ induces a map $B\text{Diff} = B\text{Diff}_d(k) \times B\text{Diff}_d(n) \rightarrow B_{\text{GL}} = B\text{GL}_k \times B\text{GL}_n$, so the picture is the following

$$\begin{array}{ccccc}
J(n, k) & \longrightarrow & E\text{Diff} \times_{\text{Diff}} \mathcal{J}_d(n, k) & & E(\text{Diff}) \times_{\text{GL}} \mathcal{J}_d(n, k) \\
\downarrow & \uparrow s_f & \downarrow & \uparrow s_{\text{Diff}} & \downarrow & \uparrow s_{\text{GL}} \\
N & \xrightarrow{\chi} & B\text{Diff} & \xrightarrow{\omega} & B_{\text{GL}}
\end{array} \tag{2.2.16}$$

Since $\Theta_d \subset \mathcal{J}_d(n, k)$, we can define the following associated subsets:

$$\Theta_{\text{Diff}} = E\text{Diff} \times_{\text{Diff}} \Theta_d,$$

$$\Theta_{\text{gl}} = E\text{Diff} \times_{\text{GL}} \Theta_d,$$

$$\Theta_N = \{(p, g) \in J(n, k); g \in \mathcal{J}_d(n, k)\}.$$

Let s_{Diff} and s_{GL} be generic sections of the second and third bundles, s_{ω} a generic section of $\omega^*(E\text{Diff} \times_{\text{GL}} \mathcal{J}_d(n, k))$, and $\Theta_{\omega} = \omega^*(\Theta_{\text{GL}})$. Since $\text{GL}_k \times \text{GL}_n$ and Diff are homotopic equivalent as topological groups, $s_{\omega}^{-1}(\Theta_{\omega})$ and $s_{\text{Diff}}^{-1}(\Theta_{\text{Diff}})$ define the same homology cycle in $B\text{Diff}$. The same is true for $s_f^{-1}(\Theta_N)$ and $\chi^*[s_{\text{Diff}}^{-1}(\Theta_{\text{Diff}})]$ in $H^*(N)$, so

$$[s_f^{-1}(\Theta_N)] = \chi^*[(s_{\text{Diff}})^{-1}(\Theta_{\text{Diff}})] = \chi^*\omega^*[s_{\text{GL}}^{-1}(\Theta_{\text{GL}})] = \chi^*\omega^*(e\text{P}[\Theta_d]). \tag{2.2.17}$$

Since B_{GL} and $B\text{Diff}$ are homotopic equivalent, $H_{\text{GL}}^* = H_{\text{Diff}}^*$, and (2.2.17) explains the connection to the dual of the cycle $Z_{\Theta_d}[f] \in N$.

Assume that the characteristic class of the pair of bundles (TN, f^*TK) over N corresponding to the bisymmetric polynomial $e\text{P}[\Theta_A]$ is not zero in $H^*(N)$. Then the variety of points $p \in N$ where the singularity A occurs is Poincaré dual to this characteristic class. This was, in fact, the definition of the Thom polynomial, and hence, using the notation introduced in the Overview in the beginning of this thesis.

$$\text{Tp}_A^{n \rightarrow k} = e\text{P}[\Theta_A, \mathcal{J}_d(n, k)], \tag{2.2.18}$$

which we will take as the definition of the Thom polynomial.

One of the natural questions to ask is how the Thom polynomials for fixed A and different pairs (n, k) are related. Denote the ring of bisymmetric polynomials in the λ s and θ s by $\mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$. We collect the known facts [1, 10, 16] in Proposition 2.2.9 below. For simplicity, we will formulate the statements for the algebra $A_d = t\mathbb{C}[t]/t^{d+1}$ we study, although essentially the same properties are satisfied by the Thom polynomials of any other contact singularity (see [16] for details).

Recall from § 2.1, that for $1 \leq d$ and $1 \leq n \leq k$, $\Theta_d = \Theta_d[n, k]$ is a non-empty subvariety of $\mathcal{J}_d(n, k)$ of codimension $d(k - n + 1)$. Consider the infinite sequence

of homogeneous polynomials $c_i \in \mathbb{C}[\boldsymbol{\lambda}, \boldsymbol{\theta}]^{\mathcal{S}_n \times \mathcal{S}_k}$, $\deg c_i = i$, defined by the generating series

$$\text{RC}(q) = 1 + c_1 q + c_2 q^2 + \cdots = \frac{\prod_{m=1}^k (1 + \theta_m q)}{\prod_{l=1}^n (1 + \lambda_l q)}; \quad (2.2.19)$$

we will call c_i the i th *relative Chern class*.

Proposition 2.2.9 ([48, 10, 16]). *Let $1 \leq d$ and $1 \leq n \leq k$. Then for each non-negative integer j , there is a polynomial $\text{TD}_d^j(b_0, b_1, b_2, \dots)$ in the indeterminates b_0, b_1, b_2, \dots with the following properties*

1. TD_d^j is homogeneous of degree d , and
2. if we set $\deg(b_i) = i$, then TD_d^j is homogeneous of degree $d(k - n + 1)$;
- 3.

$$\text{Tp}_d^{n \rightarrow k}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \text{TD}_d^{k-n}(1, c_1(\boldsymbol{\lambda}, \boldsymbol{\theta}), c_2(\boldsymbol{\lambda}, \boldsymbol{\theta}), \dots), \quad (2.2.20)$$

where the polynomials $c_i(\boldsymbol{\lambda}, \boldsymbol{\theta})$, $i = 1, \dots$, are defined by (2.2.19);

4. the polynomial TD_d^{j-1} may be obtained from TD_d^j via the following substitution:

$$\text{TD}_d^{j-1}(b_0, b_1, b_2, \dots) = \text{TD}_d^j(0, b_0, b_1, b_2, \dots),$$

It is easy to see that for fixed j and sufficiently large n and k , the polynomials $c_i(\boldsymbol{\lambda}, \boldsymbol{\theta})$, $i = 1, \dots, j$ are algebraically independent. This means that for fixed codimension j and large enough n , the Thom polynomial $\text{Tp}_d^{n \rightarrow n+j}(\boldsymbol{\lambda}, \boldsymbol{\theta})$ determines TD_d^j . On the other hand, for small values of n there are several different polynomials which could serve as expressions for the Thom polynomial. In short: $\text{Tp}_d^{n \rightarrow k}(\boldsymbol{\lambda}, \boldsymbol{\theta})$ is not well defined but TD_d^j is.

Next, following [16], observe that property (4) allows us to define a universal object, the Thom series $\text{Ts}(a_i, i \in \mathbb{Z})$, which is an infinite formal series in infinitely many variables with the following properties:

- it homogeneous of degree d ;
- setting $\deg(a_i) = i$ for $i \in \mathbb{Z}$, the series $\text{Ts}_d(a_i, i \in \mathbb{Z})$ is homogeneous of degree 0;
- the Thom-Damon series maybe expressed via the following substitution:

$$\text{TD}_d^j(b_0, b_1, b_2, \dots) = \text{Ts}_d \left(\begin{array}{l} \left\{ \begin{array}{l} a_i = b_{i+k-n+1}, \text{ if } i \geq -(k-n+1), \\ a_i = 0 \text{ otherwise.} \end{array} \right. \end{array} \right)$$

The goal of our thesis is to compute the Thom polynomials for the algebras A_d . As we mentioned in the introduction, results in this direction go back 4 decades, to the works of Thom, Porteous, Gaffney, etc (see the Introduction for references). We were particularly interested in the meaning of the coefficients of the Thom series. The known results were Porteous's formula $\text{Ts}_1 = a_0$, and Ronga's formula: $\text{Ts}_2 = a_0^2 + \sum_{i=0}^{\infty} 2^{i-1} a_i a_{-i}$.

We obtained a rather satisfactory answer, which manifestly has the structure described above; the final result is contained in Theorem 4.4.16. Computationally, this answer can be explicitly implemented for $d \leq 6$ (cf. §8).

Chapter 3

Localizing Poincaré duals

In this section, we develop the idea introduced at the end of §2.2.3: the localization of equivariant Poincaré duals based on Vergne’s integration formula. Roughly, we show that if the T -invariant subvariety $\Sigma \subset W$ is equivariantly fibered over a parameter space M , then the equivariant Poincaré dual $eP[\Sigma, W]$ may be read off from local data near fixed points of the T action on M . The final form of the statement is Proposition 3.3.1. We will start, however, with the more regular case of a smooth parameter space.

3.1 Localization in the smooth case

Let Σ be a T -invariant closed subvariety of the T -module W . Consider the following diagram:

$$\begin{array}{ccccc}
 & & & W & \longleftarrow \supset \Sigma \\
 & & & \uparrow \text{ev}_S & \\
 & & S_M & \xrightarrow{\text{ev}_M} & S \\
 S_{M^T} \hookrightarrow & & \downarrow \tau_M & & \downarrow \tau_{\text{Gr}} \\
 & & M & \xrightarrow{\phi} & \text{Gr}(m, W) \\
 \tau_T \downarrow & & \downarrow \iota_T & & \\
 M^T & \hookrightarrow & & &
 \end{array} \tag{3.1.1}$$

Here

- $\text{Gr}(m, W)$ is the Grassmannian of m -planes in W , S is the tautological bundle over $\text{Gr}(m, W)$, and $\tau_{\text{Gr}} : S \rightarrow \text{Gr}(m, W)$ is the tautological projection; we denote by ev_S the tautological evaluation map. Observe that the map $\text{ev}_S : S \rightarrow W$ is proper.

- M is a smooth compact complex manifold, endowed with a T -action; as usual, the notation M^T stands for the set $\{y \in M; Ty = y\}$ of fixed points of the T -action; suppose that M^T is a finite set of points. The embedding $M^T \hookrightarrow M$ is denoted by ι_T .
- Let $\phi : M \rightarrow \text{Gr}(m, W)$ be a T -equivariant map, and introduce the pull-back bundles $S_M = \phi^*S$ and $S_{M^T} = \iota_T^*S_M$. We denote by ev_M the induced evaluation map $S_M \rightarrow W$.
- For clarity, we indexed our spaces and maps, but these indices will be omitted whenever this does not cause confusion. For example if $p \in M$, then we will denote by S_p the fiber of the bundle S_M over the point p .

Literally, to say that Σ is fibered over M would mean that the map $\text{ev}_M : S_M \rightarrow W$ establishes a diffeomorphism of S_M with Σ . This will essentially never happen, hence we weaken this condition as follows.

Recall (see e.g. [9]) that a smooth proper map $f : X \rightarrow Y$ between connected oriented manifolds of equal dimensions has a degree: $\deg(f) \in \mathbb{Z}$. Moreover, $\deg(f) = 1$ if and only if

- for any compactly supported form μ on Y , one has $\int_X f^*\mu = \int_Y \mu$,
or, equivalently,
- there is dense open $U \subset X$ such that f restricted to U is an orientation-preserving diffeomorphism.

Remark 3.1.1. 1. Note that a holomorphic map between complex manifolds is automatically orientation-preserving.

2. The two properties above, and hence the definition of a degree-1 map may be extended in an obvious manner to the case when X and Y are not necessarily smooth, but the set of smooth points forms a connected dense submanifold in both X and Y .

3. In algebraic geometry, degree-1 maps are also called *birational maps*.

Now we are ready to formulate our first localization formula.

Proposition 3.1.2. *Assume that in diagram (3.1.1) the fixed point set M^T is finite, and ev_M establishes a degree-1 map from S_M to Σ . Then we have*

$$\text{eP}[\Sigma, W] = \sum_{p \in M^T} \frac{\text{eP}[\text{ev}_M(S_p), W]}{\text{Euler}^T(\mathbb{T}_p M)}. \quad (3.1.2)$$

Definition 3.1.3.

If the conditions of Proposition 3.1.2 hold, we say that Σ is *sliced equivariantly into linear subspaces of rank m* , or we have an *equivariant slicing* of Σ .

Remark 3.1.4. 1. The most natural situation is when M is a smooth submanifold of $\text{Gr}(m, W)$. This more general setup works, however, even when the image $\phi(M)$ is singular.

2. Since for $p \in M^T$, the space $\text{ev}_M(S_p)$ is a linear T -invariant subspace of W , the polynomial $\text{eP}[\text{ev}_M(S_p)]$ is determined by the normalization axiom: it simply equals the product of those weights of W which are not weights of $\text{ev}_M(S_p)$ (with multiplicities taken into account).

3. The equivariant Euler class in the denominator is also a product of weights (cf. (2.2.4)), hence each term in the sum is a rational function. After the summation, however, the denominators cancel, and one ends up with a polynomial result.

Proof. Vergne's integral formula, (2.2.13) combined with our assumption that $\text{ev}_M : S_M \rightarrow \Sigma$ is degree-1, implies that

$$\text{eP}[\Sigma] = \int_{S_M} \text{ev}_M^* \text{Thom}(W).$$

Integrating first along the fibers, we obtain that

$$\text{eP}[\Sigma] = \int_M \tau_* \text{ev}_M^* \text{Thom}(W),$$

where the integrand $\tau_* \text{ev}_M^* \text{Thom}(W)$ is a smooth equivariant form on M . Now we apply the Berline-Vergne equivariant integration formula (2.2.12) to this form, and obtain that

$$\text{eP}[\Sigma] = \sum_{p \in M^T} \frac{(\tau_* \text{ev}_M^* \text{Thom}(W))^{[0]}(p)}{\text{Euler}^T(\mathbb{T}_p M)}, \quad (3.1.3)$$

where, as usual, we denote by $\mu^{[0]}$ the differential-form-degree-zero part of the equivariant form μ . Since ev_M is a linear embedding on each fiber, the numerator of (3.1.3) is simply the integral $\int_{\text{ev}_M(S_p)} \text{Thom}(W)$. Now, using Vergne's formula (2.2.13) one more time, we arrive at (3.1.2). □

As a quick application, we will give yet another way of computing the ePd for the basic example introduced in §2.2.2.

Third method: localization on the projectivized cone. Consider the smooth, T -invariant projective variety $\mathbb{P}\Sigma \subset \mathbb{P}^3$ cut out by the homogeneous equation $x_1 x_3 = x_2 x_4$. In the notation of (3.1.1), we have $M = \mathbb{P}\Sigma$, $m = 1$ and $W = \mathbb{C}^4$. Then the fixed point set $\mathbb{P}\Sigma^T$ consists of the four fixed points on \mathbb{P}^3 corresponding to the four coordinate axes.

Pick one of these fixed points, say, $p = (1 : 0 : 0 : 0)$, which corresponds to the coordinate line $S_p = \{x_2 = x_3 = x_4 = 0\}$. Using the normalization axiom, we have then $\text{eP}[S_p] = \eta_2 \eta_3 \eta_4$.

Turning to the denominator in (3.1.2), it is not hard to see that

$$\text{Euler}^T(\mathbb{T}_p\mathbb{P}\Sigma) = (\eta_2 - \eta_1)(\eta_4 - \eta_1).$$

Indeed, this is the standard yoga of toric geometry: consider the parallelogram formed by the weights η_1, η_2, η_3 and η_4 ; the fixed points of the torus action correspond to the vertices of this parallelogram, and the weights at a particular fixed point are the edge-vectors emanating from the associated vertex.

The contributions at the other fixed points may be computed likewise, and the result is the following complicated formula for the equivariant Poincaré dual:

$$\begin{aligned} \text{eP}[\Sigma] = & \frac{\eta_2\eta_3\eta_4}{(\eta_2 - \eta_1)(\eta_4 - \eta_1)} + \frac{\eta_1\eta_3\eta_4}{(\eta_1 - \eta_2)(\eta_3 - \eta_2)} + \\ & \frac{\eta_1\eta_2\eta_4}{(\eta_2 - \eta_3)(\eta_4 - \eta_3)} + \frac{\eta_1\eta_2\eta_3}{(\eta_1 - \eta_4)(\eta_3 - \eta_4)}. \end{aligned} \quad (3.1.4)$$

This rational function is *not* a polynomial, however, assuming $\eta_1 + \eta_3 = \eta_2 + \eta_4$ holds, it can be easily shown to reduce to the simple form (2.2.8).

We note that this procedure may be applied, inductively, to more general toric varieties, and, again, the data may be read off the corresponding polytope. However, if the polytope is not simple, then the procedure becomes rather involved.

3.2 An interlude: the case of $d = 1$

In this paragraph, we compute the ePd of our singularity Θ_d in the case $d = 1$, and recover the classical result of Porteous.

Recall that Θ_1 is the subset of those linear maps $\mathbb{C}^n \rightarrow \mathbb{C}^k$ whose kernel is 1-dimensional. These maps may be identified with k -by- n matrices, and the weight of the action on the entry e_{ji} is equal to $\theta_j - \lambda_i$. Then the closure $\overline{\Theta}_1$ consist of those k -by- n matrices which have a nontrivial kernel:

$$\overline{\Theta}_1 = \{A \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n); \exists v \in \mathbb{C}^n, v \neq 0 : Av = 0\}. \quad (3.2.1)$$

This immediately gives us an equivariant slicing of $\overline{\Theta}_1$ over \mathbb{P}^{n-1} in the sense of Definition 3.1.3: the fiber over a point $[v] \in \mathbb{P}^{n-1}$ is the linear subspace $\{A; Av = 0\} \subset \overline{\Theta}_1$, where $[v]$ stands for the point in \mathbb{P}^{n-1} corresponding to the nonzero vector $v \in \mathbb{C}^n$.

Again, we simply need to collect our fixed-point data, and then apply (3.1.2). There are n fixed points on \mathbb{P}^{n-1} : p_1, \dots, p_n , corresponding to the coordinate axes. The weights of $\mathbb{T}_{p_i}\mathbb{P}^{n-1}$ are $\{\lambda_s - \lambda_i; s \neq i\}$. The fiber at p_i is the set of matrices A with all entries in the i th column vanishing. Again, using the normalization axiom, this shows that the ePd of the fiber at p_i is $\prod_{j=1}^k (\theta_j - \lambda_i)$, so our localization formula looks as follows:

$$\text{eP}[\Theta_1] = \sum_{i=1}^n \frac{\prod_{j=1}^k (\theta_j - \lambda_i)}{\prod_{s \neq i} (\lambda_s - \lambda_i)} \quad (3.2.2)$$

This is a local formula for fixed n , but as n increases, the number of terms also increases. We can, in fact, further “localize” this expression, and obtain a formula, which only depends on the local behavior of a function at a single point.

Indeed, consider the rational differential form

$$-\frac{\prod_{j=1}^k(\theta_j - z)}{\prod_{i=1}^n(\lambda_i - z)} dz.$$

Observe that the residues of this form at finite poles: $\{z = \lambda_i; i = 1, \dots, n\}$ exactly recover the terms of the sum (3.2.2). Applying the residue theorem, we can conclude that

$$eP[\Theta_1] = \operatorname{Res}_{z=\infty} \frac{\prod_{j=1}^k(\theta_j - z)}{\prod_{i=1}^n(\lambda_i - z)} dz.$$

Changing variables: $z = -1/q$, we obtain

$$eP[\Theta_1] = \operatorname{Res}_{q=0} \frac{\prod_{j=1}^k(1 + q\theta_j)}{\prod_{i=1}^n(1 + q\lambda_i)} \frac{dq}{q^{k-n+2}},$$

which, according to (2.2.19), is exactly the relative Chern class c_{k-n+1} . Thus we recovered the well-known Giambelli-Thom -Porteous formula ([37]; [23] Chapter I.5).

As a final remark, note that our basic example introduced in §2.2.2 is a special case of Θ_1 , corresponding to the values $n = k = 2$. Hence this computation provides us with a **fourth method** of arriving at (2.2.8). As the third method, this one also uses localization, but this constructions are different. It is a pleasant exercise to check the details.

3.3 Variations of the localization formula

We will need to amend and generalize Proposition 3.1.2 in order to be able handle the cases of Θ_d for $d > 1$.

Nonlinear fibers

Observe that, during the proof of Proposition 3.1.2, we never used the assumption of that the fibers are linear. In fact, clearly, the same formula and the same argument holds if the fibers are of S are arbitrary, possibly singular manifolds. In fact, this localization principle holds in the case of a flat family as well, but we will not need this generalization.

Passing to Euler classes

The observation of §3.3 suggests the following family version of the the formula for complete intersections (2.2.9). Assume that there is an equivariant vector bundle

E over M , and an equivariant family of polynomial maps $\gamma_p : W \rightarrow E_p$ for $p \in M$ such that the derivative $d\gamma_p$ of the map γ_p at a generic point of $\gamma_p^{-1}(0)$ is surjective to E_p . Then the set

$$\{(p, w) \in M \times W; \gamma_p(w) = 0\}$$

is a flat family the trivial bundle $M \times W$ (we can assume for simplicity that this is a subbundle), and, we assume that under the projection to W , it maps to Σ in a birational fashion. Then according to (2.2.9) we have $\text{eP}[\text{ev}_M(S_p), W] = \text{Euler}^T(E_p)$, which leads to the following variant of (3.1.2):

$$\text{eP}[\Sigma] = \sum_{p \in M^T} \frac{\text{Euler}^T(E_p)}{\text{Euler}^T(\mathbb{T}_p M)}. \quad (3.3.1)$$

Note that applying this formula to the computation of $\text{eP}[\Theta_1]$ leads to the result slightly quicker than in §3.2: in this case, E is the dual of the tautological line over \mathbb{P}^{n-1} tensored with \mathbb{C}^k .

Fibrations over a singular base

Finally, we remove the assumption that M is smooth. For brevity, below, without explicitly stating this, we will assume that every space and map is in the T -equivariant category.

Proposition 3.3.1. *Let Σ be a closed subvariety of the complex vector space W . Assume that Z is a compact, smooth complex manifold, and $M \subset Z$ is a possibly singular, closed analytic subvariety with a finite set of fixed points M^T . Let S be a fibration over Z with possibly singular fibers, and assume that S is endowed with a proper map $\text{ev}_S : S \rightarrow W$, which is an embedding in each fiber, and which establishes a degree-1 map between $\tau_Z^{-1}(M)$ and Σ . (3.1.1):*

$$\begin{array}{ccccc} S & \xrightarrow{\text{ev}_S} & W & \longleftarrow \supset & \Sigma \\ & \searrow \tau_Z & & & \\ M^T \hookrightarrow & M & \hookrightarrow & & Z \end{array} \quad (3.3.2)$$

Then

$$\text{eP}[\Sigma] = \sum_{p \in M^T} \frac{\text{eP}[\text{ev}_S(S_p)] \text{emult}_p[M, Z]}{\text{Euler}^T(\mathbb{T}_p Z)}. \quad (3.3.3)$$

Remark 3.3.2. We extend Definition 3.1.3 for the situation of Proposition 3.3.1; we say that Σ is *equivariantly sliced*, or we have an *equivariant slicing* of Σ .

Again, the proof of Proposition 3.3.1 is analogous to that of Proposition 3.1.2; when passing to (3.1.3), however, one needs to use Rossmann's integration formula (2.2.11).

Corollary 3.3.3. *Assume that, in addition, there is a vector bundle E over Z , and a family of polynomial bundle-maps $\gamma_p : W \rightarrow E_p$ satisfying $\gamma_p^{-1}(0) = \text{ev}_S(S_p)$ for each $p \in Z$, and such that the codimension of the set*

$$\{q \in S_p; d\gamma_p(q) : \text{ev}_S(S_p) \rightarrow E_p \text{ is not surjective}\}$$

in S_p is positive. Then

$$\text{eP}[\Sigma, W] = \sum_{p \in M^T} \frac{\text{Euler}^T(E_p) \text{emult}_p(M, Z)}{\text{Euler}^T(\text{T}_p Z)}. \quad (3.3.4)$$

Chapter 4

Multidegrees of A_n singularities

4.1 The test curve model

In §2.1, we described the variety Θ_d in two different ways: as a contact singularity class (cf. (2.1.2)), and as the Boardman class corresponding to the sequence $(1, 1, \dots, 1)$ (cf. Prop. 2.1.8). In this section, we recall yet another description – the so-called “test curve model” – which goes back to Porteous, Ronga, and Gaffney, [38, 41, 20]. Roughly, one generalizes (3.2.1) to $d > 1$ by declaring for $\Psi \in \mathcal{J}_d(n, k)$ the existence of a d -jet of a curve in \mathbb{C}^n which is carried to zero by Ψ . As we have not found a complete proof of this statement in the literature, we give one below.

Recall the notation $\text{Lin} : \mathcal{J}_d(n, k) \rightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^k)$ for the linear part of map germs. A d -jet of a curve in \mathbb{C}^n is simply an element of $\mathcal{J}_d(1, n)$. We will call such a curve γ *regular* if $\text{Lin}(\gamma) \neq 0$; introduce the notation $\mathcal{J}_d^{\text{reg}}(1, n)$ for the set of these curves:

$$\mathcal{J}_d^{\text{reg}}(1, n) \stackrel{\text{def}}{=} \{\gamma \in \mathcal{J}_d(1, n); \text{Lin}(\gamma) \neq 0\}. \quad (4.1.1)$$

Now define the variety

$$\Theta'_d = \{\Psi \in \mathcal{J}_d(n, k); \exists \gamma \in \mathcal{J}_d^{\text{reg}}(1, n) \text{ such that } \Psi \circ \gamma = 0\}. \quad (4.1.2)$$

In words: Θ'_d is the variety of those d -jets of maps, which take at least one regular curve to zero. Clearly, Θ'_d is an algebraic subvariety of $\mathcal{J}_d(n, k)$. Then we have:

Theorem 4.1.1. *The closures of the following two subvarieties coincide: $\overline{\Theta'_d} = \overline{\Theta_d}$.*

Proof. According to Proposition 2.1.3 and Corollary 2.1.4, Θ_d is a single \mathcal{K}_d -orbit, and hence an irreducible algebraic subvariety of $\mathcal{J}_d(n, k)$. Therefore, to prove the theorem, it is sufficient to show that

- Θ'_d is \mathcal{K}_d -invariant,
- $\Theta'_d \cap \Theta_d$ is non-empty,

- $\text{codim}(\Theta'_d) = \text{codim}(\Theta_d)$ in $\mathcal{J}_d(n, k)$, and that
- the subvariety $\Theta'_d \subset \mathcal{J}_d(n, k)$ is irreducible.

To show the \mathcal{K}_d -invariance of Θ' , observe that if $\gamma \in \mathcal{J}_d(1, n)$ is regular and $\Delta \in \text{Diff}_d(n)$, then $\Delta \circ \gamma$ is also regular. Indeed, in this case

$$\text{Lin}(\Delta \circ \gamma) = \text{Lin}(\Delta) \cdot \text{Lin}(\gamma) \neq 0.$$

Then if $\Psi \circ \gamma = 0$ for some regular γ , and $(M, \Delta) \in \mathcal{K}_d$, then we have

$$M \cdot \Psi \circ \Delta^{-1} \circ (\Delta \circ \gamma) = M \cdot \Psi \circ \gamma = 0;$$

this shows that $\Delta \circ \gamma$ is an appropriate test curve for the transformed map jet.

To find an element in the intersection of Θ_d and Θ'_d , consider the map jet

$$\Psi_0(x_1, \dots, x_n) = (0, x_2, \dots, x_d, 0, \dots, 0).$$

This obviously belongs to Θ_d ; on the other hand, for the test curve $\gamma(t) = (t, 0, \dots, 0)$, we have $\text{Lin}(\gamma) \neq 0$ and $\Psi_0 \circ \gamma = 0$ in $\mathcal{J}_d(n, k)$, hence $\Psi_0 \in \Theta'_d$.

Regarding the codimensions, we have $\text{codim}(\Theta_d) = d(k - n + 1)$ according to Proposition 2.1.1. The proof of the irreducibility of Θ'_d and the computation of its codimension (cf. Corollary 4.1.4) will follow from the more detailed study of its structure, to which we devote the rest of this section. □

Our first project is to write down the equation $\Psi \circ \gamma = 0$ in coordinates. This is a rather mechanical exercise, and we will spend some time setting up the notation.

A curve $\gamma \in \mathcal{J}_d(1, n)$ is parametrized by d vectors v_1, \dots, v_d in \mathbb{C}^n :

$$\gamma(t) = tv_1 + t^2v_2 + \dots + t^dv_d, \tag{4.1.3}$$

In this explicit form, the condition of regularity $\text{Lin}(\gamma) \neq 0$ means that $v_1 \neq 0$.

Next, we switch to a new parametrization of our space $\mathcal{J}_d(n, k)$. Separating the similar homogeneous components of the k polynomials, P_1, \dots, P_k , and thinking of a homogeneous degree- l polynomial as an element of $\text{Hom}(\text{Sym}^l \mathbb{C}^n, \mathbb{C})$, we may represent $\Psi \in \mathcal{J}_d(n, k)$ as a linear map

$$\Psi = (\Psi^1, \dots, \Psi^d) : \bigoplus_{l=1}^d \text{Sym}^l \mathbb{C}^n \rightarrow \mathbb{C}^k. \tag{4.1.4}$$

The standard basis of the vector space $\bigoplus_{l=1}^d \text{Sym}^l \mathbb{C}^n$ may be parametrized by non-decreasing sequences of positive integers, or, alternatively – and this is the language we will prefer – by *partitions*. Namely, to the partition $[i_1, \dots, i_l]$ of the integer $i_1 + \dots + i_l$ with $1 \leq i_m \leq n$, we associate the basis element $e_{i_1} \cdots e_{i_l} \in \text{Sym}^l \mathbb{C}^n$.

In what follows, certain integer characteristics of partitions will be used: for a partition $\tau = [i_1, \dots, i_l]$ of the integer $i_1 + \dots + i_l$, these are

- the *length*: $|\tau| = l$,
- the *sum*: $\text{sum}(\tau) = i_1 + \dots + i_l$,
- the *maximum*: $\text{max}(\tau) = \max(i_1, \dots, i_l)$,
- and the *number of permutations*: $\text{perm}(\tau)$, which is the number of different sequences consisting of the numbers i_1, \dots, i_l ; e.g. $\text{perm}([1, 1, 1, 3]) = 4$.

The basis elements of $\bigoplus_{l=1}^d \text{Sym}^l \mathbb{C}^n$ correspond to partitions of length at most d and maximum at most n . We denote the set of these partitions by $\Pi_d[n]$

$$\Pi_d[n] \stackrel{\text{def}}{=} \{\tau; |\tau| \leq d, \text{max}(\tau) \leq n\}.$$

Then $\dim \mathcal{J}_d(n, k) = k|\Pi_d[n]|$.

For a sequence $\mathbf{v} = (v_1, v_2, \dots)$ of vectors in \mathbb{C}^n , a partition $\tau = [i_1, \dots, i_l]$ of length l , and a map germ $\Psi \in \mathcal{J}_d(n, k)$, introduce the shorthand

$$\mathbf{v}_\tau = \prod_{j=1}^l v_{i_j} \in \text{Sym}^l \mathbb{C}^n \quad \text{and} \quad \Psi(\mathbf{v}_\tau) = \Psi^l(v_{i_1}, \dots, v_{i_l}) \in \mathbb{C}^k. \quad (4.1.5)$$

Armed with this new notation, we can write down the equation $\Psi \circ \gamma = 0$ more explicitly, as follows.

Lemma 4.1.2. *Let $\gamma \in \mathcal{J}_d(1, n)$ be given in the form (4.1.3). Then, using the notation (4.1.5), the equation $\Psi \circ \gamma = 0$ is equivalent to the following system of d linear equations with values in \mathbb{C}^k on the components Ψ^l , $l = 1, \dots, d$, of $\Psi \in \mathcal{J}_d(n, k)$:*

$$\sum_{\text{sum}(\tau)=m} \text{perm}(\tau) \Psi(\mathbf{v}_\tau) = 0, \quad m = 1, 2, \dots, d, \quad (4.1.6)$$

where the sum runs over all partitions of the number m .

Let us see what the equations (4.1.6) look like for small d . To make the formulas easier to follow, we will use the l th capital letter of the alphabet for the symmetric multi-linear map Ψ^l introduced in (4.1.4): we will write A for the linear part Ψ^1 of Ψ , B for its second order part, etc. With this convention (see also (4.1.3)), the system of equations for $d = 4$ reads as follows:

$$\begin{aligned} A(v_1) &= 0, & (4.1.7) \\ A(v_2) + B(v_1, v_1) &= 0, \\ A(v_3) + 2B(v_1, v_2) + C(v_1, v_1, v_1) &= 0, \\ A(v_4) + 2B(v_1, v_3) + B(v_2, v_2) + 3C(v_1, v_1, v_2) + D(v_1, v_1, v_1, v_1) &= 0. \end{aligned}$$

For a curve $\gamma \in \mathcal{J}_d^{\text{reg}}(1, n)$, we denote by Sol_γ the space of solutions of the system (4.1.6). Then we may write

$$\Theta'_d = \bigcup \{\text{Sol}_\gamma; \gamma \in \mathcal{J}_d^{\text{reg}}(1, n)\}.$$

Next, we collect some simple facts about the system (4.1.6).

Proposition 4.1.3. 1. Let $0 \neq v \in \mathbb{C}^n$, and assume that $\gamma \in \mathcal{J}_d^{\text{reg}}(1, n)$ is such that $\text{Lin}(\gamma)$ is parallel to v . Pick a hyperplane H in \mathbb{C}^n which is complementary to v . Then there is a unique $\delta \in \text{Diff}_d(1)$ such that

$$\gamma \circ \delta = tv + t^2v_2 + \cdots + t^dv_d \quad \text{with } v_2, v_3, \dots, v_d \in H. \quad (4.1.8)$$

2. For $\gamma \in \mathcal{J}_d^{\text{reg}}(1, n)$, the set of solutions $\text{Sol}_\gamma \subset \mathcal{J}_d(n, k)$ is a linear subspace of codimension dk .
3. If $\dim \ker(\text{Lin}(\Psi)) = 1$, then Ψ may belong to at most one of the spaces Sol_γ . More precisely,

$$\text{if } \gamma, \gamma' \in \mathcal{J}_d^{\text{reg}}(1, n), \dim(\ker \text{Lin}(\Psi)) = 1 \text{ and } \Psi \circ \gamma = \Psi \circ \gamma' = 0$$

then that there exists $\delta \in \text{Diff}_d(1)$ such that $\gamma' = \gamma \circ \delta$.

4. Given $\gamma, \gamma' \in \mathcal{J}_d^{\text{reg}}(1, n)$, we have $\text{Sol}_\gamma = \text{Sol}_{\gamma'}$ if and only if there is a $\delta \in \text{Diff}_d(1)$ such that $\gamma' = \gamma \circ \delta$.

Proof. The proofs are straightforward; we will give short sketches of the arguments, leaving the details to the reader.

For (1), write explicitly $\gamma(s) = sw_1 + \cdots + s^dw_d$ and $\delta = \lambda_1t + \cdots + \lambda_dt^d$. After making the substitution $s \mapsto \delta$ we obtain a curve $\gamma \circ \delta = tv + t^2v_2 + \cdots + t^dv_d$, where $v_l = \lambda_l w_l +$ terms with λ s which have lower indices than l ; this clearly implies the statement.

The second statement follows from the presence of the term $\Psi^l(v_1, \dots, v_1)$ in the l th equation of (4.1.6), which is clearly linearly independent of the rest of the terms in the first l equations.

To prove statement (3), we can assume that γ and γ' are normalized according to (4.1.8) with respect to some $v \in \ker(\text{Lin}(\Psi))$, and then we show that $\gamma = \gamma'$ by induction. Assume, for example, that the two curves coincide up to the third order, i.e. $v_1 = v'_1, v_2 = v'_2, v_3 = v'_3$. Then we see from (4.1.7) that $A(v_4) = A(v'_4)$. We have $A = \text{Lin}(\Psi)$ and $\ker(A) = \mathbb{C}v_1$, hence $v_4, v'_4 \in H$ and $A(v_4) = A(v'_4)$ imply $v_4 = v'_4$. This completes the inductive step.

The last statement is an immediate consequence of the second. Indeed, one only needs to observe that for any $\gamma \in \mathcal{J}_d^{\text{reg}}(1, n)$ the set of those $\Psi \in \mathcal{J}_d(n, k)$ for which $\dim \ker(\text{Lin}(\Psi)) = 1$ forms a dense Zariski open set. \square

Part (1) of the proposition describes a slice of the $\text{Diff}_d(1)$ action on $\mathcal{J}_d^{\text{reg}}(1, n)$, which can be formalized as follows. Let Q be the canonical $n - 1$ -dimensional quotient bundle over \mathbb{P}^{n-1} ; this bundle fits into the exact sequence

$$0 \longrightarrow S^1 \longrightarrow \mathbb{C}^n \longrightarrow Q \longrightarrow 0$$

where S^1 is the tautological line bundle. The slice is now $Q_{\mathbb{P}}^{d-1}$, the fiberwise direct product, and a natural isomorphism $\beta : \mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) \rightarrow Q_{\mathbb{P}}^{d-1}$ maps the curve γ given as (4.1.3) the point

$$([v_1], v_2 \bmod v_1, \dots, v_d \bmod v_1) \in Q_{\mathbb{P}}^{d-1},$$

where $[v_1]$ stands for the element of \mathbb{P}^{n-1} determined by $v_1 \in \mathbb{C}^n$; according to Proposition 4.1.3 (1), this correspondence is an isomorphism. Clearly, $\dim Q_{\mathbb{P}}^{d-1} = \dim \mathbb{P}^{n-1} + (d-1)(n-1) = d(n-1)$.

We can summarize the results of this section in the following diagram:

$$\begin{array}{ccccc}
\Theta_d & \hookrightarrow & \Theta'_d & \hookrightarrow & \mathcal{J}_d(n, k) \\
& & & \nearrow \text{ev}_{\text{Sol}} & \uparrow \text{ev } S \\
& & \text{Sol} & \longrightarrow & S \\
& & \downarrow \tau_{\mathcal{J}} & & \downarrow \tau_{Gr} \\
Q_{\mathbb{P}}^{d-1} & \xleftarrow{\beta} & \mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) & \xrightarrow{\phi_{Gr}} & \text{Gr}(\dim, \mathcal{J}_d(n, k))
\end{array} \tag{4.1.9}$$

Explanations:

- Each space in the diagram carries an action of the group $GL(k) \times GL(n)$, and the maps are equivariant with respect to this action.
- As usual, we denote the tautological bundle over the Grassmannian by S (cf. diagram 3.1.1). The rank of the bundle S equals to $\dim(\mathcal{J}_d(n, k)) - dk$; we simply wrote “dim” for this number in the diagram.
- The map ϕ_{Gr} is the correspondence $\gamma \mapsto \text{Sol}_{\gamma}$ introduced after (4.1.7); it is an embedding according to Proposition 4.1.3 (4). We denoted by Sol the total space of all solutions, hence $\text{Sol} = \phi_{Gr}^* S$.
- The map ev_S is the tautological evaluation map, and ev_{Sol} is the fiberwise embedding of the solution spaces. By definition, we have $\text{im}(\text{ev}_{\text{Sol}}) = \Theta'_d$, and according to Proposition 4.1.3(3), the map ev_{Sol} is injective when restricted to a subset which is Zariski open in each fiber; this implies that ev_{Sol} is birational.

Now we can formulate two important corollaries of Proposition 4.1.3.

Corollary 4.1.4. 1. Θ'_d is irreducible subvariety of $\mathcal{J}_d(n, k)$.

2. $\text{codim}(\Theta_d) = \text{codim}(\Theta'_d) = d(k - n + 1)$.

Indeed, as observed above, the map ev_{Sol} is birational to Θ'_d from a vector bundle whose base is irreducible; this implies (1). The fibers of this vector bundle are codimension dk vector spaces in $\mathcal{J}_d(n, k)$, while the base $Q_{\mathbb{P}}^{d-1}$ is of dimension $d(n-1)$, hence the codimension of Θ'_d equals $dk - d(n-1) = d(k-n+1)$. \square

The situation described on diagram (4.1.9) is quite close to the setup of Proposition 3.1.2 and diagram (3.1.1). The most immediate obstacle to the application of the localization principles is the fact that the base $\mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) \cong Q_{\mathbb{P}}^{d-1}$ is not compact. Our diagram suggests a reasonable compactification, however: the closure of $\phi_{\text{Gr}}(\mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1))$ in the Grassmannian. The choice of the compactification is very important from the point of view of the efficiency of the resulting formulas, and we will be very careful in constructing one. This is the subject of the next section.

4.1.1 Nonreductive quotients

The standard construction of quotient spaces in algebraic geometry uses Mumford's Geometric Invariant Theory (GIT) [34]. Given a *reductive* algebraic group G acting linearly on a quasi-projective variety X , GIT identifies two important G -invariant subsets of X : the stable locus, X^s , a Zariski open subset whose topological quotient space X^s/G is an algebraic variety, and the semistable locus X^{ss} , a Zariski open subset which contains the stable locus and maps to the 'GIT quotient' of X , denoted $X//G$, such that $X//G$ is a 'categorical' quotient of X^{ss} .

Mumford's GIT construction of quotients $X//G$ depends in a crucial way, however, on the group G being reductive (that is, G is the complexification of a maximal compact subgroup). On the other hand, there are many natural geometric problems where the group G is not reductive, so that GIT does not apply, and these include the diffeomorphism groups like $\text{Diff}_d(1)$.

On the other hand, the Frances Kirwan and Brent Doran have been independently developing a generalisation of GIT for non-reductive group actions, including various notions of stable and semistable points and compactified quotients [11]. They transfer the problem to conventional GIT by choosing a reductive group G which contains as a subgroup the non-reductive group H acting on X , and consider the associated reductive G -actions on projective compactifications $\overline{G \times_H X}$ of $G \times_H X$. The GIT quotients $\overline{G \times_H X} // G$ provide compactifications of X^s/H , and (in principle, at least) existing localisation methods can be applied to the action of G on $\overline{G \times_H X}$ to study the topology of these compactified quotients of the non-reductive action. For more details, see [11]

At the moment, the methods of Kirwan and Doran do not handle the difficulties related to localizations on nonreductive quotients. We hope, that our work represents a step in the direction of creating an effective theory of localization on nonreductive quotients.

4.2 The compactification

As we observed at the end of the previous section, the embedding ϕ_{Gr} in diagram (4.1.9) may be used to compactify $\mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1)$, and this would allow us to apply the localization techniques of §3. The resulting formulas turn out to be intractable, however, and the purpose of this section is to replace the Grassmanian by a “smaller” space, which provides us with a better compactification and, hopefully, more efficient formulas.

We will employ two ideas. The first is straightforward: we note that not every dk -codimensional linear subspace of the Grassmanian may appear as the solution space of a system of our equations. Indeed, the system of equations (4.1.6) has a special form respecting a certain filtration.

The second idea is a bit more involved: we remove a certain part of the space of regular curves, thus breaking the $\text{Diff}_d(1)$ -symmetry, and fiber the remainder over the space of full flags of d -dimensional subspaces of \mathbb{C}^n . This leads to a double fibration, whose study we can reduce to that of a single fiber.

4.2.1 Embedding into the space of equations

We start by rewriting the linear system $\Psi \circ \gamma = 0$ associated to $\gamma \in \mathcal{J}_d(1, n)$ in a dual form (cf. Lemma 4.1.2). The system is based on the standard composition map (2.1.1):

$$\mathcal{J}_d(n, k) \times \mathcal{J}_d(1, n) \longrightarrow \mathcal{J}_d(1, k),$$

which, in view of $\mathcal{J}_d(n, k) = \mathcal{J}_d(n, 1) \otimes \mathbb{C}^k$, is derived from the map

$$\mathcal{J}_d(n, 1) \times \mathcal{J}_d(1, n) \longrightarrow \mathcal{J}_d(1, 1)$$

via tensoring with \mathbb{C}^k . Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

$$\psi : \mathcal{J}_d(1, n) \longrightarrow \text{Hom}(\mathcal{J}_d(1, 1)^*, \mathcal{J}_d(n, 1)^*). \quad (4.2.1)$$

To present this map explicitly, we recall (cf. (4.1.3)) that a d -jet of a curve $\gamma \in \mathcal{J}_d(1, n)$ is given by a sequence of d vectors in \mathbb{C}^n , and thus we can

$$\text{identify } \mathcal{J}_d(1, n) \text{ with } \text{Hom}(\mathbb{C}^d, \mathbb{C}^n) \quad (4.2.2)$$

as vector spaces. Also, according to (4.1.4) the dual of $\mathcal{J}_d(n, 1)$ is the vector space $\text{Sym}_d^\bullet \mathbb{C}^n = \bigoplus_{l=1}^d \text{Sym}^l \mathbb{C}^n$. Then a system of equations on $\mathcal{J}_d(n, 1)$ may be thought of as a linear map $\varepsilon \in \text{Hom}(\mathbb{C}^d, \text{Sym}_d^\bullet \mathbb{C}^n)$, and the solution set of this system is the linear subspace orthogonal to the image of ε : $\text{im}(\varepsilon)^\perp \subset \mathcal{J}_d(n, 1)$.

Using these identifications, we can recast the map ψ in (4.2.1) as

$$\psi : \text{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \longrightarrow \text{Hom}(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n), \quad (4.2.3)$$

which may be written out explicitly as follows (cf. (4.1.7)):

$$\psi : (v_1, \dots, v_d) \longmapsto \left(v_1, v_2 + v_1^2, v_3 + 2v_1v_2 + v_1^3, \dots, \sum_{\text{sum}(\tau)=m} \text{perm}(\tau) \mathbf{v}_\tau, \dots \right).$$

Note that in (4.2.3) – somewhat looking ahead – we marked the two copies of \mathbb{C}^d with different indices: L for left and R for (cf. **Convention** after Lemma 4.2.1 below).

The constructions of this section will be based on the observation that the spaces of map germs $\mathcal{J}_d(n, 1)$ and $\mathcal{J}_d(1, 1)$ – and hence their duals – have natural filtrations, and these filtrations are preserved by the map ψ .

The filtration on the dual of $\mathcal{J}_d(n, 1)$ (cf. (4.1.4)) is

$$\text{Sym}_d^\bullet \mathbb{C}^n = \bigoplus_{l=1}^d \text{Sym}^l \mathbb{C}^n \supset \bigoplus_{l=1}^{d-1} \text{Sym}^l \mathbb{C}^n \supset \dots \supset \mathbb{C}^n \oplus \text{Sym}^2 \mathbb{C}^n \supset \mathbb{C}^n; \quad (4.2.4)$$

setting $n = 1$, this reduces to \mathbb{C}^d with the standard filtration:

$$\mathbb{C}^d \supset \bigoplus_{l=1}^{d-1} \mathbb{C}e_l \supset \dots \supset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \supset \mathbb{C}e_1. \quad (4.2.5)$$

Now introduce the space of filtration-preserving maps

$$\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n) = \{\varepsilon \in \text{Hom}(\mathbb{C}^d, \text{Sym}_d^\bullet \mathbb{C}^n); \varepsilon(e_l) \in \bigoplus_{m=1}^l \text{Sym}^m \mathbb{C}^n, l = 1, \dots, d\}, \quad (4.2.6)$$

and its subspaces of nondegenerate systems

$$\mathcal{F}_d(n) = \{\varepsilon \in \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n); \ker(\varepsilon) = 0\} \quad (4.2.7)$$

and regular nondegenerate systems

$$\mathcal{F}_d^{\text{reg}}(n) = \{\varepsilon \in \text{Hom}(\mathbb{C}^d, \text{Sym}_d^\bullet \mathbb{C}^n) \cap \mathcal{F}_d(n); \varepsilon(e_l) \notin \bigoplus_{m=1}^{l-1} \text{Sym}^m \mathbb{C}^n, l = 1, \dots, d\}. \quad (4.2.8)$$

The following properties of the map ψ are manifest (cf. Proposition 4.1.3(2)):

Lemma 4.2.1. *The correspondence ψ given in (4.2.3) takes values in $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$. Moreover, for $\gamma \in \mathcal{J}_d^{\text{reg}}(1, n)$ we have $\psi(\gamma) \in \mathcal{F}_d^{\text{reg}}(n)$.*

Convention: The group of linear automorphisms of \mathbb{C}^d will be denoted, as usual by GL_d , its subgroup of diagonal matrices by T_d , and its subgroup of upper-triangular maps by B_d . In what follows, the two (left and right) copies of \mathbb{C}^d appearing in (4.2.3) will play a rather different but important role. To avoid any confusion we will use the following notation for the corresponding groups:

$$T_L \subset B_L \subset \text{GL}_L \quad \text{and} \quad T_R \subset B_R \subset \text{GL}_R.$$

The space $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$ carries a left action of GL_n , and also a right action of the Borel subgroup B_R of GL_R preserving the filtration (4.2.5):

$$B_R = \{b \in \text{Hom}^\Delta(\mathbb{C}_R^d, \mathbb{C}_R^d); b \text{ invertible}\}. \quad (4.2.9)$$

Lemma 4.2.2. *The subspace $\mathcal{F}_d(n) \subset \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$ is invariant under both GL_n and B_R , and the quotient $\widetilde{\mathcal{F}}_d(n) = \mathcal{F}_d(n)/B_R$ is a compact smooth manifold endowed with a GL_n -action; $\mathcal{F}_d^{\text{reg}}(n)$ is B_R -invariant and the quotient $\widetilde{\mathcal{F}}_d^{\text{reg}}(n) = \mathcal{F}_d^{\text{reg}}(n)/B_R$ is an open cell in $\widetilde{\mathcal{F}}_d(n)$.*

Proof. To check the invariance with respect to the group actions is straightforward. The rest of the Lemma follows from an identification of $\widetilde{\mathcal{F}}_d(n)$ with a Schubert variety.

Indeed, by definition,

$$\widetilde{\mathcal{F}}_d(n) = \mathcal{F}_d(n)/B_R \subset \text{Hom}^{\text{reg}}(\mathbb{C}^d, \text{Sym}_d^\bullet \mathbb{C}^n)/B_R = \text{Flag}_d(\text{Sym}_d^\bullet \mathbb{C}^n), \quad (4.2.10)$$

where $\text{Hom}^{\text{reg}}(\mathbb{C}^d, \text{Sym}_d^\bullet \mathbb{C}^n)$ denotes the maps of rank d , and $\text{Flag}_d(\text{Sym}_d^\bullet \mathbb{C}^n)$ is the compact partial flag manifold of full flags of d -dimensional subspaces of $\text{Sym}_d^\bullet \mathbb{C}^n$:

$$\text{Flag}(\text{Sym}_d^\bullet \mathbb{C}^n) = \{0 = F_0 \subset F_1 \subset \dots \subset F_d \subset \text{Sym}_d^\bullet \mathbb{C}^n, \dim F_l = l\}.$$

For simplicity, we use temporarily the notation $\dim_i = \dim(\oplus_{l=1}^i \text{Sym}^l \mathbb{C}^n)$ for $i = 1, \dots, d$ and $\dim_0 = 1$. Let

$$\mathbb{C} \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^{\dim_d} = \text{Sym}_d^\bullet \mathbb{C}^n \quad (4.2.11)$$

be a filtration, preserving (4.2.4), i.e $\mathbb{C}^{\dim_i} = \oplus_{l=1}^i \text{Sym}^l \mathbb{C}^n$ for $i = 1, \dots, d$.

We can identify $\widetilde{\mathcal{F}}_d^{\text{reg}}(n)$ and $\widetilde{\mathcal{F}}_d(n)$ in $\text{Flag}_d(\text{Sym}_d^\bullet \mathbb{C}^n)$ via the following rank conditions:

$$\begin{aligned} \widetilde{\mathcal{F}}_d^{\text{reg}}(n) &= \{(F_1 \subset F_2 \subset \dots \subset F_d); \#\{k \leq i; \dim_k \leq l\} \leq \dim(F_i \cap \mathbb{C}^l) \leq \\ &\leq \#\{k \leq i; \dim_{k-1} \leq l\}\} \end{aligned} \quad (4.2.12)$$

and

$$\widetilde{\mathcal{F}}_d^{\text{reg}}(n) = \{(F_1 \subset F_2 \subset \dots \subset F_d); \dim(F_i \cap \mathbb{C}^l) \geq \#\{k \leq i; \dim_k \leq l\}\} \quad (4.2.13)$$

If $1 \leq w_1 \leq \dots \leq w_d \leq \dim_d$ is a sequence of integers

$$\text{Sch}_{w_1, \dots, w_d} = \{(F_1 \subset F_2 \subset \dots \subset F_d); \dim(F_i \cap \mathbb{C}^l) = \#\{k \leq i; d_k \leq l\}\} \quad (4.2.14)$$

defines an open Schubert cell of the partial flag variety $\text{Flag}(\text{Sym}_d^\bullet \mathbb{C}^n)$. Therefore (4.2.12) is the defining property of the union of some open Schubert cells and (4.2.13) is the closure of their union, which is a closed Schubert variety.

Schubert varieties of this type were shown to be smooth by Lakshmibai and Sandhya in [28] (see also [21], Theorem 1.1). \square

Remark 4.2.3. 1. $\widetilde{\mathcal{F}}_d(n)$ can also be described as the total space of a tower of d fibrations as follows. The base of the tower is $\mathbb{P}(\mathbb{C}^n)$, and a fiber of the first fibration over a line $l_1 \in \mathbb{P}(\mathbb{C}^n)$ is $\mathbb{P}((\mathbb{C}^n \oplus \text{Sym}^2 \mathbb{C}^n)/l_1)$. Next, the fiber of the second fibration over a point $(l_1, l_2) \in (\mathbb{P}(\mathbb{C}^n), P(\mathbb{C}^n \oplus \text{Sym}^2 \mathbb{C}^n/l_1))$ is $\mathbb{P}(\mathbb{C}^n \oplus \text{Sym}^2 \mathbb{C}^n \oplus \text{Sym}^3 \mathbb{C}^n)/(l_1 + l_2)$, etc.

2. The smooth Schubert variety $\tilde{\mathcal{F}}_d(n)$ is also a Bott-Samelson type resolution of a singular Schubert variety in the Grassmannian of d -planes in $\text{Sym}_d^\bullet \mathbb{C}^n$.

We introduce some notation associated with the quotient in Lemma 4.2.2.

Definition 4.2.4. For $\varepsilon \in \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$, thought of as a system of equations, introduce the notation

- Sol_ε for the solution set $\text{im}(\varepsilon)^\perp \otimes \mathbb{C}^k \subset \mathcal{J}_d(n, k)$, and
- $\tilde{\varepsilon}$ for the point in $\tilde{\mathcal{F}}_d(n)$ corresponding to ε .
- Clearly, $\text{Sol}_\varepsilon = \text{Sol}_{\varepsilon b}$ for $\varepsilon \in \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$ and $b \in B_R$, hence to each element $\tilde{\varepsilon} \in \tilde{\mathcal{F}}_d(n)$ we can associate a solution space $\text{Sol}_{\tilde{\varepsilon}}$, and
- these spaces form a vector bundle $\text{Sol}_{\tilde{\mathcal{F}}}$ over $\tilde{\mathcal{F}}_d(n)$
- This bundle induces a map (cf. (4.1.9))

$$\alpha : \tilde{\mathcal{F}}_d(n) \rightarrow \text{Gr}(\dim, \mathcal{J}_d(n, k)). \quad (4.2.15)$$

We can identify the bundle $\text{Sol}_{\tilde{\mathcal{F}}}$ as an associated bundle over $\tilde{\mathcal{F}}_d(n)$ as follows.

Lemma 4.2.5. Consider the bundle V over $\tilde{\mathcal{F}}_d(n)$ associated to the standard representation of B_R : $V = \mathcal{F}_d(n) \times_{B_R} \mathbb{C}_R^d$. Then the canonical pairing

$$\mathcal{F}_d(n) \times \mathcal{J}_d(n, 1) \rightarrow \text{Hom}(\mathbb{C}_R^d, \mathbb{C})$$

induces a linear bundle map from the trivial bundle with fiber $\mathcal{J}_d(n, 1)$:

$$s : \tilde{\mathcal{F}}_d(n) \rightarrow \text{Hom}(\mathcal{J}_d(n, 1), V^*)$$

such that for $\tilde{\varepsilon} \in \tilde{\mathcal{F}}_d(n)$, we have $\ker(s(\tilde{\varepsilon})) \otimes \mathbb{C}^k = \text{Sol}_{\tilde{\varepsilon}} \subset \mathcal{J}_d(n, k)$.

The proof is an exercise in linear algebra, and will be omitted.

Comparing α to the embedding ϕ_{Gr} of diagram (4.1.9), we would like to argue that the map ψ induces an embedding of $\mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1)$ into $\tilde{\mathcal{F}}_d(n)$. This seems reasonable since the natural map $\mathcal{J}_d(1, n) \rightarrow \text{Gr}(\dim, \mathcal{J}_d(n, k))$ clearly factors through the map α . There is a subtlety here, however: the map (4.2.15) is not injective, thus we have to be a bit more careful. For example, let $d = 3$, and take the points

$$\varepsilon_1 = (v_1, v_2, v_1^2) \text{ and } \varepsilon_2 = (v_1, v_1^2, v_2)$$

in $\tilde{\mathcal{F}}_3(n)$. Then $\text{Sol}_{\varepsilon_1} = \text{Sol}_{\varepsilon_2}$, but $\tilde{\varepsilon}_1 \neq \tilde{\varepsilon}_2$. Observe, however, that ε_1 and ε_2 are not in the image $\psi(\mathcal{J}_d^{\text{reg}}(1, n))$.

The following two lemmas solve the problem.

Lemma 4.2.6. *We have*

- $\psi(\mathcal{J}_d^{\text{reg}}(1, n)) \subset \mathcal{F}_d^{\text{reg}}(n)$, and
- the map α (defined in (4.2.15)) restricted to $\tilde{\mathcal{F}}_d^{\text{reg}}(n)$ is an embedding.

Proof. If $(v_1, \dots, v_d) \in \mathcal{J}_d^{\text{reg}}(1, n)$ then $v_1 \neq 0$. Since

$$\psi((v_1, \dots, v_d))(e_d) = v_1^d + (d-1)v_1^{d-2}v_2 + \dots$$

and $v_1^d \neq 0$, the first part is proved. The second part is easy linear algebra. \square

Lemma 4.2.7. *Let X, Y be two sets acting on by the groups G_X, G_Y , respectively, and Z an arbitrary set. Suppose we have maps $\rho : X \rightarrow Y, \mu : X \rightarrow Z$ and $\nu : Y \rightarrow Z$ such that $\mu = \nu \circ \rho$. If μ and ν are constant on the orbits and injective on the orbit spaces, i.e they descend to embeddings $\tilde{\mu} : X/G_X \hookrightarrow Z, \tilde{\nu} : Y/G_Y \hookrightarrow Z$, then ρ descends to an embedding $\tilde{\rho} : X/G_X \hookrightarrow Y/G_Y$.*

Proof. An easy exercise, left to the reader. \square

Proposition 4.2.8. *The map ψ in (4.2.3) induces an embedding*

$$\phi_{\tilde{\mathcal{F}}} : \mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) \hookrightarrow \tilde{\mathcal{F}}_d(n), \text{ and } \phi_{\tilde{\mathcal{F}}}^*(\text{Sol}_{\tilde{\mathcal{F}}}) = \text{Sol}.$$

Proof. Using Lemma 5.2.26, we can apply Lemma 5.2.27 with $X = \mathcal{J}_d^{\text{reg}}(1, n), Y = \mathcal{F}_d^{\text{reg}}(n), Z = \text{Gr}(\dim, \mathcal{J}_d(n, k))$, and $\rho = \psi, \tilde{\mu} = \phi_{\text{Gr}}, \tilde{\nu} = \alpha$. \square

We can summarize the situation in the following diagram:

$$\begin{array}{ccccc}
 \text{Sol} & & \text{Sol}_{\tilde{\mathcal{F}}} \subset & \mathcal{J}_d(n, k) & \xrightarrow{s} & V^* \otimes \mathbb{C}^k \\
 & \searrow & \downarrow & \downarrow & \swarrow & \\
 & & \mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) & \xrightarrow{\phi_{\tilde{\mathcal{F}}}} & \tilde{\mathcal{F}}_d(n) & \\
 & & \downarrow \phi_{\text{Gr}} & & \downarrow \alpha & \\
 & & & & \text{Gr}(\dim, \mathcal{J}_d(n, k)) &
 \end{array} \tag{4.2.16}$$

The main point is that we managed to factor the map ϕ_{Gr} through the map α , and, at the same time, we identified the bundle Sol as the pull-back of an associated bundle.

4.2.2 Fibration over the flag variety

In the previous paragraph we took advantage of the special “filtered” form of the system (4.1.6) to replace the Grassmannian from (4.1.9) with the space of linear systems $\tilde{\mathcal{F}}_d(n)$. In this second part of the section, we further refine this construction.

We start with a closer look at the “natural” identification (4.2.2). In fact, the two objects are rather different: $\mathcal{J}_d(1, n)$ is a module over $\text{Diff}_d(n) \times \text{Diff}_d(1)$ while $\text{Hom}(\mathbb{C}^d, \mathbb{C}^n)$ is a module over $\text{GL}_n \times \text{GL}_d$; in addition, note that we have the following rather odd inclusions:

$$\text{Diff}_d(1) \subset \text{GL}_d, \quad \text{GL}_n \subset \text{Diff}_d(n). \quad (4.2.17)$$

By a straightforward computation, the first of the two inclusions may be made more precise as follows.

Lemma 4.2.9. *Under the identification (4.2.2), a substitution*

$$\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_d t^d \in \text{Diff}_d(1)$$

corresponds to the upper-triangular matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_d \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{d-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{d-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^d \end{pmatrix};$$

the coefficient in the i th row and j th column is

$$\sum_{\text{sum}(\tau)=j, |\tau|=i} \text{perm}(\tau) \alpha_\tau,$$

where the notation $\alpha_\tau = \prod_{i \in \tau} \alpha_i$ was used. This correspondence establishes an isomorphism of $\text{Diff}_d(1)$ with a d -dimensional subgroup H_d of the Borel subgroup $B_d \subset \text{GL}_d$.

Lemma 4.2.9, at least formally, allows us to fiber $\mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1)$ over the set $\text{Hom}(\mathbb{C}_L^d, \mathbb{C}^n)/B_L$. To make this precise, consider the subspace of injective linear maps:

$$\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) = \{\gamma \in \text{Hom}(\mathbb{C}_L^d, \mathbb{C}^n); \ker(\gamma) = 0\} \quad (4.2.18)$$

The following statements are standard:

Lemma 4.2.10. • *Under the identification (4.2.2), the space $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$ is a dense, open subset of $\mathcal{J}_d^{\text{reg}}(1, n)$.*

- *The action of B_L on $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$ is free, and the quotient $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/B_L$ is the compact, smooth variety of full flags of d -dimensional subspaces of \mathbb{C}^n :*

$$\text{Flag}_d(\mathbb{C}^n) = \{0 = F_0 \subset F_1 \subset \dots \subset F_d \subset \mathbb{C}^n, \dim F_l = l\}.$$

- The residual action of GL_n on $\text{Flag}_d(\mathbb{C}^n)$ is transitive.

Since fibrations over $\text{Flag}_d(\mathbb{C}^n)$ will play a major role in what follows, we introduce some notation related to the quotient described in Lemma 4.2.10.

Definition 4.2.11. • Denote by γ_{ref} the *reference* sequence

$$\gamma_{\text{ref}} = (e_1, \dots, e_d) \in \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n),$$

and by \mathbf{f}_{ref} the corresponding flag in $\text{Flag}_d(\mathbb{C}^n)$.

- For a manifold X endowed with a left B_L -action, denote by $\text{Ind}(X)$ the induced space $\text{Ind}(X) = \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} X$. In particular, we have $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) = \text{Ind}(B_L \gamma_{\text{ref}})$.

Identifying the fiber of the fibration $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L \rightarrow \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/B_L$ over \mathbf{f}_{ref} with B_L/H_L , we can write

$$\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L = \text{Ind}(B_L/H_L).$$

So the fibration of (an open part of) $\mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1)$ over $\text{Flag}_d(\mathbb{C}^n)$ may be presented as follows

$$\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L = \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} (B_L/H_L).$$

This suggests investigating the systems of equations (4.1.6) along the fiber $B_L \gamma_{\text{ref}}$ of $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$ over the point $\mathbf{f}_{\text{ref}} \in \text{Flag}_d(\mathbb{C}^n)$. To inspect these systems, we will write them down in the standard basis of $\text{Sym}_d^{\bullet} \mathbb{C}^n$:

$$e_{\tau} = e_{i_1} \cdots e_{i_m}, \quad \text{where } \tau = [i_1, \dots, i_m] \text{ and } \max(\tau) \leq n;$$

as before, the corresponding components are denoted by

$$\Psi_{\tau} = \Psi^m(e_{i_1}, \dots, e_{i_m}).$$

We start with the *reference system* $\varepsilon_{\text{ref}} = \psi(\gamma_{\text{ref}})$:

$$\varepsilon_{\text{ref}} = \left\{ \sum_{\text{sum}(\tau)=l} \text{perm}(\tau) \Psi_{\tau} = 0, \quad l = 1, 2, \dots, d \right\}. \quad (4.2.19)$$

With the convention of using the m th capital letter of the alphabet for Ψ^m , the first four equations of ε_{ref} look as follows:

$$\begin{aligned} A_1 &= 0 & (4.2.20) \\ A_2 + B_{11} &= 0 \\ A_3 + 2B_{12} + C_{111} &= 0 \\ A_4 + 2B_{13} + B_{22} + 3C_{112} + D_{1111} &= 0 \end{aligned}$$

Now consider a general element of $B_L\gamma_{\text{ref}}$, a test curve over the reference flag:

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \cdot \\ 0 & \beta_{22} & \beta_{23} & \cdot \\ 0 & 0 & \beta_{33} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \cdot \gamma_{\text{ref}} = (\beta_{11}e_1, \beta_{22}e_2 + \beta_{12}e_1, \beta_{33}e_3 + \beta_{23}e_2 + \beta_{13}e_1 \dots).$$

The first 3 equations of the corresponding system (4.1.6) are

$$\begin{aligned} \beta_{11}A_1 &= 0 & (4.2.21) \\ \beta_{22}A_2 + \beta_{12}A_1 + (\beta_{11})^2B_{11} &= 0 \\ \beta_{33}A_3 + \beta_{23}A_2 + \beta_{13}A_1 + 2\beta_{11}\beta_{22}B_{12} + 2\beta_{11}\beta_{12}B_{11} + (\beta_{11})^3C_{111} &= 0; \end{aligned}$$

these are thus of the form

$$\begin{aligned} u_1^1A_1 &= 0 & (4.2.22) \\ u_2^2A_2 + u_1^2A_1 + u_{11}^2B_{11} &= 0 \\ u_3^3A_3 + u_2^3A_2 + u_1^3A_1 + 2u_{12}^3B_{12} + u_{11}^3B_{11} + u_{111}^3C_{111} &= 0, \end{aligned}$$

with some complex coefficients of the form u_τ^m , where m is the ordinal number of the equation, while τ marks the component of Ψ . Studying which components of the mapjet Ψ actually appear in this system, we can conclude the following:

Lemma 4.2.12. *The system of equations (4.1.6) corresponding to a test curve $\gamma \in B_L\gamma_{\text{ref}}$ is of the form*

$$\sum_{\text{sum}(\tau) \leq l} \text{perm}(\tau) u_\tau^l \Psi_\tau = 0, \quad l = 1, 2, \dots, d, \quad (4.2.23)$$

where u_τ^l , $\text{sum}(\tau) \leq l \leq d$, are some complex coefficients.

In other words, in the l th equations of these systems, only the components Ψ_τ satisfying $\text{sum}(\tau) \leq l$ will appear.

We can formalize this simple point as follows: introduce a new filtered vector space $\text{Ym}^\bullet\mathbb{C}_L^d$:

$$\text{Ym}^\bullet\mathbb{C}_L^d = \bigoplus_{\text{sum}(\tau) \leq d} \mathbb{C}\gamma_\tau \supset \bigoplus_{\text{sum}(\tau) \leq d-1} \mathbb{C}\gamma_\tau \supset \dots \supset \mathbb{C}e_2 \oplus \mathbb{C}e_1^2 \oplus \mathbb{C}e_1 \supset \mathbb{C}e_1; \quad (4.2.24)$$

the notation is motivated by the fact that $\text{Ym}^\bullet\mathbb{C}_L^d$ is a truncation of $\text{Sym}_d^\bullet\mathbb{C}^n$. Observe that the space $\text{Ym}^\bullet\mathbb{C}_L^d$ is a left-right representation of $B_L \times B_R$, and consider the diagram

$$\begin{array}{ccc} & \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet\mathbb{C}^n) & \\ & \nearrow \psi & \uparrow \kappa \\ \text{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) & \xrightarrow{\phi} & \text{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet\mathbb{C}_L^d) \end{array} \quad (4.2.25)$$

where the horizontal arrow is the correspondence $\gamma \mapsto (\gamma, \varepsilon_{\text{ref}})$, while the map κ is obtained by composing the linear map $\mathbb{C}_R^d \rightarrow \text{Ym}^\bullet \mathbb{C}_L^d$ with the substitution $\mathbb{C}_L^d \rightarrow \mathbb{C}^n$. This latter map is clearly invariant under $\text{GL}_{\mathbb{C}_L^d}$, and hence under B_L . One can easily see that the diagram (4.2.25) is commutative, and hence it establishes a factorization of the map ψ defined in (4.2.1).

Now we introduce an analog of $\tilde{\mathcal{F}}_d(n)$ as follows:

$$\mathcal{E} = \{\varepsilon \in \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d); \ker(\varepsilon) = 0\}, \quad (4.2.26)$$

Proposition 4.2.13. *1. The subspace $\mathcal{E} \subset \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$ is invariant under the left-right action of $B_L \times B_R$.*

2. The Zariski closure of the quotient $\tilde{\mathcal{E}} = \mathcal{E}/B_R$ is a smooth, compact Schubert variety endowed with a left action of B_L .

3. The map κ in diagram (4.2.25) is B_R -equivariant, and induces a map $\tilde{\kappa} : \text{Ind}(\tilde{\mathcal{E}}) \rightarrow \tilde{\mathcal{F}}_d(n)$.

4. The horizontal map in the diagram descends to a map $\phi_{\tilde{\mathcal{E}}} : \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L \hookrightarrow \text{Ind}(\tilde{\mathcal{E}})$.

5. the restriction of the embedding

$$\phi_{\tilde{\mathcal{F}}} : \mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) \hookrightarrow \mathcal{F}_d(n)$$

to $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L$ factorizes as $\tilde{\kappa} \circ \phi_{\tilde{\mathcal{E}}}$.

Proof. The first part is obvious, while the second can be proved the same way as Lemma 4.2.2, applied to the filtration (4.2.24). If $\dim_1, \dots, \dim_d = \dim \text{Ym}^\bullet \mathbb{C}_L^d$ denote the dimensions of the subspaces in the filtration (4.2.24), then the flags in $\tilde{\mathcal{E}}$ satisfy the same relations (4.2.13).

The B_R -equivalence of κ is clear, and $\ker(\varepsilon) = 0$ implies that $\ker(\kappa(\varepsilon)) = 0$, so κ induces a map $\tilde{\kappa} : \text{Ind}(\tilde{\mathcal{E}}) \rightarrow \tilde{\mathcal{F}}_d(n)$.

Consider the map

$$B_L \hookrightarrow \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$$

defined by $b \mapsto (e_1, \dots, e_d) \cdot b$ for $b \in B_L$. Since $B_L \subset \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) \subset \mathcal{J}_d^{\text{reg}}(1, n)$, the embedding $\phi_{\tilde{\mathcal{F}}} : \mathcal{J}_d^{\text{reg}}(1, n)/\text{Diff}_d(1) \rightarrow \tilde{\mathcal{F}}_d(n)$ of Corollary 4.2.8 restricts to $\phi_{B_L/H_L} : B_L/H_L \hookrightarrow \tilde{\mathcal{F}}_d(n)$, which is induced from ψ of the diagram (4.2.25). But $\psi(B_L) \subset \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$, so $\text{im}(\phi_{B_L/H_L}) \subset \tilde{\mathcal{E}}$. This defines the embedding

$$\phi_{\tilde{\mathcal{E}}} : \text{Ind}(B_L/H_L) \hookrightarrow \text{Ind}(\tilde{\mathcal{E}})$$

We can identify $\text{Ind}(B_L/H_L)$ with $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L$, and we get the fourth statement. The last statement comes from the definition of ψ . □

Remark 4.2.14. 1. The closure of the image of the embedding $B_L/H_L \hookrightarrow \tilde{\mathcal{E}}$ defined in the proof of Proposition 4.2.13 is a subvariety of a Schubert variety in $\tilde{\mathcal{E}}$.

2. In the proof of Lemma 4.3.10 we construct an affine cover of the Schubert variety $\tilde{\mathcal{E}}$, and use this cover to describe the weights of the T_L action at the fixed points.

The existence of the embedding

$$\phi_{\tilde{\mathcal{E}}} : \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L \hookrightarrow \text{Ind}(\tilde{\mathcal{E}})$$

of Proposition 4.2.13 means that ϕ maps H_L -orbits to the same point, i.e H_L is part of the stabilizer of $\tilde{\varepsilon}_{\text{ref}} = \text{pr}(\varepsilon_{\text{ref}}) \in \tilde{\mathcal{E}}$ under the action of B_L on $\tilde{\mathcal{E}}$. Since different H_L orbits map to different points, the stabilizer not bigger than H_L , and we have

Corollary 4.2.15. *Let $\tilde{\varepsilon}_{\text{ref}} \in \tilde{\mathcal{E}}$ be the reference point $\text{pr}(\varepsilon_{\text{ref}})$, where $\text{pr} : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ is the projection. The stabilizer of the B_L -action on $\tilde{\mathcal{E}}$ of the point $\tilde{\varepsilon}_{\text{ref}}$ is the subgroup $H_L \subset B_L$.*

Combining the results of Proposition 4.2.13 with diagram (4.2.16), we arrive at the following picture:

$$\begin{array}{ccccc}
 B_L/H_L & \hookrightarrow & \tilde{\mathcal{E}} & & \text{Sol}_{\tilde{\mathcal{F}}} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L & \xrightarrow{\phi_{\tilde{\mathcal{E}}}} & \text{Ind}(\tilde{\mathcal{E}}) & \xrightarrow{\tilde{\kappa}} & \tilde{\mathcal{F}}_d(n) & \leftarrow & \mathcal{J}_d(n, k) \quad (4.2.27) \\
 \searrow^{\tau_{\text{FI}}} & & \searrow^{\tau_{\text{FI}}} & & \swarrow & & \downarrow \\
 & & \text{Flag}_d(\mathbb{C}^n) & & V^* \otimes \mathbb{C}^k & & \downarrow \\
 & & & & & & s \\
 & & & & & & \downarrow \\
 & & & & & & V^* \otimes \mathbb{C}^k
 \end{array}$$

We can formulate our model as follows.

- Consider the fibered product $V = \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} \mathcal{E} \times_{B_R} \mathbb{C}_R^d$, resulting in the double fibration

$$\text{Flag}_d(\mathbb{C}^n) \longleftarrow \text{Ind}(\tilde{\mathcal{E}}) \xleftarrow{\tau_V} V$$

where \mathcal{E} is defined in (4.2.26), and $\tilde{\mathcal{E}} = \mathcal{E}/B_R$.

- Let

$$\begin{array}{ccc}
 \text{Sol}_{\tilde{\mathcal{E}}} & \xrightarrow{\text{ev}_{\tilde{\mathcal{E}}}} & \mathcal{J}_d(n, k) \xrightarrow{s} V^* \otimes \mathbb{C}^k \\
 \searrow^{\tau_{\tilde{\mathcal{E}}}} & & \swarrow \\
 & & \text{Ind}(\tilde{\mathcal{E}})
 \end{array}$$

be a short exact sequence of vector bundles, where the map s is obtained as the adjoint of the natural composition map $V \rightarrow \mathrm{Sym}_d^\bullet \mathbb{C}^n$ tensored with the identity on \mathbb{C}^k .

Proposition 4.2.16. *The evaluation map $\mathrm{ev}_{\tilde{\mathcal{E}}}$ establishes a degree-1 map between $\tau_{\tilde{\mathcal{E}}}^{-1}(\mathrm{Ind}(B_L \tilde{\varepsilon}_{\mathrm{ref}}))$ and the singularity locus Θ_d .*

Proof. The map $\mathrm{ev}_{\mathrm{Sol}}$ of diagram (4.1.9) is a birational map between Sol and Θ_d . By Lemma 4.2.10, $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L$ is a dense, open subset of $\mathcal{J}_d^{\mathrm{reg}}(1, n)/\mathrm{Diff}_d(1)$, so $\mathrm{Sol}|_{\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L}$ is also birational to Θ_d . So it is enough to show, that the image of $\mathrm{Hom}^{\mathrm{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/H_L$ under the embedding $\phi_{\tilde{\mathcal{E}}}$ is $\mathrm{Ind}(B_L \tilde{\varepsilon}_{\mathrm{ref}}) \subset \mathrm{Ind}(\tilde{\mathcal{E}})$. This follows from Corollary 4.2.15, and the fact, that the image of the map

$$\phi : \mathrm{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \rightarrow \mathrm{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} \mathrm{Hom}^\Delta(\mathbb{C}_R^d, \mathrm{Ym}^\bullet \mathbb{C}_L^d)$$

of diagram (4.2.25) is by definition $\mathrm{Hom}(\mathbb{C}_L^d, \mathbb{C}^n) \times_{B_L} (B_L \varepsilon_{\mathrm{ref}}) = \mathrm{Ind}(B_L \varepsilon_{\mathrm{ref}})$. \square

4.3 Application of the localization formulas

Recall that our aim is the computation of the equivariant Poincaré dual $\mathrm{eP}[\Theta_d]$ of the subvariety $\Theta_d \subset \mathcal{J}_d(n, k)$ representing the A_d -singularity (cf. § 2.1). The symmetry group of the problem is the product of matrix groups $\mathrm{GL}_n \times \mathrm{GL}_k$; the respective subgroups of diagonal matrices are T_n with weights $(\lambda_1, \dots, \lambda_n)$ and T_k with weights $(\theta_1, \dots, \theta_k)$, hence $\mathrm{eP}[\Theta_d]$ is a bisymmetric polynomial in these two sets of variables.

In this section, we apply the localization techniques of §3 to the computation of $\mathrm{eP}[\Theta_d]$, using the model described in §4.2.2. As our model is a double fibration, the application of the localization formula is a 2-step process.

4.3.1 Localization in $\mathrm{Flag}_d(\mathbb{C}^n)$

The model of Proposition 4.2.16 is an equivariant fibration over the smooth homogeneous space $\mathrm{Flag}_d(\mathbb{C}^n)$, hence, in this case, we can use Proposition 3.1.2 with the extension (cf. § 3.3) that the fibers of S are not necessarily linear and smooth.

The data needed for formula (3.1.2) is

- the fixed point set of the T_n action on $\mathrm{Flag}_d(\mathbb{C}^n)$,
- the weights of this action on the tangents spaces $T_p \mathrm{Flag}_d(\mathbb{C}^n)$ at these fixed points,
- the equivariant Poincaré duals of the fibers at these fixed points.

The following general statement will be helpful in organizing our fixed point data. Its proof is straightforward and will be omitted.

Lemma 4.3.1. *Assume that the torus action in Proposition 3.1.2 is obtained by a restriction of a GL_n -action to its subgroup of diagonal matrices T_n . Then the Weyl group of permutation matrices \mathcal{S}_n acts on M^{T_n} , and we have*

$$\mathrm{eP}[S_{\sigma,p}, W] = \sigma \cdot \mathrm{eP}[S_p, W] \text{ and } \mathrm{Euler}^{T_n}(\mathrm{T}_{\sigma,p}M) = \sigma \cdot \mathrm{Euler}^{T_n}(\mathrm{T}_pM).$$

for all $\sigma \in \mathcal{S}_n$ and $p \in M_{T_n}$.

Our situation is fortunate in the sense that the action of \mathcal{S}_n on the fixed point set is transitive. Indeed, the fixed point set $\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}$ is the set of partial flags obtained from sequences of d elements of the basis (e_1, \dots, e_n) of \mathbb{C}^n ; in particular, $|\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}| = n(n-1) \dots (n-d+1)$.

Recall the notation $\mathbf{f}_{\mathrm{ref}}$ for the reference flag associated to the sequence (e_1, \dots, e_d) . The stabilizer subgroup of $\mathbf{f}_{\mathrm{ref}}$ in \mathcal{S}_n is the subgroup \mathcal{S}_{n-d} permuting the numbers starting with $d+1$, and the map $\sigma \mapsto \sigma \cdot \mathbf{f}_{\mathrm{ref}}$ induces a bijection between $\mathrm{Flag}_d(\mathbb{C}^n)^{T_n}$ and the quotient $\mathcal{S}_n/\mathcal{S}_{n-d}$.

According to Lemma 4.3.1, it is sufficient for us to compute the equivariant Poincaré dual of the fiber and the weights of the tangent space at the reference flag $\mathbf{f}_{\mathrm{ref}}$. The weights of $\mathrm{T}_{\mathbf{f}_{\mathrm{ref}}}\mathrm{Flag}_d(\mathbb{C}^n)$ are well-known:

$$\mathrm{Weights}_{\mathbf{f}_{\mathrm{ref}}} = \{\lambda_i - \lambda_m; 1 \leq m \leq d, m < i \leq n\};$$

the weights at the other fixed points are obtained by applying the corresponding permutation to this set.

The numerators of the summands of (3.1.2) in our case are much harder to compute, although, thanks to Lemma 4.3.1, it is sufficient to know the numerator for the fixed point $\mathbf{f}_{\mathrm{ref}}$. According to Proposition 4.2.16, the fiber over $\mathbf{f}_{\mathrm{ref}}$ is the set $\mathrm{ev}_{\tilde{\mathcal{E}}}(\tau_{\tilde{\mathcal{E}}}^{-1}(B_L \tilde{\varepsilon}_{\mathrm{ref}}))$, which consists of all possible solutions of the systems of equations of the form $B_L \varepsilon_{\mathrm{ref}}$. We wrote down these systems explicitly in (4.2.21), and saw in § 4.2.2 that all these systems are in \mathcal{E} .

Recall (cf. (2.2.3)) that we defined the equivariant Poincaré dual of a variety to be that of its closure. It will be convenient to pass to the closure

$$\mathcal{O} = \overline{B_L \tilde{\varepsilon}_{\mathrm{ref}}} \subset \tilde{\mathcal{E}}, \tag{4.3.1}$$

and write the numerator in question as the equivariant Poincaré dual

$$\mathrm{eP}[\mathrm{ev}_{\tilde{\mathcal{E}}}(\tau_{\tilde{\mathcal{E}}}^{-1}(\mathcal{O})), \mathcal{J}_d(n, k)]; \tag{4.3.2}$$

this is a polynomial in two sets of variables: $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$. Note that it is symmetric in the θ s only. The following statement is straightforward.

Lemma 4.3.2. *The equivariant Poincaré dual (4.3.2) of the fiber $\mathrm{ev}_{\tilde{\mathcal{E}}}(\tau_{\tilde{\mathcal{E}}}^{-1}(\mathcal{O}))$ does not depend on the last $n-d$ basic λ -weights: $\lambda_{d+1}, \dots, \lambda_n$.*

Indeed, systems of equations in \mathcal{E} impose conditions only on those components of Ψ which do not have indices higher than d . \square

As a consequence of Lemma 4.3.2, the equivariant Poincaré dual (4.3.2) may be considered as being taken with respect to the group $T_L \times T_k$, which has weights $\mathbf{z} = (z_1, \dots, z_d)$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$.

Now we can summarize the results of this paragraph as follows.

Corollary 4.3.3. *We have*

$$\mathrm{eP}[\Theta_d] = \sum_{\sigma \in \mathcal{S}_n / \mathcal{S}_{n-d}} \frac{Q_{\mathrm{Fl}}(\lambda_{\sigma \cdot 1}, \dots, \lambda_{\sigma \cdot d}, \boldsymbol{\theta})}{\prod_{1 \leq m \leq d} \prod_{i=m+1}^n (\lambda_{\sigma \cdot i} - \lambda_{\sigma \cdot m})}, \quad (4.3.3)$$

where

$$Q_{\mathrm{Fl}}(\mathbf{z}, \boldsymbol{\theta}) = \mathrm{eP}[\mathrm{ev}_{\tilde{\mathcal{E}}}(\tau_{\tilde{\mathcal{E}}}^{-1}(\mathcal{O})), \mathcal{J}_d(n, k)]_{T_L \times T_k}. \quad (4.3.4)$$

4.3.2 Residue formula for the cohomology pairings of $\mathrm{Flag}_d(\mathbb{C}^n)$

Usually formulas such as (4.3.3) are difficult to use: even though they have the form of a finite sum of rational functions, each term is singular, and only after summing these terms and performing some cancellations we obtain a polynomial, which obscures the result. Moreover, the number of terms of the sum grows very quickly with n and d .

In this paragraph, we derive an efficient residue formula for the right hand side of (4.3.3). While the geometric meaning of this formula is not entirely clear, our summation procedure yields an effective, “truly” localized formula; by this we mean that for its evaluation one only needs to know the behavior of a certain function at a single point, rather than at a large, albeit finite number of points.

To describe this formula, we will need the notion of an *iterated residue* (cf. e.g. [46]) at infinity. Let $\omega_1, \dots, \omega_N$ be affine linear forms on \mathbb{C}^d ; denoting the coordinates by z_1, \dots, z_d , this means that we can write $\omega_i = a_i^0 + a_i^1 z_1 + \dots + a_i^d z_d$. We will use the shorthand $h(\mathbf{z})$ for a function $h(z_1, \dots, z_d)$, and $d\mathbf{z}$ for the holomorphic d -form $dz_1 \dots dz_d$. For a Laurent polynomial $h(\mathbf{z})$ and define

$$\mathrm{Res}_{z_1=\infty} \dots \mathrm{Res}_{z_d=\infty} \frac{h(\mathbf{z}) d\mathbf{z}}{\prod_{i=1}^N \omega_i} \stackrel{\mathrm{def}}{=} \int_{|z_1|=R_1} \dots \int_{|z_d|=R_d} \frac{h(\mathbf{z}) d\mathbf{z}}{\prod_{i=1}^N \omega_i}, \quad (4.3.5)$$

where $1 \ll R_1 \ll \dots \ll R_d$. At first sight this is the same integral as the one defining the residue at the origin, but here we give the opposite orientation of the cycle $|z_i| = R_i$, e.g. $\int_{|z|=R} \frac{1}{z} dz = -1$.

In practice, one can compute this iterated residue by writing for each i

$$\frac{1}{\omega_i} = \sum_{j=0}^{\infty} (-1)^j \frac{(a_i^0 + a_i^1 y_i + \dots + a_i^{q(i)-1} z_{q(i)-1})^j}{(a_i^{q(i)} z_{q(i)})^{j+1}}, \quad (4.3.6)$$

where $q(i)$ is the largest value of m for which $a_i^m \neq 0$. Then we can obtain the iterated residue (4.3.5) by multiplying the product of these expressions with $h(z_1, \dots, z_d)$, and then taking the coefficient of $z_1^{-1} \dots z_d^{-1}$ in the resulting Laurent series with the sign $(-1)^d$.

For simplicity we introduce the shortened notation $\text{Res}_{\mathbf{z}=\infty}$ for $\text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \dots \text{Res}_{z_d=\infty}$. We have the following *iterated residue theorem*.

Proposition 4.3.4. *For a polynomial $Q(\mathbf{z})$ on \mathbb{C}^d , we have*

$$\sum_{\sigma \in \mathcal{S}_n / \mathcal{S}_{n-d}} \frac{Q(\lambda_{\sigma \cdot 1}, \dots, \lambda_{\sigma \cdot d})}{\prod_{1 \leq m \leq d} \prod_{i=m+1}^n (\lambda_{\sigma \cdot i} - \lambda_{\sigma \cdot m})} = \text{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m < l \leq d} (z_m - z_l) Q(\mathbf{z}) d\mathbf{z}}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)} \quad (4.3.7)$$

Proof. We compute the iterated residue (4.3.7) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. The first residue, which is taken with respect to z_d , is a contour integral, whose value is minus the sum of the z_d -residues of the form in (4.3.7). These poles are at $z_d = \lambda_j$, $j = 1, \dots, n$, and the resulting sum is

$$\sum_{j=1}^n \frac{\prod_{1 \leq m < l \leq d-1} (z_m - z_l) \prod_{l=1}^{d-1} (z_l - \lambda_j) Q_{\text{Fl}}(z_1, \dots, z_{d-1}, \lambda_j) dz_1 \dots dz_{d-1}}{\prod_{l=1}^{d-1} \prod_{i=1}^n (\lambda_i - z_l) \prod_{i \neq j}^n (\lambda_i - \lambda_j)}.$$

After cancellation and exchanging the sum and the residue operation, at the next step, we have

$$(-1)^{d-1} \sum_{j=1}^n \text{Res}_{z_{d-1}=\infty} \frac{\prod_{1 \leq m < l \leq d-1} (z_m - z_l) Q_{\text{Fl}}(z_1, \dots, z_{d-1}, \lambda_j) dz_1 \dots dz_{d-1}}{\prod_{i \neq j}^n \left((\lambda_i - \lambda_j) \prod_{l=1}^{d-1} (\lambda_i - z_l) \right)}.$$

Now we can repeat our previous trick of applying the Residue Theorem; the only difference is now that the pole $z_{d-1} = \lambda_j$ has been eliminated. As a result, after converting the second residue to a sum, we obtain

$$(-1)^{2d-3} \sum_{j=1}^n \sum_{s=1, s \neq j}^n \frac{\prod_{1 \leq m < l \leq d-2} (z_l - z_m) Q_{\text{Fl}}(z_1, \dots, z_{d-2}, \lambda_s, \lambda_j) dz_1 \dots dz_{d-2}}{(\lambda_s - \lambda_j) \prod_{i \neq j, s}^n \left((\lambda_i - \lambda_j) (\lambda_i - \lambda_s) \prod_{l=1}^{d-1} (\lambda_i - z_l) \right)}.$$

Iterating this process, we arrive at a sum very similar to (4.3.3). The difference between the two sums will be the sign: $(-1)^{d(d-1)/2}$, and that the $d(d-1)/2$ factors of the form $(\lambda_{\sigma(i)} - \lambda_{\sigma(m)})$ with $1 \leq m < i \leq d$ in the denominator will have the opposite signs. These two differences cancel each other, and this completes the proof. \square

Remark 4.3.5. Usually, changing the order of the variables in iterated residues changes the result. In this case, however, because all the poles are normal crossing, we would have obtained the same result with residues taken in any other order.

4.3.3 Localization in the fiber

Combining Corollary 4.3.3 with Proposition 4.3.4, we arrive at the formula

$$\mathrm{eP}[\Theta_d, \mathcal{J}_d(n, k)] = \mathrm{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m < l \leq d} (z_m - z_l) Q_{\mathrm{Fl}}(\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z}}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)}, \quad (4.3.8)$$

where $Q_{\mathrm{Fl}}(\mathbf{z}, \boldsymbol{\theta})$ is defined in (4.3.4), and, therefore, we turn to the computation of this polynomial.

Let us briefly review the construction of $Q_{\mathrm{Fl}}(\mathbf{z}, \boldsymbol{\theta})$ (cf. diagram (4.2.2), Lemma 4.3.2, Corollary 4.3.3). This polynomial is an equivariant Poincaré dual taken with respect to the group $T_L \times T_k$ which has weights (z_1, \dots, z_d) and $(\theta_1, \dots, \theta_k)$. Consider the $B_L \times B_R$ -module $\mathrm{Hom}^\Delta(\mathbb{C}_R^d, \mathrm{Ym}^\bullet \mathbb{C}_L^d)$, and endow it with coordinates $u_\tau^l \in \mathrm{Hom}^\Delta(\mathbb{C}_R^d, \mathrm{Ym}^\bullet \mathbb{C}_L^d)^*$, indexed by pairs $(\tau, l) \in \Pi \times \mathbb{Z}_{>0}$ satisfying $\mathrm{sum}(\tau) \leq l \leq d$. We will consider this dual space as carrying a *right* action of $T_L \times T_k$; accordingly,

$$\text{the weight of } u_\tau^l = (z_{i_1} + z_{i_2} + \dots + z_{i_m}, \theta_l), \text{ where } \tau = [i_1, i_2, \dots, i_m]. \quad (4.3.9)$$

For each nondegenerate system $\varepsilon \in \mathcal{E} \subset \mathrm{Hom}^\Delta(\mathbb{C}_R^d, \mathrm{Ym}^\bullet \mathbb{C}_L^d)$ denote the image in the quotient $\tilde{\mathcal{E}} = \mathcal{E}/B_R$ by $\tilde{\varepsilon}$; in particular, we have a reference point $\tilde{\varepsilon}_{\mathrm{ref}} \in \tilde{\mathcal{E}}$ corresponding to the system $\varepsilon_{\mathrm{ref}}$ given by

$$u_\pi^l(\tilde{\varepsilon}_{\mathrm{ref}}) = \begin{cases} 1, & \text{if } \mathrm{sum}(\tau) = l \\ 0, & \text{otherwise.} \end{cases} \quad (4.3.10)$$

The stabilizer subgroup of $\tilde{\varepsilon}_{\mathrm{ref}} \in \tilde{\mathcal{E}}$ under the B_L -action is a d -dimensional subgroup $H_L \in B_L$, hence the orbit $B_L \tilde{\varepsilon}_{\mathrm{ref}} \subset \tilde{\mathcal{E}}$ is a subvariety of dimension $d(d-1)/2$; we denoted the closure of this subvariety by \mathcal{O} .

Next consider the vector bundle

$$V = \mathcal{E} \times_{B_R} \mathbb{C}_R^d \longrightarrow \tilde{\mathcal{E}} = \mathcal{E}/B_R$$

associated to the standard representation of B_R , and the $T_L \times T_k$ -equivariant linear bundle map from a trivial bundle

$$s : \tilde{\mathcal{E}} \times \mathcal{J}_d(n, k) \longrightarrow V^* \otimes \mathbb{C}^k$$

defined by the natural compositions. Then, according to Proposition 4.2.16, the polynomial $Q_{\mathrm{Fl}}(\mathbf{z}, \boldsymbol{\theta})$ is the *equivariant Poincaré dual of the projection to $\mathcal{J}_d(n, k)$ of the restriction $\ker(s)|_{\mathcal{O}}$ of the vector bundle $\ker(s)$* . This is, naturally, identical to the variety $\mathrm{ev}_{\tilde{\varepsilon}}(\tau_{\tilde{\varepsilon}}^{-1}(\mathcal{O}))$ which appears in the definition (4.3.4).

While the variety \mathcal{O} is highly singular, the set of T_L -fixed points of \mathcal{O} is finite – as we will see shortly – and hence we can apply here the localization principle based on Rossmann’s integration formula: Proposition 3.3.3. The result is (cf. (3.3.4)):

$$Q_{\mathrm{Fl}}(\mathbf{z}, \boldsymbol{\theta}) = \sum_{p \in \mathcal{O}^{T_d}} \frac{\mathrm{Euler}^{T_d \times T_k}(V_p^* \otimes \mathbb{C}^k) \mathrm{emult}(\hat{T}_p \mathcal{O}, \tilde{\mathcal{E}})}{\mathrm{Euler}^{T_d \times T_k}(T_p \tilde{\mathcal{E}})}. \quad (4.3.11)$$

Our task thus has reduced to the identification and computation of the objects in this formula. These are:

- The set \mathcal{O}^{T_d} of T_d -fixed points in $\mathcal{O} \subset \tilde{\mathcal{E}}$,
- the weights of the T_d -action on the fibers V_p for $p \in \mathcal{O}^{T_d}$,
- the weights of the T_d -action on the tangent spaces $T_p \tilde{\mathcal{E}}$ for $p \in \mathcal{O}^{T_d}$,
- the equivariant multiplicities of \mathcal{O} in $\tilde{\mathcal{E}}$ at each fixed point $p \in \mathcal{O}^{T_d}$.

The most immediate problem we face is that we do not have an effective description of the set \mathcal{O}^{T_d} of T_d -fixed points in \mathcal{O} . There is a formal way around this: we replace the fixed point set \mathcal{O}^{T_d} with the larger set $\tilde{\mathcal{E}}^{T_d}$ and define the equivariant multiplicity $\text{emult}(\hat{T}_p \mathcal{O}, \tilde{\mathcal{E}})$ to be zero in the case when $p \in \tilde{\mathcal{E}}^{T_d} \setminus \mathcal{O}^{T_d}$.

The set of fixed points $\tilde{\mathcal{E}}^{T_d}$ is fairly easy to determine: these fixed points are given by those nondegenerate systems $\varepsilon \in \mathcal{E} \subset \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$ for which the tensors $\varepsilon(e_m) \in \text{Ym}^\bullet \mathbb{C}_L^d$, $m = 1, \dots, d$ are of pure T_d -weight. These, in turn, may be enumerated as follows.

Definition 4.3.6. We will call a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \Pi_d^{\times d}$ *admissible* if

- $\text{sum}(\pi_l) \leq l$ for $l = 1, \dots, d$ and
- $\pi_l \neq \pi_m$ for $1 \leq l \neq m \leq d$.

We will denote the set of admissible sequences by $\mathbf{\Pi}_d$.

As an example, consider the case $d = 3$. The set of admissible partition sequences of length 3 consists of the following 8 elements:

$$\mathbf{\Pi}_3 = \{([1], [2], [3]), ([1], [2], [1, 2]), ([1], [2], [1, 1]), ([1], [2], [1, 1, 1]), \\ ([1], [1, 1], [3]), ([1], [1, 1], [1, 1, 1]), ([1], [1, 1], [2]), ([1], [1, 1], [1, 2])\};$$

For an admissible $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \mathbf{\Pi}_d$ introduce the system ε_π given by

$$u_\tau^l(\varepsilon_\pi) = \begin{cases} 1 & \text{if } \tau = \pi_l, \\ 0 & \text{otherwise.} \end{cases}$$

As usual, the point corresponding to ε_π in $\tilde{\mathcal{E}}$ will be denoted by $\tilde{\varepsilon}_\pi$.

The following statement is a straightforward exercise.

Lemma 4.3.7. • *The correspondence $\boldsymbol{\pi} \mapsto \tilde{\varepsilon}_\pi$ establishes a bijection between the set $\mathbf{\Pi}_d$ of admissible sequences of partitions and the fixed point set $\tilde{\mathcal{E}}^{T_d}$.*

- For $\tau \in \Pi$ and an integer i , denote by $\text{mult}(i, \tau)$ the number of times i occurs in τ , and let $z_\tau = \sum_{i \in \tau} \text{mult}(i, \tau) z_i$. Then, given an admissible sequence $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, the weight of the T_d -action on the fiber of V at the fixed point $\tilde{\varepsilon}_\boldsymbol{\pi}$ is

$$(z_{\pi_1}, \dots, z_{\pi_d}).$$

Corollary 4.3.8. *The weights of the $T_d \times T_k$ action on fiber $V_{\tilde{\varepsilon}_\boldsymbol{\pi}}^* \otimes \mathbb{C}^k$ are given as $\{\theta_j - z_{\pi_m}; m = 1, \dots, d, j = 1, \dots, k\}$.*

Next we turn to the 3rd item on our list: the weights of the T_d -action on tangent space of $\tilde{\mathcal{E}}$ at the fixed points $\tilde{\varepsilon}_\boldsymbol{\pi}$; we will use the simplified notation $T_\boldsymbol{\pi} \tilde{\mathcal{E}}$ for this tangent space.

We need to introduce a few additional objects indexed by the finite set $\mathbf{\Pi}_d$. Recall from Proposition 4.2.13 that $\tilde{\mathcal{E}}$ is a Schubert variety in the partial flag manifold

$$\text{Flag}_d(\text{Ym}^\bullet \mathbb{C}_L^d) = \{0 = F_0 \subset F_1 \subset \dots \subset F_d \subset \text{Ym}^\bullet \mathbb{C}_L^d, \dim F_l = l\}. \quad (4.3.12)$$

We use the notation $\tilde{\text{pr}} : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ for the projection to the quotient, as introduced earlier.

Definition 4.3.9. For each $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \mathbf{\Pi}_d$ introduce the affine-linear subspace $\mathcal{N}_\boldsymbol{\pi} \subset \mathcal{E}$ given by

$$\mathcal{N}_\boldsymbol{\pi} = \left\{ \varepsilon \in \mathcal{E}; u_{\pi_l}^m(\varepsilon) = \begin{cases} 1 & \text{if } m = l \\ 0 & \text{if } m > l \end{cases} \text{ for } 1 \leq l \leq d \right\};$$

Clearly, $\varepsilon_\boldsymbol{\pi} \in \mathcal{N}_\boldsymbol{\pi}$, and considering this special point to be the origin, we will think of $\mathcal{N}_\boldsymbol{\pi}$ as *linear* space. Then $\mathcal{N}_\boldsymbol{\pi}$ is endowed with a natural set of coordinates:

$$\hat{u}_{\tau|\boldsymbol{\pi}}^l = u_\tau^l|_{\mathcal{N}_\boldsymbol{\pi}}, \text{sum}(\tau) \leq l \leq d, \tau \neq \pi_1, \dots, \pi_l. \quad (4.3.13)$$

Proposition 4.3.10. *Let $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ be an admissible partition. Then*

- the restriction of the projection $\tilde{\text{pr}} : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ to $\mathcal{N}_\boldsymbol{\pi}$ is an embedding and the collection $\{\tilde{\text{pr}}(\mathcal{N}_\boldsymbol{\pi}); \boldsymbol{\pi} \in \mathbf{\Pi}_d\}$ forms an open cover of $\tilde{\mathcal{E}}$.
- If $\text{defect}(\boldsymbol{\pi}) = 0$, then $\tilde{\text{pr}}(\mathcal{N}_\boldsymbol{\pi}) \subset \tilde{\mathcal{E}}$ is B_d -invariant, and the induced B_d -action on $\mathcal{N}_\boldsymbol{\pi}$ is linear.
- for any $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, the image $\tilde{\text{pr}}(\mathcal{N}_\boldsymbol{\pi}) \subset \tilde{\mathcal{E}}$ is T_d -invariant, and the induced T_d -action on $\mathcal{N}_\boldsymbol{\pi}$ is linear, and diagonal with respect to the coordinates (4.3.13). Again considering T_d acting on the right on these coordinates,

$$\text{the weight of } \hat{u}_{\tau|\boldsymbol{\pi}}^l = z_\tau - z_{\pi_l}. \quad (4.3.14)$$

Proof. We first show that $\cup \{\tilde{\text{pr}}(\mathcal{N}_\pi); \pi \in \mathbf{\Pi}_d\} = \tilde{\mathcal{E}}$. This means that for an arbitrary element $\varepsilon \in \mathcal{E}$, we have to find an admissible partition $\pi \in \mathbf{\Pi}_d$ and an upper-triangular matrix $b_R = b_R(\varepsilon, \pi) \in B_R$ such that $\varepsilon \cdot b_R \in \mathcal{N}_\pi$. This can be done by elementary column operations: consider ε as a $\dim(\text{Ym}^\bullet \mathbb{C}_L^d) \times d$ matrix whose columns are linearly independent, and whose rows are indexed by partitions. The only nonzero entry in the first column corresponds to the trivial partition [1]; then we can multiply the first column by a constant to rescale this entry to 1, and then annihilate all other entries in the same row by adding multiples of the first column to the others. Next, since ε is nonsingular, we can pick a nonzero entry in the second column of the resulting matrix – this entry will correspond to a partition π_2 – and, again, using column operations, we annihilate all entries in this row starting from column 3 and so on. Continuing this process, we obtain an admissible $\pi = (\pi_1, \dots, \pi_d)$, and the described sequence of column operations produces an upper-triangular $b_R \in B_R$ such that $\varepsilon \cdot b_R \in \mathcal{N}_\pi$.

More formally, for each $\pi \in \mathbf{\Pi}_d$ introduce the map

$$\alpha_\pi : \text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d) \rightarrow \text{Mat}^{d \times d}$$

which associates to each system ε its $d \times d$ minor corresponding to partitions in π . The process described above finds an appropriate $\pi \in \mathbf{\Pi}_d$ for each ε , and brings $\alpha_\pi(\varepsilon)$ to lower-triangular form.

Now we turn to the second part of Proposition 4.3.10. By Definition 4.3.9,

$$\mathcal{N}_\pi = \{\varepsilon \in \mathcal{E}; \alpha_\pi(\varepsilon) \in U_-\} \quad (4.3.15)$$

where U_- is unipotent subgroup of lower triangular $d \times d$ matrices with 1s on the diagonal; hence,

$$\tilde{\text{pr}}^{-1}(\tilde{\text{pr}}(\mathcal{N}_\pi)) = \mathcal{N}_\pi B_R = \{\varepsilon \in \mathcal{E} : \alpha_\pi(\varepsilon) \in U_- B_R\}.$$

Note that $U_- B_R \cong U_- \times B_R$ is an open cell in the group of invertible matrices, and, whenever $a \in U_- B_R$, denote by a_R the projection on to the upper-triangular component; thus $a \cdot a_R^{-1} \in U_-$.

This allows us to rewrite the B_d -invariance as the following statement:

$$\text{if } \text{defect}(\pi) = 0 \text{ and } \varepsilon \in \mathcal{N}_\pi, \text{ then for } b_L \in B_L \text{ we have } \alpha_\pi(b_L \varepsilon) \in U_- B_R. \quad (4.3.16)$$

Indeed, observe that under these assumptions, the filtration preserving property implies that $\alpha_\pi(\varepsilon)$ is upper-triangular. This, in turn, allows us to conclude that if the diagonal entries of $b \in B_d$ are $\exp(z_1), \dots, \exp(z_d)$, then

$$\alpha_\pi(b_L \varepsilon) = \text{diag}[\exp(z_{\pi_1}), \dots, \exp(z_{\pi_d})] \cdot \alpha_\pi(\varepsilon), \quad (4.3.17)$$

where $\text{diag}[w_1, \dots, w_d]$ denotes the diagonal matrix with the corresponding entries. This immediately implies the B_d -invariance and linearity.

In fact, now we have an explicit description of the induced linear action of B_d on \mathcal{N}_π when $\text{defect}(\pi) = 0$. Indeed, for any $\pi \in \mathbf{\Pi}_d$ define

$$\beta_\pi : B_d \rightarrow \text{GL}(\mathcal{N}_\pi), \quad \beta_\pi(b) \varepsilon = b_L \cdot \varepsilon \cdot (\alpha_\pi(b_L \cdot \varepsilon)_R)^{-1}. \quad (4.3.18)$$

The third statement of Proposition 4.3.10 now also easily follows. The action (4.3.18) is not defined for all $b \in B_d$ if the defect of π is not zero, because, in general, we cannot guarantee that $\alpha_\pi(b_L \cdot \varepsilon)_R \in U_- B_R$. This problem does not arise, however, if $b \in T_d$, i.e. when b is diagonal, because then trivially $\alpha_\pi(b_L \cdot \varepsilon)_R \in B_R$.

To compute the weights observe that T_d acts diagonally on \mathcal{E} ; the weight of the coordinate u_τ^l is z_τ under this action. Combining (4.3.18) with formula (4.3.17), we see that the T_d -weight of u_τ^l is (see the remark after this proof)

$$z_\tau - z_{\pi_l}. \quad (4.3.19)$$

Recall that these coordinates on \mathcal{E} are indexed by pairs $(\tau, l) \in \Pi \times \mathbb{Z}_{>0}$ satisfying $\text{sum}(\tau) \leq l \leq d$. The open chart \mathcal{N}_π is defined as the subspace $\{\varepsilon : u_{\tau_m}^l(\varepsilon) = 0\}$ for $d \geq l \geq m$, and these coordinates remain 0 after any diagonal action, so $\rho_\pi(T_d)\mathcal{N}_\pi = \mathcal{N}_\pi$.

The weight of $\rho_\pi(T_d)$ on $\hat{u}_{\tau|\pi}^l$ is $z_\tau - z_{\pi_l}$ by (4.3.19), and the third part of Lemma 4.3.10 is proved. \square

Remark 4.3.11. If we have a left torus action on a vector space, the corresponding action on the dual vector space can be defined either as a left or as right action, whose weights are the same with opposite sign. We choose the sign (4.3.19) at the price of the less convenient consequence, that we have to define the T_L action as a right action, T_R as a left action on the dual of \mathcal{E} , and on the dual coordinates u_τ^l and $\hat{u}_{\tau|\pi}^l$.

An important outcome of Lemma 4.3.10 is that we managed to linearize the T_d -action on $\tilde{\mathcal{E}}$ near every fixed point. Since the equivariant multiplicity is defined as the equivariant Poincaré dual of the subvariety in a local linearization, we have the following important implications:

Corollary 4.3.12. Denote by \mathcal{O}_π the piece of the orbit closure $\mathcal{O} \subset \tilde{\mathcal{E}}$ in the chart \mathcal{N}_π , i.e.

$$\mathcal{O}_\pi = (\tilde{\text{pr}}|_{\mathcal{N}_\pi})^{-1}(\mathcal{O}) = \overline{B_L \varepsilon_{\text{ref}} B_R} \cap \mathcal{N}_\pi. \quad (4.3.20)$$

Then for every $\pi \in \mathbf{\Pi}_d$, we have

$$\text{emult}(\hat{T}_\pi \mathcal{O}, \tilde{\mathcal{E}}) = \text{eP}[(\mathcal{O}_\pi, \mathcal{N}_\pi)]. \quad (4.3.21)$$

and

$$\text{Euler}^{T_d}(\text{T}_\pi \tilde{\mathcal{E}}) = \prod_{l=1}^d \prod_{\text{sum}(\tau) \leq l, \tau \neq \pi_1, \dots, \pi_l} (z_\tau - z_{\pi_l}) \quad (4.3.22)$$

Substituting the appropriate expressions into (4.3.11) according to Corollaries 4.3.8 and 4.3.12, we obtain:

$$Q_{\text{Fl}}(\boldsymbol{\lambda}, \boldsymbol{\theta}) = \sum_{\boldsymbol{\pi} \in \mathbf{\Pi}_d} \frac{\prod_{m=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) Q_{\boldsymbol{\pi}}(z_1, \dots, z_d)}{\prod_{l=1}^d \prod_{\tau \in \Pi_{\leq l} \setminus \{\pi_1, \dots, \pi_l\}} (z_{\tau} - z_{\pi_l})}, \quad (4.3.23)$$

where $\Pi_{\leq l} = \{\pi; \text{sum}(\pi) \leq l\}$, and

$$Q_{\boldsymbol{\pi}} = \begin{cases} \text{eP}[(\mathcal{O}_{\boldsymbol{\pi}}, \mathcal{N}_{\boldsymbol{\pi}}] \text{ if } \tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}, \\ 0 \text{ if } \tilde{\varepsilon}_{\boldsymbol{\pi}} \notin \mathcal{O}. \end{cases} \quad (4.3.24)$$

For notational simplicity, introduce the set

$$\mathbf{\Pi}_{\mathcal{O}} = \{\boldsymbol{\pi} \in \mathbf{\Pi}_d; \tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}\} \quad (4.3.25)$$

Now we can combine this with (4.3.7), and we arrive at our first formula for $\text{eP}[\Theta_d]$:

$$\text{eP}[\Theta_d] = \text{Res}_{\mathbf{z}=\infty} \frac{\prod_{m < l} (z_m - z_l) dz_1 \dots dz_d}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)} \sum_{\boldsymbol{\pi} \in \mathbf{\Pi}_d} \frac{\prod_{m=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m}) Q_{\boldsymbol{\pi}}(\mathbf{z})}{\prod_{l=1}^d \prod_{\tau \in \Pi_{\leq l} \setminus \{\pi_1, \dots, \pi_l\}} (z_{\tau} - z_{\pi_l})} \quad (4.3.26)$$

Now observe that the sum in this formula is finite, hence we are free to exchange the summation with the residues. Rearranging our the formula accordingly, we arrive at the following statement.

Proposition 4.3.13. *For each admissible series $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$ of d partitions, denote by $Q_{\boldsymbol{\pi}}$ the equivariant Poincaré dual of the part $\mathcal{O}_{\boldsymbol{\pi}}$ of the closure \mathcal{O} of the B_d -orbit of $\tilde{\varepsilon}_{\text{ref}} \in \mathcal{E}$ in the chart $\mathcal{N}_{\boldsymbol{\pi}}$ (cf. (4.3.21), (4.3.24)). Then*

$$\text{eP}[\Theta_d] = \sum_{\boldsymbol{\pi} \in \mathbf{\Pi}_d} \text{Res}_{\mathbf{z}=\infty} \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m < l} (z_m - z_l) \prod_{m=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m})}{\prod_{l=1}^d \prod_{\tau \in \Pi_{\leq l} \setminus \{\pi_1, \dots, \pi_l\}} (z_{\tau} - z_{\pi_l}) \prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)} dz. \quad (4.3.27)$$

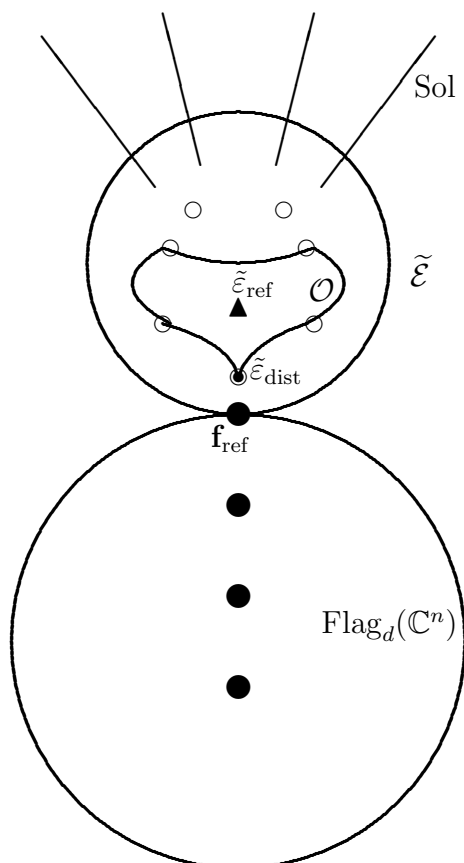
This formula has the pleasant feature that the dependence on the three parameters of our problem, n, k and d , have been separated. The first fraction here only depends on d , the denominator of the second only depends on n and the numerator controls the k -dependence, with some interference from the sequence $\boldsymbol{\pi}$.

While this formula is a step forward, it is rather difficult to compute, since the number of terms and factors grows with d as the the number of elements in $\mathbf{\Pi}_d$. The other unpleasant thing about this formula is that it does not give us the Thom

polynomial as an expression in terms of the relative Chern classes in any explicit form.

It turns out that after careful study, this formula goes through two dramatic simplifications, which makes it easy to compute for small values of d .

Before proceeding, we present a schematic diagram of the main objects of our constructions, which we hope, will help the reader to understand our constructions.



Explanations:

- The lower circle is the flag variety $\text{Flag}_d(\mathbb{C}^n)$; the fat dots inside represent the fixed flags in $\text{Flag}_d(\mathbb{C}^n)$.
- The upper circle is the fiber $\tilde{\mathcal{E}}$ over the reference flag \mathbf{f}_{ref} . The small circles inside represent the T_d -fixed points in $\tilde{\mathcal{E}}$. One of these fixed points: $\tilde{\epsilon}_{\text{dist}} \in \tilde{\mathcal{E}}$ is distinguished. This fixed point will play an important in what follows.
- The region bounded by the curly pentagon represents the B_d -orbit of the reference point $\tilde{\epsilon}_{\text{ref}}$, which is marked by a triangle. The closure of the orbit is \mathcal{O} , which is singular subvariety of $\tilde{\mathcal{E}}$; it contains a certain subset of fixed points, which was denoted by $\mathbf{\Pi}_{\mathcal{O}}$, but not all of them. \mathcal{O} contains $\tilde{\epsilon}_{\text{ref}}$.

- The straight lines on top are the linear fibers of the solutions of over a particular system of equations in $\tilde{\mathcal{E}}$. The union of these linear fibers lying over those points of the fiber bundle induced by $\tilde{\mathcal{E}}$ which correspond to \mathcal{O} is the set Θ_d .

4.4 Vanishing residues and the main result

The terms on the right hand side of formula (4.3.27) are enumerated by admissible sequences. There is a simplest one among these:

$$\boldsymbol{\pi}_{\text{dist}} = ([1], [2], \dots, [d]), \quad (4.4.1)$$

which we will call *distinguished*. To avoid double indices, below, we will use the simplified notation Q_{dist} instead of $Q_{\boldsymbol{\pi}_{\text{dist}}}$, and similarly $\tilde{\varepsilon}_{\text{dist}}, \mathcal{N}_{\text{dist}}, \mathcal{O}_{\text{dist}}$, etc.

The following remarkable vanishing result holds.

Proposition 4.4.1. *Assume that $d \ll n \leq k$. Then all terms of the sum in (4.3.27) vanish except for the term corresponding to the sequence of partitions $\boldsymbol{\pi}_{\text{dist}} = ([1], [2], \dots, [d])$. Hence, formula (4.3.27) reduces to*

$$\text{eP}[\Theta_d] = \text{Res}_{\mathbf{z}=\infty} \frac{Q_{\text{dist}}(z_1, \dots, z_n) \prod_{m < l} (z_m - z_l) d\mathbf{z}}{\prod_{l=1}^d \prod \{(z_\tau - z_l); \text{sum}(\tau) \leq l, |\tau| > 1\}} \frac{\prod_{l=1}^d \prod_{j=1}^k (\theta_j - z_l)}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)}, \quad (4.4.2)$$

where $Q_{\text{dist}} = \text{eP}[\mathcal{O}_{\text{dist}}, \mathcal{N}_{\text{dist}}]$.

Before turning to the proof, we make a few remarks. First, note that this simplification is dramatic: the number of terms in (4.3.27) grows exponentially with d , and of this sum now a single term survives. This is fortunate, because computing all the polynomials $Q_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \mathbf{\Pi}_d$ seems to be an insurmountable task; at the moment, we do not even have an algorithm to determine when $Q_{\boldsymbol{\pi}} = 0$, i.e. when $\tilde{\varepsilon}_{\boldsymbol{\pi}} \in \mathcal{O}$.

Our second observation is that after replacing in (4.4.2) z_l by $-z_l, l = 1, \dots, d$, we can rewrite the formula as

$$\text{eP}[\Theta_d] = \text{Res}_{\mathbf{z}=\infty} \frac{(-1)^d \prod_{m < l} (z_m - z_l) Q_{\text{dist}}(z_1, \dots, z_n)}{\prod_{l=1}^d \prod \{(z_\tau - z_l); \text{sum}(\tau) \leq l, |\tau| > 1\}} \prod_{l=1}^d \text{RC} \left(\frac{1}{z_l} \right) z_l^{k-n} dz_l, \quad (4.4.3)$$

where $\text{RC}(z)$ is the generating series of the relative Chern classes introduced in (2.2.19). This means that our formulas explicitly conform to the framework of Thom-Damon, Proposition 2.2.9 (3): we have obtained an explicit of the Thom polynomial of the A_d -singularity in terms of the relative Chern classes.

Most of the present section will be taken up by the proof of Proposition 4.4.1. In § 4.4.2, we derive a criterion (Proposition 4.4.3) for the vanishing of iterated residues of the form (4.3.5). Applying this criterion to the right hand side of (4.3.27) reduces Proposition 4.4.1 to a statement about the factors of the polynomials $Q_{\boldsymbol{\pi}}, \boldsymbol{\pi} \in \mathbf{\Pi}_d$: Proposition 4.4.4. According to Lemma 2.2.2, such divisibility properties follow from

the existence of relations of a certain form in the ideal of the subvariety $\mathcal{O}_\pi \subset \mathcal{N}_\pi$. We find a family of such relations in § 4.4.3 (see (4.4.11)), and then convert the condition in Lemma 2.2.2 into a combinatorial condition on π (cf. Lemma 4.4.12). At the end of § 4.4.3, we show that if a sequence π does not satisfy this combinatorial condition, then it is either π_{dist} or it does not belong to $\mathbf{\Pi}_{\mathcal{O}}$, thus completing the proof of Proposition 4.4.1.

In the course of this proof, we obtain a rather efficient, albeit incomplete criterion for a sequence $\pi \in \mathbf{\Pi}_d$ *not* to belong to $\mathbf{\Pi}_{\mathcal{O}}$; we explain this criterion in § 4.4.4. Finally, in §4.4.16, we further simplify (4.4.3), and formulate our main result, Theorem 4.4.16.

Before embarking on this rather tortuous route, we give a few examples below in §4.4.1, which demonstrate the localization formulas and the vanishing property explicitly. Note that we devote the last section of our thesis to the detailed study of (4.4.3) for small values of d , and hence the proofs in §4.4.1 will be omitted.

4.4.1 The localization formulas for $d = 2, 3$

The situation for $d = 2$ and 3 is simplified by the fact, that in these cases the closure of the Borel-orbit $\mathcal{O} = \overline{B_d \tilde{\mathcal{E}}_{\text{ref}}} \subset \tilde{\mathcal{E}}$ is smooth. Thus in these cases, we can use the Berline-Vergne localization formula (2.2.12) instead of Rossmann's formula, and instead of (4.3.23) we can work with an explicit expression, not containing equivariant multiplicities which need to be computed.

This allows us to write down the fixed point formula for $eP[\Theta_d]$ obtained by substituting a simplified version of (4.3.23) into (4.3.3), and then compare it to the residue formula (4.4.2). In these cases we can describe the set $\mathbf{\Pi}_{\mathcal{O}}$ easily as well.

The formulas below are justified in §5.1.

For $d = 2$, we have $\mathcal{O} = \tilde{\mathcal{E}} \cong \mathbb{P}^1$. There are two fixed points in $\tilde{\mathcal{E}}$:

$$\mathbf{\Pi}_{\mathcal{O}} = \mathbf{\Pi}_{d2} = \{([1], [2]), ([1], [1, 1])\}.$$

Then our fixed point formula reads as follows:

$$\begin{aligned} eP[\Theta_2] &= \sum_{s=1}^n \sum_{t \neq s}^n \frac{1}{\prod_{i \neq s}^n (\lambda_i - \lambda_s) \prod_{i \neq s, t}^n (\lambda_i - \lambda_t)} \\ &\quad \times \left(\frac{\prod_{j=1}^k (\theta_j - \lambda_s) \prod_{j=1}^k (\theta_j - \lambda_t)}{2\lambda_s - \lambda_t} + \frac{\prod_{j=1}^k (\theta_j - \lambda_s) \prod_{j=1}^k (\theta_j - 2\lambda_s)}{\lambda_t - 2\lambda_s} \right). \end{aligned}$$

This is equal to the residue (4.3.27):

$$\begin{aligned} \text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} &\frac{z_1 - z_2}{\prod_{i=1}^n (\lambda_i - z_1) \prod_{i=1}^n (\lambda_i - z_2)} \\ &\times \left(\frac{\prod_{j=1}^k (\theta_j - z_1) \prod_{j=1}^k (\theta_j - z_2)}{2z_1 - z_2} + \frac{\prod_{j=1}^k (\theta_j - z_1) \prod_{j=1}^k (\theta_j - 2z_1)}{z_2 - 2z_1} \right). \end{aligned}$$

Proposition 4.4.1 states that the residue of the second term vanishes; this is easy to check by hand.

For $d = 3$, the orbit closure \mathcal{O} is a smooth 3-dimensional hypersurface in $\tilde{\mathcal{E}}$. There are 6 fixed points in \mathcal{O} , namely

$$\mathbf{\Pi}_{\mathcal{O}} = \{([1], [2], [3]), ([1], [2], [1, 2]), ([1], [2], [1, 1]), \\ ([1], [1, 1], [3]), ([1], [1, 1], [1, 1, 1]), ([1], [1, 1], [2])\};$$

the remaining 2 fixed points in $\tilde{\mathcal{E}}$ do not belong to \mathcal{O} (see Proposition 4.4.14):

$$([1], [2], [1, 1, 1]), ([1], [1, 1], [1, 2]) \notin \mathbf{\Pi}_{\mathcal{O}}.$$

Hence the corresponding fixed point formula has 6 terms:

$$\begin{aligned} \text{eP}[\Theta_3] &= \sum_{s=1}^n \sum_{t \neq s}^n \sum_{u \neq s, t}^n \frac{\prod_{j=1}^k (\theta_j - \lambda_s)}{\prod_{i \neq s}^n (\lambda_i - \lambda_s) \prod_{i \neq s, t}^n (\lambda_i - \lambda_t) \prod_{i \neq s, t, u}^n (\lambda_i - \lambda_u)} \cdot \left[\frac{\prod_{j=1}^k (\theta_j - \lambda_t)}{2\lambda_s - \lambda_t} \right. \\ &\left(\frac{\prod_{j=1}^k (\theta_j - \lambda_u)}{(2\lambda_s - \lambda_u)(\lambda_s + \lambda_t - \lambda_u)} + \frac{\prod_{j=1}^k (\theta_j - \lambda_s - \lambda_t)}{(\lambda_u - \lambda_s - \lambda_t)(2\lambda_s - \lambda_s - \lambda_t)} + \frac{\prod_{j=1}^k (\theta_j - 2\lambda_s)}{(\lambda_u - 2\lambda_s)(\lambda_s + \lambda_t - 2\lambda_s)} \right) + \\ &\left. \frac{\prod_{j=1}^k (\theta_j - 2\lambda_s)}{\lambda_t - 2\lambda_s} \cdot \left(\frac{\prod_{j=1}^k (\theta_j - \lambda_u)}{(\lambda_t - \lambda_u)(3\lambda_s - \lambda_u)} + \frac{\prod_{j=1}^k (\theta_j - 3\lambda_s)}{(\lambda_u - 3\lambda_s)(\lambda_t - 3\lambda_s)} + \frac{\prod_{j=1}^k (\theta_j - \lambda_t)}{(\lambda_u - \lambda_t)(3\lambda_s - \lambda_t)} \right) \right]. \end{aligned}$$

The corresponding residue formula (4.3.27) also has 6 terms:

$$\begin{aligned} \text{eP}[\Theta_3] &= \text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \text{Res}_{z_3=\infty} \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \prod_{j=1}^k (\theta_j - z_1)}{\prod_{i=1}^n (\lambda_i - z_1) \prod_{i=1}^n (\lambda_i - z_2) \prod_{i=1}^n (\lambda_i - z_3)} \times \left[\frac{\prod_{j=1}^k (\theta_j - z_2)}{2z_1 - z_2} \right. \\ &\left(\frac{\prod_{j=1}^k (\theta_j - z_3)}{(2z_1 - z_3)(z_1 + z_2 - z_3)} + \frac{\prod_{j=1}^k (\theta_j - z_1 - z_2)}{(z_3 - z_1 - z_2)(2z_1 - z_1 - z_2)} + \frac{\prod_{j=1}^k (\theta_j - 2z_1)}{(z_3 - 2z_1)(z_1 + z_2 - 2z_1)} \right) + \\ &\left. \frac{\prod_{j=1}^k (\theta_j - 2z_1)}{z_2 - 2z_1} \cdot \left(\frac{\prod_{j=1}^k (\theta_j - z_3)}{(z_2 - z_3)(3z_1 - z_3)} + \frac{\prod_{j=1}^k (\theta_j - 3z_1)}{(z_3 - 3z_1)(z_2 - 3z_1)} + \frac{\prod_{j=1}^k (\theta_j - z_2)}{(z_3 - z_2)(3z_1 - z_2)} \right) \right]. \end{aligned}$$

Here, again, the last 5 terms vanish, and only the first one remains, the one at the distinguished fixed point $([1], [2], [3])$, leaving us with (4.4.2).

For $d > 3$, the variety $\mathcal{O}_d \subset \tilde{\mathcal{E}}_d$ is singular. This means that the analogs of these formulas involve computations of equivariant multiplicities, which is a rather difficult problem. We present some results in § 5.1.

4.4.2 The vanishing of residues

In this paragraph, we describe the conditions under which iterated residues of type appearing in the sum in (4.3.27) vanish.

We start with the 1-dimensional case, where the residue at infinity is defined by (4.3.5) with $d = 1$. By bounding the integral representation along a contour $|z| = R$ with R large, one can easily prove

Lemma 4.4.2. *Let $p(z), q(z)$ be polynomials of one variable. Then*

$$\operatorname{Res}_{z=\infty} \frac{p(z) dz}{q(z)} = 0 \quad \text{if } \deg(p(z)) + 1 < \deg(q).$$

Consider now the multidimensional situation. Let $p(\mathbf{z}), q(\mathbf{z})$ be polynomials in the d variables z_1, \dots, z_d , and assume that $q(\mathbf{z})$ is the product of linear factors $q = \prod_{i=1}^N L_i$, as in (4.4.2). Introduce the notation $d\mathbf{z} = dz_1 \dots dz_d$. We would like to formulate conditions under which the iterated residue

$$\operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=\infty} \dots \operatorname{Res}_{z_d=\infty} \frac{p(\mathbf{z}) d\mathbf{z}}{q(\mathbf{z})} \tag{4.4.4}$$

vanishes. Introduce the following notation:

- For a set of indices $S \subset \{1, \dots, d\}$, denote by $\deg(p(\mathbf{z}); S)$ the degree of the one-variable polynomial $p_S(t)$ obtained from p via the substitution $z_m \rightarrow \begin{cases} t & \text{if } m \in S, \\ 1 & \text{if } m \notin S. \end{cases}$
- For a nonzero linear function $L = \sum_{l=1}^d \alpha_l z_l$, denote by $\operatorname{coeff}(L, z_l)$ the coefficient α_l ;
- finally, for $1 \leq m \leq d$, set

$$\operatorname{lead}(q(\mathbf{z}); m) = \#\{i; \max\{l; \operatorname{coeff}(L_i, z_l) \neq 0\} = m\},$$

which is the number of those factors L_i in which the le

Thus we group the N linear factors of $q(\mathbf{z})$ according to the nonvanishing coefficient with the largest index; in particular, for $1 \leq m \leq d$ we have

$$\deg(q(\mathbf{z}); m) \geq \operatorname{lead}(q(\mathbf{z}); m), \quad \text{and} \quad \sum_{m=1}^d \operatorname{lead}(q(\mathbf{z}); m) = N.$$

Now applying Lemma 4.4.2 to the first residue in (4.4.4), we see that

$$\operatorname{Res}_{z_d=\infty} \frac{p(z_1, \dots, z_{d-1}, z_d) d\mathbf{z}}{q(z_1, \dots, z_{d-1}, z_d)} = 0$$

whenever $\deg(p(\mathbf{z}); d) + 1 < \deg(q(\mathbf{z}), d)$; in this case, of course, the entire iterated residue (4.4.4) vanishes.

Now we suppose the residue with respect to z_d does not vanish, and we look for conditions of vanishing of the next residue:

$$\operatorname{Res}_{z_{d-1}=\infty} \operatorname{Res}_{z_d=\infty} \frac{p(z_1, \dots, z_{d-2}, z_{d-1}, z_d) dz}{q(z_1, \dots, z_{d-2}, z_{d-1}, z_d)}. \quad (4.4.5)$$

Note that now the condition $\deg(p(\mathbf{z}); d-1) + 1 < \deg(q(\mathbf{z}), d-1)$ is *insufficient*; indeed, we need to require

$$\deg(p(\mathbf{z}); d-1, d) + 2 < \deg(q(\mathbf{z}); d-1, d)$$

since, by expanding an inverse of a linear form L in which both z_{d-1} and z_d have nonzero coefficients, we can increase the z_{d-1} -degree in the numerator at the expense of increasing the z_d -degree in the denominator.

There is, however, another way to ensure the vanishing of (4.4.5): suppose that for $i = 1, \dots, N$, every time we have $\operatorname{coeff}(L_i, z_{d-1}) \neq 0$, we also have $\operatorname{coeff}(L_i, z_d) = 0$. This is equivalent to the condition $\deg(q(\mathbf{z}), d-1) = \operatorname{lead}(q(\mathbf{z}); d-1)$. If this holds, then the compensating effect we described above cannot take place, and we can conclude the vanishing of (4.4.5) as long as

$$\deg(p(\mathbf{z}), d-1) + 1 < \deg(q(\mathbf{z}), d-1)$$

holds. This argument generalizes to the following statement.

Proposition 4.4.3. *Let $p(\mathbf{z})$ and $q(\mathbf{z})$ be polynomials in the variables z_1, \dots, z_d , and assume that $q(\mathbf{z})$ is a product of linear factors: $q(\mathbf{z}) = \prod_{i=1}^N L_i$; set $d\mathbf{z} = dz_1 \dots dz_d$. Then*

$$\operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=\infty} \dots \operatorname{Res}_{z_d=\infty} \frac{p(\mathbf{z}) dz}{q(\mathbf{z})} = 0$$

if for some $l \leq d$, either of the following two options hold:

- $\deg(p(\mathbf{z}); d, d-1, \dots, l) + d - l + 1 < \deg(q(\mathbf{z}); d, d-1, \dots, l)$,
- or
- $\deg(p(\mathbf{z}); l) + 1 < \deg(q(\mathbf{z}); l) = \operatorname{lead}(q(\mathbf{z}); l)$.

For us the second option will be more important. In terms of the linear factors of $q(\mathbf{z})$, the equality $\deg(q(\mathbf{z}); l) = \operatorname{lead}(q(\mathbf{z}); l)$ means that

$$\text{for each } i = 1, \dots, N \text{ and } m > l, \operatorname{coeff}(L_i, z_l) \neq 0 \text{ implies } \operatorname{coeff}(L_i, z_m) = 0. \quad (4.4.6)$$

Recall that our goal is to show that all the terms of the sum in (4.3.27) vanish except for the one corresponding to $\boldsymbol{\pi}_{\text{dist}} = ([1], \dots, [d])$. Let us apply our new-found tool, Proposition 4.4.3, to the terms of this sum, and see what happens.

Fix a sequence $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d) \in \mathbf{\Pi}_d$, and consider the iterated residue corresponding to it on the right hand side of (4.3.27). The expression under the residue is the product of two fractions:

$$\frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} \cdot \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})},$$

where

$$\frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} = \frac{Q_{\boldsymbol{\pi}}(\mathbf{z}) \prod_{m < l} (z_m - z_l)}{\prod_{l=1}^d \prod_{\text{sum}(\tau) \leq l, \tau \neq \pi_1, \dots, \pi_l} (z_\tau - z_{\pi_l})} \quad \text{and} \quad \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})} = \frac{\prod_{m=1}^d \prod_{j=1}^k (\theta_j - z_{\pi_m})}{\prod_{l=1}^d \prod_{i=1}^n (\lambda_i - z_l)}.$$

Note that $p(\mathbf{z})$ is a polynomial, while $q(\mathbf{z})$ is a product of linear forms, and that $p_1(\mathbf{z})$ and $q_1(\mathbf{z})$ are independent of n and k , and depend on d only.

As a warm-up, we show that if the last element of the sequence is not the trivial partition, i.e. if $\pi_d \neq [d]$, then already the first residue in the corresponding term on the right hand side of (4.3.27) – the one with respect to z_d – vanishes. Indeed, if $\pi_d \neq [d]$, then $\deg(q_2(\mathbf{z}); d) \geq n$, while z_d does not appear in $p_2(\mathbf{z})$. Then, assuming that $d \ll n$, we have $\deg(p(\mathbf{z}); d) \ll \deg(q(\mathbf{z}); d)$, and this, in turn, implies the vanishing of the residue with respect to z_d (see Proposition 4.4.3).

We can thus assume that $\pi_d = [d]$, and proceed to the study of the next residue, the one taken with respect to z_{d-1} . Again, assume that $\pi_{d-1} \neq [d-1]$. As in the case of z_d above, $d \ll n$ implies $\deg(p(\mathbf{z}); d-1) \ll \deg(q(\mathbf{z}); d-1)$. However, now we cannot use the first option in Proposition 4.4.3, because $\deg(p_2(\mathbf{z}); d-1, d) = k \geq n$. In order to apply the second option, we have to exclude all linear factors from $q_1(\mathbf{z})$ which have nonzero coefficients in front of both z_{d-1} and z_d . The fact that $\pi_d = [d]$, and the restrictions $\text{sum}(\pi_l) \leq l$, $l = 1, \dots, d$, tell us that there are two troublesome factors: $(z_d - z_{d-1})$ and $(z_d - z_{d-1} - z_1)$ which come from the two partitions: $\tau = [d-1]$ and $\tau = [d-1, 1]$ in the $l = d$ part of $q_1(\mathbf{z})$. The first of the two fortunately cancels with a factor in the Vandermonde determinant in the numerator; as for the second factor: our only hope is to find it as a factor in the polynomial $Q_{\boldsymbol{\pi}}$.

Continuing this argument by induction, we can reduce Proposition 4.4.1 to the following statement about the equivariant multiplicities $Q_{\boldsymbol{\pi}}$, $\boldsymbol{\pi} \in \mathbf{\Pi}_d$.

Proposition 4.4.4. *Let $l \geq 1$, and let $\boldsymbol{\pi}$ be an admissible sequence of partitions of the form (4.4.8), where $\pi_l \neq [l]$. Then for $m > l$, and every partition τ such that $l \in \tau$, $\text{sum}(\tau) \leq m$, and $|\tau| > 1$, we have*

$$(z_\tau - z_m) | Q_{\boldsymbol{\pi}}. \tag{4.4.7}$$

This statement will be proved in the next paragraph: §4.4.3. For now, we will assume that it is true, and give a quick proof of the result with which we started

this section.

Proof of Proposition 4.4.1: Let $\boldsymbol{\pi} \neq \boldsymbol{\pi}_{\text{dist}}$ be an admissible sequence of partitions. This means that there is $l > 1$ such that $\pi_l \neq [l]$, but $\pi_m = [m]$ for $m > l$:

$$\boldsymbol{\pi} = (\pi_1, \dots, \pi_l, [l+1], [l+2], \dots, [d]). \quad (4.4.8)$$

Note that l does not appear anywhere in $\boldsymbol{\pi}$, and thus we can conclude $\deg(p(\mathbf{z}); l) \ll \deg(q(\mathbf{z}); l)$ from $d \ll n$, as usual. This allows us to apply the second option of Proposition 4.4.3 to the residue taken with respect to z_l as long as we can cancel from $q_2(\mathbf{z})$ all factors which do not satisfy condition (4.4.6).

These factors are of the form $z_\tau - z_m$, where $m > l$ and $l \in \tau$. If $|\tau| = 1$, i.e. if $\tau = [l]$, then we can find this factor in the Vandermonde determinant in the numerator. We can use Proposition 4.4.4 to cancel the rest of the factors, as long as we make sure that such factors occur in $q_1(\mathbf{z})$ with multiplicity 1. This is straightforward in our case, since the variable z_m with $m \geq l$ may appear only in the m th factor of $q_1(\mathbf{z})$. \square

4.4.3 The homogeneous ring of $\tilde{\mathcal{E}}$ and factorization of $Q_\boldsymbol{\pi}$

Now we turn to the proof of Proposition 4.4.4. Let $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ be an admissible sequence of partitions. Recall (cf. (4.3.24)) that $Q_\boldsymbol{\pi}$ is the T_d -equivariant Poincaré dual of the part $\mathcal{O}_\boldsymbol{\pi} = \tilde{\text{pr}}^{-1}(\mathcal{O}) \cap \mathcal{N}_\boldsymbol{\pi}$ of the orbit closure \mathcal{O} in the linear chart $\mathcal{N}_\boldsymbol{\pi}$ (cf. (4.3.21)); this latter linear space is endowed with coordinates $\hat{u}_{\tau|\boldsymbol{\pi}}^l$ defined in (4.3.13).

Our plan is to use Lemma 2.2.2, which, when applied to our situation, says that the divisibility relation (4.4.7) follows if we find a relation in the ideal of the subvariety $\mathcal{O}_\boldsymbol{\pi} \subset \mathcal{N}_\boldsymbol{\pi}$ expressing the appropriate variable $\hat{u}_{\tau|\boldsymbol{\pi}}^m$ as a polynomial of the rest of the variables.

We will lift the calculation from $\tilde{\mathcal{E}}$ to the linear space $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$ endowed with coordinates u_τ^l , $1 \leq l \leq d$, $\text{sum}(\tau) \leq l$. In order to write down the appropriate relations, we need to make some preparations.

Observe that if we find a polynomial $Z \in I_\mathcal{O}$, then the restriction $Z|_{\mathcal{N}_\boldsymbol{\pi}}$ will be an element of the vanishing ideal of $\tilde{\text{pr}}^{-1}(\mathcal{O}) \cap \mathcal{N}_\boldsymbol{\pi} \in \mathcal{N}_\boldsymbol{\pi}$. This restricted polynomial is expressed in terms of the $\hat{u}_{\cdot|\boldsymbol{\pi}}$ variables according to the following prescription:

Lemma 4.4.5. *Let $Z \in I_\mathcal{O}$ be a polynomial in the variables u_τ^l . Then the restriction $Z|_{\mathcal{N}_\boldsymbol{\pi}}$ in terms of the coordinates $\hat{u}_{\cdot|\boldsymbol{\pi}}$ is obtained by*

- setting $u_{\pi_l}^l$ to 1, for $l = 1, \dots, d$,
- setting $u_{\pi_l}^m$ to 0, for $1 \leq l \leq m \leq d$,
- replacing the remaining variables u_τ^l by $\hat{u}_{\tau|\boldsymbol{\pi}}^l$.

Proof. This follows from Definition 4.3.9, and the way (4.3.13) we defined the local coordinates. \square

Denote by $\mathbb{C}[u^\bullet]$ the ring of polynomial functions on the vector space $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$, i.e. the space of polynomials in the variables u_τ^l , $1 \leq l \leq d$, $\text{sum}(\tau) \leq l$. This ring carries a right action of the group B_L , and a left action of the group B_R . In particular, it has two multigradings induced from the T_L and T_R actions: $\text{Deg}_L(u_\pi^l)$ is the vector of multiplicities $(\text{mult}(i, \pi), i = 1, \dots, d)$, while $\text{Deg}_R(u_\pi^l)$ is the l th basis vector in \mathbb{Z}^d . A combination of these gradings will be particularly important for us: $\text{defect}(u_\pi^l) = l - \text{sum}(\pi)$; this induces a $\mathbb{Z}^{\geq 0}$ -grading on $\mathbb{C}[u^\bullet]$.

Recall that the projection $B_d \rightarrow T_d$ is a group homomorphism, whose kernel is the subgroup of unipotent matrices. We denote the corresponding nilpotent Lie algebras of strictly upper-triangular matrices by \mathfrak{n}_R and \mathfrak{n}_L for B_R and B_L , respectively..

The two Lie algebras, \mathfrak{n}_L and \mathfrak{n}_R are generated by the simple root vectors

$$\Delta_L = \{E_{l,l+1}^L; l = 1, \dots, d-1\}, \text{ and } \Delta_R = \{E_{l,l+1}^R; l = 1, \dots, d-1\},$$

respectively, where $E_{l,l+1}$ is the matrix whose only nonvanishing entry is a 1 in the l th row and $l+1$ st column. Let us write down the action of these root vectors on $\mathbb{C}[u^\bullet]$ in the coordinates u_τ^l , $|\tau| \leq l \leq d$. We first define a few operations on partitions:

- given a positive integer m and a partition $\tau \in \Pi$, denote by $\tau \cup m$ the partition with m added to τ , e.g. $[2, 3, 4] \cup 3 = [2, 3, 3, 4]$
- if $m \in \tau$, then denote by $\tau - m$ the partition τ with one of the m s deleted, e.g. $[2, 4, 4, 5, 5, 5, 6] - 5 = [2, 4, 4, 5, 5, 6]$;
- more generally, we will write $[2, 4, 5, 5] \cup [3, 4] = [2, 3, 4, 4, 5, 5]$, and $[2, 4, 5, 5] - [4, 5] = [2, 5]$.

Returning to the Lie algebra actions, observe that both \mathfrak{n}_R and \mathfrak{n}_L decrease the defect; more precisely, we have

$$\begin{cases} \mathfrak{n}_R u_\tau^l = u_\tau^l \mathfrak{n}_L = 0, & \text{if } \text{sum}(\tau) = l, \\ E_{m,m+1}^R u_\tau^l = \delta_{l,m+1} u_\tau^{l-1}, & u_\tau^l E_{m,m+1}^L = \text{mult}(m, \tau) u_{\tau-m \cup m+1}^l, & \text{if } \text{sum}(\tau) < l. \end{cases} \quad (4.4.9)$$

where $\delta_{a,b}$ is the Kronecker delta.

Now we are ready to describe the ideal of polynomials in $\mathbb{C}[u^\bullet]$ vanishing on the preimage of $\mathcal{O} \subset \tilde{\mathcal{E}}$.

Definition 4.4.6. Let

$$R_{\tilde{\mathcal{E}}} = \{Z \in \mathbb{C}[u^\bullet]; \mathfrak{n}_R Z = 0\} \text{ and } I_{\mathcal{O}} = \{Z \in R_{\tilde{\mathcal{E}}}; [Z \mathfrak{n}_L^N](\varepsilon_{\text{ref}}) = 0 \text{ for } N = 0, 1, 2, \dots\},$$

where \mathfrak{n}_L^N is the subset $\{X_1 \cdots X_N; X_i \in \mathfrak{n}_L, i = 1, \dots, N\}$ of the universal enveloping algebra of \mathfrak{n}_L .

Proposition 4.4.7. 1. Both $R_{\tilde{\mathcal{E}}}$ and $I_{\mathcal{O}}$ are direct sums of $T_R \times T_L$ -invariant subspaces.

2. $R_{\tilde{\mathcal{E}}}$ is a ring and $I_{\mathcal{O}} \subset R_{\tilde{\mathcal{E}}}$ is an ideal.

3. For $\varepsilon \in \mathcal{E}$ we have $\tilde{\varepsilon} \in \mathcal{O} \Leftrightarrow Z(\varepsilon) = 0$ for all $Z \in I_{\mathcal{O}}$.

4. In the definition of $R_{\tilde{\mathcal{E}}}$, we can restrict ourselves to the simple root vectors:

$$R_{\tilde{\mathcal{E}}} = \{Z \in \mathbb{C}[u^\bullet]; E_{l_i, l_i+1}^R Z = 0, l = 1, \dots, d-1\}.$$

5. One has $Z(\varepsilon_{\text{ref}}) = 0$ for any polynomial $Z \in \mathbb{C}[u^\bullet]$ of pure $T_R \times T_L$ -weight for which $\text{defect}(Z) > 0$.

6. Let $Z \in R_{\tilde{\mathcal{E}}}$ be a polynomial of pure $T_R \times T_L$ weight with $\text{defect}(Z) = N$. Then

$$Z \in I_{\mathcal{O}} \Leftrightarrow Z E_{l_1, l_1+1}^L \dots E_{l_N, l_N+1}^L = 0 \text{ for any } 1 \leq l_i \leq d-1, i = 1, \dots, N.$$

Proof. The first statement is obvious, while the second follows from the Leibniz rule. To prove (3), note that the \mathfrak{n}_R -invariance of Z and the fact that \mathcal{O} is the closure of the B_L -orbit of $\tilde{\varepsilon}_{\text{ref}}$ allow us to assume, without loss of generality, that $\varepsilon \in B_L \varepsilon_{\text{ref}}$. Also, according to (1), it is sufficient to consider polynomials $Z \in I_{\mathcal{O}}$ of pure $T_R \times T_L$ weight; for such Z , clearly, $Z(\varepsilon) = 0 \Leftrightarrow t_R Z t_L(\varepsilon)$ for $t_L \in T_L, t_R \in T_R$. Now, since $\ker(B_L \rightarrow T_L) = \exp(\mathfrak{n}_L)$, the definition of $I_{\mathcal{O}}$ also implies $Z(b\varepsilon_{\text{ref}}) = 0$ for all $b \in B_L$. This proves the \Rightarrow direction of the statement.

For the reverse direction, we need to show that for every ε with $\tilde{\varepsilon} \notin \mathcal{O}$, there exists a $Z \in I_{\mathcal{O}}$ such that $Z(\varepsilon) \neq 0$. Recall that $\tilde{\mathcal{E}}$ is a projective variety, and $\mathcal{O} \subset \tilde{\mathcal{E}}$ is a Zariski closed subvariety. Thus we need to argue that there is a polynomial – and therefore analytic – function \check{Z} on $\tilde{\mathcal{E}}$ vanishing on \mathcal{O} , but $\check{Z}(\varepsilon) \neq 0$. Since $\check{Z}|_{\mathcal{O}} \equiv 0, \check{Z} \in I_{\mathcal{O}}$, which is a contradiction.

The fourth statement again follows from the Leibniz rule since every element of \mathfrak{n}_L is the polynomial of the simple root vectors in the enveloping algebra. Each monomial term of a pure $T_R \times T_L$ -weight $Z \in \mathbb{C}[u^\bullet]$ with $\text{defect}(Z) > 0$ contains a factor u_π^l , whose defect is positive, and for this $u_\pi^l(\varepsilon_{\text{ref}}) = 0$, by definition, so the fifth statement follows. Finally, the sixth statement follows from the fifth and the fact, that each simple root vector decreases the defect of Z by one. \square

The geometric meaning of the two spaces thus is as follows. Clearly, a polynomial Z of homogeneous T_R -weight λ may be thought of as a section of the line bundle $\mathcal{E} \times_{B_R} \mathbb{C}_\lambda$, where B_R acts on \mathbb{C}_λ via the homomorphism $B_R \rightarrow T_R$. Hence $R_{\tilde{\mathcal{E}}}$ maybe thought of as the ring of sections of the these type of line bundles over $\tilde{\mathcal{E}}$, and according to the third statement of Proposition 4.4.7, the space $I_{\mathcal{O}}$ is the vanishing ideal of $\mathcal{O} \in \tilde{\mathcal{E}}$ in this ring.

If we find a polynomial $Z \in I_{\mathcal{O}}$, then the restriction $Z|_{\mathcal{N}_\pi}$ will be an element of the vanishing ideal of $\tilde{\text{pr}}^{-1}(\mathcal{O}) \cap \mathcal{N}_\pi \in \mathcal{N}_\pi$. This restricted polynomial is expressed in terms of the $\hat{u}_{|\pi}$ variables according to the prescription in Lemma 4.4.5.

We will be looking for such polynomials Z in a particular subspace of $\mathbb{C}[u^\bullet]$. To describe this space, introduce for each $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ the monomial

$$\mathbf{u}^\boldsymbol{\pi} = \prod_{l=1}^d u_{\pi_l}^l; \text{ these satisfy } \mathbf{u}_\boldsymbol{\pi}(\varepsilon_{\boldsymbol{\pi}'}) = \begin{cases} 1, & \text{if } \boldsymbol{\pi} = \boldsymbol{\pi}' \\ 0, & \text{otherwise.} \end{cases} \quad (4.4.10)$$

Now consider the linear span of these monomials:

$$\Lambda = \left\{ \sum_{\boldsymbol{\pi} \in \mathbf{\Pi}_d} \alpha_\boldsymbol{\pi} \mathbf{u}^\boldsymbol{\pi} \in \mathbb{C}[u^\bullet]; \alpha_\boldsymbol{\pi} \in \mathbb{C} \right\}.$$

Our task is thus constructing elements of $\Lambda \cap I_{\mathcal{O}}$.

To write down the appropriate formulas, we need to introduce two operations on $\mathbf{\Pi}_d$. For a sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$ and a permutation $\sigma \in \mathcal{S}_d$ define the the permuted sequence

$$\boldsymbol{\pi} \cdot \sigma = (\pi_{\sigma(1)}, \dots, \pi_{\sigma(d)});$$

this defines a natural right action of \mathcal{S}_d on $\Pi[d]^{\times d}$. Note that permuting an admissible sequence $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ does not necessarily result in an admissible sequence.

The second operation modifies just one entry of $\boldsymbol{\pi}$: for $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ and $\tau \in \Pi[d]$, define

$$\boldsymbol{\pi} \cup_m \tau = (\pi_1, \dots, \pi_{m-1}, \pi_m \cup \tau, \pi_{m+1}, \dots, \pi_d).$$

Now we are ready to write down our relations.

Proposition 4.4.8. *Let $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ be an admissible sequence of partitions and let $\tau \in \Pi[d]$ be any partition. Then following polynomial is an element of $I_{\mathcal{O}}$:*

$$\text{Rel}(\boldsymbol{\pi}, \tau) = \sum \text{sign}(\sigma) \mathbf{u}^{\boldsymbol{\pi} \cdot \sigma \cup_m \tau}, 1 \leq m \leq d, \sigma \in \mathcal{S}_d, \boldsymbol{\pi} \cdot \sigma \cup_m \tau \in \mathbf{\Pi}_d, \quad (4.4.11)$$

Remark 4.4.9. The sum in (4.4.11) may be empty. This happens when there are no pairs (σ, m) satisfying the conditions in (4.4.11). Note, however, that no two terms of this sum may cancel each other.

Proof. We begin by noting that $\text{Rel}(\boldsymbol{\pi}, \tau)$ is of pure $T_R \times T_L$ weight. Indeed, the torus T_R acts on the whole space Λ with the same weight $(1, 1, \dots, 1)$, while the l th component of the T_L -weight of a term of $\text{Rel}(\boldsymbol{\pi}, \tau)$ is equal to $\text{mult}(l, \tau) + \sum_{m=1}^d \text{mult}(l, \pi_m)$.

Next, we show that

$$E_l^R \text{Rel}(\boldsymbol{\pi}, \tau) = 0, \quad l = 1, \dots, d-1, \quad (4.4.12)$$

which, according to Proposition 4.4.7 (4), implies that $\text{Rel}(\boldsymbol{\pi}, \tau) \in R_{\tilde{\mathcal{E}}}$. Let us fix l ; the terms of $\text{Rel}(\boldsymbol{\pi}, \tau)$ in (4.4.11) are indexed by pairs (σ, m) , and we can ignore those pairs for which $\text{sum}(\pi_{l+1}) + \delta_{m, l+1} \text{sum}(\tau) = l+1$, since in this case $E_l^R \mathbf{u}^{\boldsymbol{\pi} \cdot \sigma \cup_m \tau} = 0$. Then the vanishing (4.4.12) clearly follows if, on the set of the

remaining pairs contributing to (4.4.11), we find an involution $(\sigma, m) \mapsto (\sigma', m')$ such that

$$E_{l,l+1}^R \mathbf{u}^{\pi \cdot \sigma \cup_m \tau} = E_{l,l+1}^R \mathbf{u}^{\pi \cdot \sigma' \cup_{m'} \tau} \text{ and } \text{sign}(\sigma') = -\text{sign}(\sigma).$$

Indeed, it is easy to check that this holds for the involution

$$(\sigma', m') = (\sigma \cdot \langle l \leftrightarrow l+1 \rangle, \langle l \leftrightarrow l+1 \rangle(m)),$$

where $\langle l \leftrightarrow l+1 \rangle \in \mathcal{S}_d$ is the transposition of l and $l+1$.

Now introduce the linear space

$$I'_\mathcal{O} = \{ Z \in \mathbb{C}[u^\bullet]; [Z \mathbf{n}_L^N] (\varepsilon_{\text{ref}}) = 0 \text{ for } N = 0, 1, \dots \}.$$

Again, because of the Leibniz rule, we see that $I'_\mathcal{O} \subset \mathbb{C}[u^\bullet]$ is an ideal. Clearly, we have $I_\mathcal{O} = I'_\mathcal{O} \cap R_{\tilde{\varepsilon}}$, hence it is now sufficient to show that $\text{Rel}(\boldsymbol{\pi}, \tau) \in I'_\mathcal{O}$.

First we show that for partitions $\rho, \tau \in \Pi[d]$ and $m \geq \text{sum}(\rho) + \text{sum}(\tau)$ the polynomial

$$Z_{\rho\tau}^m = u_{\rho \cup \tau}^m - \sum u_\rho^t u_\tau^r, \quad t+r = m, \quad t \geq \text{sum}(\rho), \quad r \geq \text{sum}(\tau) \quad (4.4.13)$$

is in $I'_\mathcal{O}$. Indeed, a quick computation produces the equality

$$Z_{\rho\tau}^m E_{l,l+1}^L = \text{mult}(l, \rho) Z_{\rho'\tau}^m + \text{mult}(l, \tau) Z_{\rho\tau'}^m, \quad \text{where } \rho' = \rho - l \cup [l+1], \quad \tau' = \tau - l \cup [l+1].$$

This equality implies that it is sufficient for us to prove $Z_{\rho\tau}^m(\varepsilon_{\text{ref}}) = 0$ for the case $m = \text{sum}(\rho) + \text{sum}(\tau)$. In this case we have

$$Z_{\rho\tau}^m = u_{\rho \cup \tau}^m - u_\rho^{\text{sum}(\rho)} u_\tau^{\text{sum}(\tau)}, \quad (4.4.14)$$

and this polynomial clearly vanishes on ε_{ref} , because all three coordinates appearing in this relation are equal to 1 according to (4.3.10).

Now we return to the proof of $\text{Rel}(\boldsymbol{\pi}, \tau) \in I'_\mathcal{O}$. Using the fact that $Z_{\rho\tau}^m$ is in the ideal $I'_\mathcal{O}$, modulo the $I'_\mathcal{O}$, we can replace all the factors of the form $u_{\pi_{\sigma(m)} \cup \tau}^m$ in all the terms of $\text{Rel}(\boldsymbol{\pi}, \tau)$ by the appropriate sum of quadratic terms in (4.4.13). Our claim is that the resulting polynomial is identically zero, which implies that $\text{Rel}(\boldsymbol{\pi}, \tau) \in I'_\mathcal{O}$.

Indeed, let us perform this substitution; the terms of the resulting sum are parametrized by a triple (σ, m, r) , which is obtained by applying (4.4.13) to the term of $\text{Rel}(\boldsymbol{\pi}, \tau)$ indexed by (σ, m) and taking the term corresponding to r in (4.4.13). The correspondence is thus

$$(\sigma, m, r) \longrightarrow u_{\pi_{\sigma(1)}}^1 \dots u_{\pi_{\sigma(m-1)}}^{m-1} u_{\pi_{\sigma(m)}}^{m-r} u_\tau^r u_{\pi_{\sigma(m+1)}}^{m+1} \dots u_{\pi_{\sigma(d)}}^d. \quad (4.4.15)$$

Just as above, we can see that the involution $(\sigma, m, r) \mapsto (\sigma \cdot \langle m \leftrightarrow m-r \rangle, m, r)$ provides us with a complete pairing of the terms of the sum described above; each pair consists of identical monomials with opposite signs. This implies that indeed, the result is zero, hence $\text{Rel}(\boldsymbol{\pi}, \tau)$ vanishes modulo $I'_\mathcal{O}$, i.e. $\text{Rel}(\boldsymbol{\pi}, \tau) \in I'_\mathcal{O}$. This completes the proof. \square

Armed with these relations, we are ready to *prove Proposition 4.4.4*. Recall that according to the strategy described at the beginning of this paragraph, given $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, l and τ as in Proposition 4.4.4, we need to find a relation of the form $\text{Rel}(\cdot, \cdot)$, which, when restricted to $\mathcal{N}_{\boldsymbol{\pi}}$ expresses the variable $\hat{u}_{\tau|\boldsymbol{\pi}}^l$ in terms of the rest of the variables.

Thus the first thing to check is when $\hat{u}_{\tau|\boldsymbol{\pi}}^l$ appears the restriction of a monomial of the form $\mathbf{u}^{\boldsymbol{\pi}'}$. The following statement immediately follows from the prescription (4.4.5).

Lemma 4.4.10. *Given $\boldsymbol{\pi} \in \mathbf{\Pi}_d$, $1 \leq m \leq d$, and $\tau \in \Pi[d] \setminus \{\pi_1, \dots, \pi_m\}$ satisfying $\text{sum}(\tau) \leq m$. Then we have $\mathbf{u}^{\boldsymbol{\pi}'}|_{\mathcal{N}_{\boldsymbol{\pi}}} = \hat{u}_{\tau|\boldsymbol{\pi}}^l$ for some $\boldsymbol{\pi}' \in \mathbf{\Pi}_d$ if and only if $\tau \neq \pi_{m+1}, \dots, \pi_d$ and*

$$\boldsymbol{\pi}' = (\pi_1, \dots, \pi_{l-1}, \tau, \pi_{l+1}, \dots, \pi_d).$$

The next step is to find out for which $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ the monomial $\mathbf{u}^{\boldsymbol{\pi}}$ appears in one of the relations (4.4.11). Taking into account Remark 4.4.9, the criterion is easy to find:

Definition 4.4.11. We will call an admissible sequence of partitions $\boldsymbol{\pi} = (\pi_1, \dots, \pi_d)$ *complete* if for every $l \in \{1, \dots, d\}$ and every nontrivial subpartition $\tau \subset \pi_l$, there is $m \in \{1, \dots, d\}$ such that $\pi_m = \tau$.

Lemma 4.4.12. *A monomial $\mathbf{u}^{\boldsymbol{\pi}}$ appears in a relation $\text{Rel}(\boldsymbol{\rho}, \tau)$ for some $\boldsymbol{\rho} \in \mathbf{\Pi}_d$ and $\tau \in \Pi[d]$ if and only if $\boldsymbol{\pi}$ is not complete.*

Comparing Lemmas 4.4.10 and 4.4.12 to the conditions of Proposition 4.4.4, and keeping in mind our strategy, we see that the desired divisibility property (4.4.7) is reduced to the following statement: given $1 \leq l < m \leq d$ and $\tau \in \Pi[d]$ satisfying

$$\text{sum}(\tau) \leq m, l \in \tau \text{ and } |\tau| > 1,$$

and a sequence $\boldsymbol{\pi}$ of the form (4.4.8) with $\pi_l \neq [l]$, we need to show that the sequence

$$(\pi_1, \dots, \pi_l, [l+1], [l+2], \dots, [m-1], \tau, [m+1], \dots, [d-1], [d])$$

is in $\mathbf{\Pi}_d$ but is not complete. This immediately follows, however, from the fact that $[l]$ is a proper subpartition of τ , which cannot be equal to any of the other partitions in this sequence. This completes the proof of Proposition 4.4.4 and thus also the proof of Proposition 4.4.1. \square

4.4.4 The fixed points of the T_L -action on \mathcal{O}

As a small detour, based on the results of the previous paragraph, we obtain a rather powerful criterion for $\boldsymbol{\pi} \in \mathbf{\Pi}_d$ *not* to belong to $\mathbf{\Pi}_{\mathcal{O}}$, i.e. we will construct a large number of T_L -fixed points which do not lie in \mathcal{O} . We note, however, that describing the set $\mathbf{\Pi}_{\mathcal{O}}$ remains an interesting open problem. Our starting point is (4.4.10).

Lemma 4.4.13. *If the monomial \mathbf{u}^π appears with nonzero coefficient in a polynomial from $\Lambda \cap I_{\mathcal{O}}$, then the fixed point $\tilde{\varepsilon}_\pi \notin \mathcal{O}$, i.e. $\pi \notin \mathbf{\Pi}_{\mathcal{O}}$.*

Proof. Indeed, let Z be such a polynomial. According to Proposition 4.4.7 (3), a polynomial in $I_{\mathcal{O}}$ vanishes at all points of \mathcal{O} . On the other hand, it is clear from (4.4.10) that all but exactly one of the terms of Z vanishes at ε_π , and hence $Z(\varepsilon_\pi) \neq 0$. \square

Combining this statement with Lemma 4.4.12 we have the following.

Proposition 4.4.14. *If $\pi \in \mathbf{\Pi}_{\mathcal{O}}$. i.e. if $\tilde{\varepsilon}_\pi \in \mathcal{O}$, then the sequence π is complete.*

This Proposition provides us a rather strict necessary, although, as an example below shows, not sufficient condition for π to be in $\mathbf{\Pi}_{d\mathcal{O}}$.

Example 4.4.15. 1. The sequence

$$([1], [2], \dots, [d-1], [l, m]), \quad \text{where } l + m \leq d.$$

is complete, and, in fact, it corresponds to a fixed point.

2. For $d = 3, 4$, the reverse of Proposition 4.4.14 holds: if π is complete then the fixed point $\tilde{\varepsilon}_\pi$ lies in the orbit closure \mathcal{O}_d , see section §5.1.
3. The completeness of π is a necessary but not sufficient condition for π to be in $\mathbf{\Pi}_{\mathcal{O}}$. An example is the following zero-defect sequence of partitions: let $d = 60$, $\tau = [1, 12, 12, 15, 20]$ and set

$$\pi_l = \begin{cases} \rho, & \text{if } \rho \subset \tau \text{ and } \text{sum}(\rho) = l, \\ [l], & \text{otherwise.} \end{cases}$$

By definition, this is a complete sequence of partitions, but it is not in \mathcal{O} , which is left as an exercise.

4.4.5 The distinguished fixed point and the main result

Now we turn our attention to our much simplified formula (4.4.2) for the Thom polynomial of the A_d -singularity.

The proof of the vanishing of the contributions to (4.3.27), naturally, fails at the fixed point $\tilde{\varepsilon}_{\text{dist}}$. Indeed, for the distinguished sequence π_{dist} , we have $\deg(p_2(\mathbf{z}); l) > \deg(q_2(\mathbf{z}); l)$ for $l = 1, \dots, d$, and hence we cannot apply Proposition 4.4.3.

The factorization arguments of §4.4.3 may be partially saved, however. Indeed, for the case of the distinguished partition π_{dist} , each T_L -weight $z_\tau - z_l$ of $\mathcal{N}_{\text{dist}}$ appears with multiplicity one (cf. end of §4.4.2). Then, again, we can apply Lemmas 2.2.2, 4.4.10 and 4.4.12 to conclude that for $|\tau| > 1$,

$$(z_\tau - z_l) \mid Q_{\text{dist}} \quad \text{if } ([1], [2], \dots, [l-1], \tau, [l+1], \dots, [d-1], [d]) \text{ is not complete.}$$

Clearly, such a sequence is complete if and only if $|\tau| = 2$, and this means that in the fraction on the right hand side of (4.4.3), we can cancel all factors between the numerator and the denominator corresponding to partitions τ with $|\tau| > 2$. This reduces the denominator to the product of the factors with $|\tau| = 2$:

$$\prod (z_m + z_r - z_l), \quad 1 \leq m \leq r, \quad m + r \leq l \leq d,$$

while Q_{dist} is replaced by a polynomial \widehat{Q}_d , whose degree is much smaller than that of Q_{dist} . Note that in this case no factors of the Vandermonde in the numerator are canceled; the fraction in (4.4.3) simplifies to

$$\frac{(-1)^d \prod_{m < l} (z_m - z_l) \widehat{Q}_d(z_1, \dots, z_d)}{\prod_{l=1}^d \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m, l-m)} (z_m + z_r - z_l)}$$

The polynomial \widehat{Q}_d , just as Q_{dist} , only depends on d ; we mark its d -dependence explicitly. Before we formulate our main theorem, we describe the geometric meaning of this cancellation, and that of the polynomial \widehat{Q}_d itself.

First, note that π_{dist} is of the defect-0 type, hence $\varepsilon_{\text{ref}} \in \mathcal{N}_{\text{dist}}$, and by Lemma 4.3.10 $\mathcal{N}_{\text{dist}}$ is invariant under the induced linear B_L -action. Moreover, $\mathcal{O}_{\text{dist}} = \overline{B_d \cdot \varepsilon_{\text{ref}}}$.

To simplify our notation, we will write \hat{u}_τ^m instead of $\hat{u}_{\tau|\text{dist}}^m$ for the coordinates on the space $\mathcal{N}_{\text{dist}}$. The cancellation we described above follows from the fact that some of these variables \hat{u}_τ^m may be expressed through the rest, and, according to Remark 2.2.3, the polynomial \widehat{Q}_d may be thought of as an equivariant Poincaré dual as follows.

Define the linear subspace

$$\widehat{\mathcal{N}}_d = \{\varepsilon \in \mathcal{N}_{\text{dist}}; \hat{u}_\tau^m(\varepsilon) = 0 \text{ for } |\tau| > 2\} \subset \mathcal{N}_{\text{dist}}, \quad (4.4.16)$$

and let $\widehat{\text{pr}} : \mathcal{N}_{\text{dist}} \rightarrow \widehat{\mathcal{N}}_d$ be the natural projection. Then (cf. Remark 2.2.3) we can conclude that

$$\widehat{Q}_d = \text{eP}[\widehat{\mathcal{O}}_d, \widehat{\mathcal{N}}_d] \quad \text{where} \quad \widehat{\mathcal{O}}_d = \widehat{\text{pr}}(\widehat{\mathcal{O}}_{\text{dist}}). \quad (4.4.17)$$

In addition, by definition (5.2.14), the length of τ is preserved under the actions of both B_L and B_R , hence the subspace $\widehat{\mathcal{N}}_d \subset \mathcal{N}_{\text{dist}}$ is B_L -invariant, and, in fact, the projection $\widehat{\text{pr}}$ commutes with this B_L -action. This implies that

$$\widehat{\mathcal{O}}_d = \overline{B_d \hat{\varepsilon}_{\text{ref}}}, \quad \text{where} \quad \hat{\varepsilon}_{\text{ref}} = \widehat{\text{pr}}(\varepsilon_{\text{ref}}).$$

With these preparations, we can formulate our main result, where we write all this down explicitly.

Theorem 4.4.16. *Let $T_d \subset B_d \subset \text{GL}_d$ be the subgroups of invertible diagonal and upper-triangular matrices, respectively; denote the diagonal weights of T_d by z_1, \dots, z_d . Consider the GL_d -module of 3-tensors $\text{Hom}(\mathbb{C}^d, \text{Sym}^2 \mathbb{C}^d)$; identifying the*

weight- $(z_m + z_r - z_l)$ symbols q_l^{mr} and q_l^{rm} , we can write a basis for this space as follows:

$$\mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2 \mathbb{C}^d) = \bigoplus \mathbb{C} q_l^{mr}, \quad 1 \leq m, r, l \leq d.$$

Consider the reference element

$$\hat{\varepsilon}_{\mathrm{ref}} = \sum_{m=1}^d \sum_{r=1}^{d-m} q_{mr}^{m+r},$$

in the B_d -invariant subspace

$$\hat{\mathcal{N}}_d = \bigoplus_{1 \leq m+r \leq l \leq d} \mathbb{C} q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^d, \mathrm{Sym}^2 \mathbb{C}^d). \quad (4.4.18)$$

Set the notation $\hat{\mathcal{O}}_d$ for the orbit closure $\overline{B_d \hat{\varepsilon}_{\mathrm{ref}}} \subset \hat{\mathcal{N}}_d$, and consider its T_d -equivariant Poincaré dual

$$\hat{Q}_d(z_1, \dots, z_d) = \mathrm{eP}[\hat{\mathcal{O}}_d, \hat{\mathcal{N}}_d]_{T_d},$$

which is a homogeneous polynomial of degree $\dim(\hat{\mathcal{N}}_d) - \dim(\hat{\mathcal{O}}_d)$.

Then for arbitrary integers $n \leq k$, the Thom polynomial for the A_d -singularity with n -dimensional source space and k -dimensional target space is given by the following iterated residue formula:

$$\mathrm{eP}[\Theta_d] = \mathrm{Res}_{\mathbf{z}=\infty} \frac{(-1)^d \prod_{m < l} (z_m - z_l) \hat{Q}_d(z_1, \dots, z_d)}{\prod_{l=1}^d \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m, l-m)} (z_m + z_r - z_l)} \prod_{l=1}^d \mathrm{RC} \left(\frac{1}{z_l} \right) z_l^{k-n} dz_l, \quad (4.4.19)$$

where $\mathrm{RC}(\cdot)$ is the generating function of the relative Chern classes given in (2.2.19).

The proof of this theorem is thus almost complete: to finish it, we need to make two small remarks. First the identification of the space $\hat{\mathcal{N}}_d$ introduced in (4.4.16) as a B_d -module in (4.4.18) is a simple exercise using (4.3.18).

The second point is that the statement of the theorem seems more general than to which we seem to be entitled: Proposition 4.4.1 includes the assumption $d \ll n$, while here we claim that our statement holds for any d and $n \leq k$. The reason is Proposition 2.2.9, which, in particular, says that if an expression of a Thom polynomial in the relative Chern classes holds for large n , then the same expression works for any n .

In fact, it is easy to see that formula (4.4.19) manifestly satisfies all properties listed in Proposition 2.2.9. In particular, it only depends on the codimension $k - n$, and reducing the codimension by 1 leads to shifting the indices of the relative Chern classes. We also see that the geometric meaning for the so-called *Thom series* defined in [16] is concentrated in the key object \hat{Q}_d .

A detailed study of the polynomial \hat{Q}_d will be given in a later publication [2]. In the final section of our thesis, we turn to examples, and explicit calculations.

Chapter 5

Calculation of \widehat{Q}_d

5.1 Calculation for small values of d . Explicit formulas for Thom polynomials

Theorem 4.4.16 reduces the computation of the Thom polynomials of the algebra A_d to that of the polynomial \widehat{Q}_d , which is the equivariant Poincaré dual of a B_d -orbit in a certain B_d -invariant subspace of 3-tensors in d dimensions. Note that the parameters n and k do not enter this picture; in particular, \widehat{Q}_d only depends on d .

Clearly, in principle, the computation of \widehat{Q}_d is a finite problem in commutative algebra, which, for each value of d , can be handled by a computer algebra package such as Macaulay. However, the number of variables and the degree of \widehat{Q}_d grow rather quickly: they are of order d^3 . More importantly, computer algebra programs have difficulties dealing with parametrized subvarieties already in very small examples.

At this point, we do not have an efficient method of computation for \widehat{Q}_d in general. The purpose of this section is twofold: to show how to compute \widehat{Q}_d for small degrees: $d = 2, 3, 4, 5, 6$, and we would like to describe the extra information we have uncovered about the orbit $B_d\hat{\varepsilon}_{\text{ref}} \subset \widehat{\mathcal{N}}_d$ in Section 5.2.

5.1.1 The degree of \widehat{Q}_d

The degree of the polynomial \widehat{Q}_d is the codimension of the orbit $B_d\varepsilon_{\text{ref}}$, or that of its closure $\widehat{\mathcal{O}}_d$, in $\widehat{\mathcal{N}}_d$.

Recall that $\widehat{\mathcal{N}}_d$ has a basis indexed by the set of indices $\{m + r \leq l \leq d\}$. An elementary computation shows that $\dim \widehat{\mathcal{N}}_d$ is given by a cubic quasi-polynomial in d with leading term $d^3/24$

On the other hand, we have

$$\dim(B_d\hat{\varepsilon}_{\text{ref}}) = \dim(B_d) - \dim(H_d) = \binom{d+1}{2} - d = \binom{d}{2}.$$

Next, denote by $\widehat{\mathcal{N}}_d^0$ the *minimal* or defect-zero part of $\widehat{\mathcal{N}}_d$ spanned by the vectors $\{q_{mr}^l; m+r=l \leq d\}$, and let $\text{pr}_0 : \widehat{\mathcal{N}}_d \rightarrow \widehat{\mathcal{N}}_d^0$ be the natural projection; note that $\hat{e}_{\text{ref}} \in \widehat{\mathcal{N}}_d^0$. Recall that $B_d = T_d U_d$, where $U_d \subset B_d$ is the subgroup of unipotent matrices, which have 1s on the diagonal. It is easy to check that U_d acts trivially on $\widehat{\mathcal{N}}_d^0$, and its action commutes with the projection pr_0 . This motivates the introduction of the toric orbit $T_d \hat{e}_{\text{ref}} \subset \widehat{\mathcal{N}}_d^0$ and its closure $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$.

Lemma 5.1.1. *The projection pr_0 restricted to the orbit $B_d \hat{e}_{\text{ref}}$ establishes a fibration over the toric orbit $T_d \hat{e}_{\text{ref}}$. This map extends to a map between the closures $\widehat{\mathcal{O}} \rightarrow \widehat{\mathcal{T}}$, where $\widehat{\mathcal{T}} = \overline{T_d \hat{e}_{\text{ref}}}$.*

The proof is straightforward; the details will appear elsewhere. One can show that the polynomial $\text{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_d^0]$ gives a certain “piece” of $\widehat{Q}_d = \text{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$. This is relevant because there are standard algorithms to compute the equivariant Poincaré dual of a toric orbit – we presented some of these in the example of the toric orbit in §2.2.2 – but no such algorithm is known for Borel orbits. In section 5.2 we compute a simple relation between $\text{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_d^0]$ and $\text{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$.

Lemma 5.1.1 implies, in particular, that the codimension of $B_d \hat{e}_{\text{ref}}$ is the sum of the codimensions of $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}_d^0$ and the codimension in the fiberwise directions. We collect the appropriate numeric values in the following table:

d	$\dim \widehat{\mathcal{O}} = \binom{d}{2}$	$\dim \widehat{\mathcal{N}}_d$	$\deg \widehat{Q}_d = \text{codim}(\widehat{\mathcal{O}})$	$\dim(\widehat{\mathcal{T}}) = d - 1$	$\dim \widehat{\mathcal{N}}_d^0$	$\text{codim}(\widehat{\mathcal{T}})$
1	0	0	0	0	0	0
2	1	1	0	1	1	0
3	3	3	0	2	2	0
4	6	7	1	3	4	1
5	10	13	3	4	6	2
6	15	22	7	5	9	4

The first 3 columns describe the codimension of the closure of the Borel orbit $\widehat{\mathcal{O}}$ in $\widehat{\mathcal{N}}_d$, while the last three - the codimension of the closure of the toric orbit $\widehat{\mathcal{T}}$ in $\widehat{\mathcal{N}}_d^0$.

Now we are ready for the computations.

5.1.2 The cases $d=1,2,3$

In these cases $\deg \widehat{Q}_d = 0$ and thus $\widehat{Q}_d = 1$; geometrically, this means that $\mathcal{O}_d = \widetilde{\mathcal{E}}_d$, and thus $\widehat{\mathcal{O}}_d = \widehat{\mathcal{N}}_d$. The case of $d = 1$ was described in §3.2.

For $d = 2$ we obtain

$$\text{eP}[\Theta_2] = \text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \frac{z_1 - z_2}{2z_1 - z_2} \text{RC} \left(\frac{1}{z_1} \right) \text{RC} \left(\frac{1}{z_2} \right) z_1^{k-n} z_2^{k-n} dz_1 dz_2. \quad (5.1.1)$$

Expanding the iterated residue, one immediately recovers Ronga's formula [40]:

$$\mathrm{eP}[\Theta_2] = c_{k-n+1}^2 + \sum_{i=1}^{k-n+1} 2^{i-1} c_{k-n+1-i} c_{k-n+1+i}.$$

For $d = 3$, the formula is

$$\begin{aligned} \mathrm{eP}[\Theta_3] = & (-1) \operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=\infty} \operatorname{Res}_{z_3=\infty} \frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)} \\ & \operatorname{RC}\left(\frac{1}{z_1}\right) \operatorname{RC}\left(\frac{1}{z_2}\right) \operatorname{RC}\left(\frac{1}{z_3}\right) z_1^{k-n} z_2^{k-n} z_3^{k-n} dz_1 dz_2 dz_3. \end{aligned} \quad (5.1.2)$$

This is a much more compact and conceptual formula for $\mathrm{eP}[\Theta_3]$ than the one given in [4].

5.1.3 The basic equations in general

As our table in §5.2.1 shows, the polynomial \widehat{Q}_d is not trivial when $d > 3$. As a step towards its computation, we describe a set of equations satisfied by $\widehat{\mathcal{O}} \subset \widehat{\mathcal{N}}_d$ and $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$. We will call these equations “basic”.

The equations will be written in terms of the local variables \hat{u}_τ^l introduced in §4.4.1, where now we assume that $|\tau| = 2$. Clearly, these variables form a dual basis to the basis $\{q_{mr}^l\}$ of $\widehat{\mathcal{N}}_d$. We will streamline our notation by writing \hat{u}_{mr}^l instead of $\hat{u}_{[m,r]}^l$; naturally, we have $\hat{u}_{mr}^l = \hat{u}_{rm}^l$, and $r + m \leq l$.

The construction is as follows. If $i + j + m \leq l$, then the sequence

$$\boldsymbol{\pi}(i, j, m; l) = ([1], [2], \dots, [l-1], [i, j, m], [l+1], \dots, [d-1], [d])$$

is admissible but not complete, hence $\mathbf{u}^{\boldsymbol{\pi}(i,j,m)}$ will appear as a term of some of the relations $\operatorname{Rel}(\boldsymbol{\rho}, \tau)$ introduced in Proposition 4.4.8. In fact, it appears in three different relations:

$$\text{for } \tau = [i], \rho_l = [j, m], \text{ for } \tau = [j], \rho_l = [i, m], \text{ and for } \tau = [m], \rho_l = [i, j];$$

in all cases $\rho_r = [r]$ for $r \neq l$. Next, we reduce the relation $\operatorname{Rel}(\boldsymbol{\rho}, \tau)$ according to the prescription of Lemma 5.2.18. After the reduction, only the terms corresponding to the identity permutation and those corresponding to the transpositions of the form (s, l) survive; for example, in the case $\tau = [m]$, we obtain the “localized” relation

$$\hat{u}_{ijm}^l = \sum_{s=j+m}^{l-i} \hat{u}_{jm}^s \hat{u}_{is}^l. \quad (5.1.3)$$

Note that the number of terms on the right hand side is $l - (i + j + m) + 1$, which is the defect of \hat{u}_{ijm}^l plus 1.

We obtain two other expressions for \hat{u}_{ijm}^l when we choose τ to be $[j]$ or $[i]$, and the resulting equalities provide us with quadratic relations among our variables \hat{u}_{mr}^l , $m + r \leq l \leq d$.

Proposition 5.1.2. *Let $(i, j, m; l)$ be a quadruple of nonnegative integers satisfying $i + j + m \leq l \leq d$. Then the ideal of the variety $\widehat{\mathcal{O}} \subset \widehat{\mathcal{N}}_d$ contains the relations*

$$R(i, j, m; l) : \sum_{s=j+m}^{l-i} \hat{u}_{jm}^s \hat{u}_{is}^l = \sum_{s=i+m}^{l-j} \hat{u}_{im}^s \hat{u}_{js}^l = \sum_{s=i+j}^{l-m} \hat{u}_{ij}^s \hat{u}_{ms}^l. \quad (5.1.4)$$

Remark 5.1.3. • In general, the quadruple $(i, j, m; l)$ gives us 2 relations. If $i = j \neq m$, then the number of relations reduces to 1, and if $i = j = m$, then (5.1.4) is vacuous.

- The equalities $R(i, j, m; l)$ with $i + j + m = l$ are relations of the toric orbit closure $\widehat{\mathcal{T}} \subset \widehat{\mathcal{N}}_d^0$. We will call these equations *toric*.

5.1.4 Basic equations and commuting matrices

The basic equations (5.1.4) have a particularly simple, compact form introducing the following d matrices. For $i = 1, 2, \dots, d$ let

$$(M_k)_{i,j} = \begin{cases} \hat{u}_{ik}^j, & \text{if } i + k \leq j \leq d, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1.5)$$

Remark 5.1.4. • All these matrices are upper-triangular, moreover, the main diagonal and the neighbouring $i - 1$ diagonals of M_i is 0.

- Since $\hat{u}_{ik}^j = \hat{u}_{ki}^j$, for $i \neq k$ the i th row of M_k is equal to the k th row of M_i .

Theorem 5.1.5. *The basic equations can be written as $M_i M_j = M_j M_i$ for any $1 \leq i \leq j \leq d$.*

So M_1, \dots, M_d are commuting matrices. When $d = 2$, adding a stability condition this is exactly the Nakajima quiver description of the Hilbert scheme of points on \mathbb{C}^2 , see [36], Theorem 1.14. The non-basic equations, however do not have such a simple description.

5.1.5 $d=4,5,6$

This is the first nontrivial case: here $\deg \widehat{Q}_4 = 1$, i.e. $\widehat{\mathcal{O}}_4 = \overline{B_4 \widehat{\mathcal{E}}_{\text{ref}}}$ is a hypersurface in $\widehat{\mathcal{N}}_4$. Checking the table at the end of § 5.2.1, we see that in this case the codimension of the toric piece $\widehat{\mathcal{T}}_4$ in $\widehat{\mathcal{N}}_4^0$ is the same as the codimension of $\widehat{\mathcal{O}}_4$ in $\widehat{\mathcal{N}}_4$. This means that $\widehat{\mathcal{O}}_4 = \text{pr}_0^* \widehat{\mathcal{T}}_4$, which implies that $\widehat{Q}_4 = \text{eP}[\widehat{\mathcal{T}}_4, \widehat{\mathcal{N}}_4^0]$.

It is not surprising then to find that the only basic equation is a toric one, corresponding to the quadruple $(1, 1, 2, 4)$:

$$R(1, 1, 2; 4) : \hat{u}_{11}^2 \hat{u}_{22}^4 = \hat{u}_{12}^3 \hat{u}_{13}^4. \quad (5.1.6)$$

We note that this toric hypersurface is essentially our basic example introduced in §2.2.2. The variety defined by (5.1.6) is irreducible, therefore it coincides with $\widehat{\mathcal{T}}$. We have already determined the equivariant Poincaré dual in this case in a number of ways: it is the sum of the weights of any of the monomials in the equation. Applying this here, we arrive at the formula

$$\widehat{Q}_4(z_1, z_2, z_3, z_4) = (2z_1 - z_2) + (2z_2 - z_4) = 2z_1 + z_2 - z_4. \quad (5.1.7)$$

As a result we obtain

$$\begin{aligned} \text{eP}[\Theta_4] &= \text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \text{Res}_{z_3=\infty} \text{Res}_{z_4=\infty} \prod_{l=1}^4 \text{RC} \left(\frac{1}{z_l} \right) z_l^{k-n} dz_l \\ &= \frac{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4)(2z_1 + z_2 - z_4)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)(z_1 + z_3 - z_4)(2z_2 - z_4)(z_1 + z_2 - z_4)(2z_1 - z_4)}. \end{aligned}$$

d=5: Again, we consult our table. We have $\dim \widehat{\mathcal{N}}_5 = 13$ and $\text{codim } \widehat{\mathcal{O}}_5 = 3$, while $\dim \widehat{\mathcal{N}}_5^0 = 6$ and $\text{codim } \widehat{\mathcal{T}}_5 = 2$.

Let us list our variables.

$$\begin{aligned} 6 \text{ toric} &: \hat{u}_{14}^5, \hat{u}_{23}^5, \hat{u}_{13}^4, \hat{u}_{22}^4, \hat{u}_{12}^3, \hat{u}_{11}^2 \\ 4 \text{ defect-1} &: \hat{u}_{13}^5, \hat{u}_{22}^5, \hat{u}_{12}^4, \hat{u}_{11}^3, \\ 2 \text{ defect-2} &: \hat{u}_{12}^5, \hat{u}_{11}^4, \text{ and} \\ 1 \text{ defect-3} &: \hat{u}_{11}^5. \end{aligned}$$

There are 3 toric equations, which necessarily involve the toric variables only:

$$\begin{aligned} R(1, 1, 2; 4) &: \hat{u}_{12}^3 \hat{u}_{13}^4 = \hat{u}_{11}^2 \hat{u}_{22}^4 \\ R(1, 1, 3; 5) &: \hat{u}_{14}^5 \hat{u}_{13}^4 = \hat{u}_{23}^5 \hat{u}_{11}^2 \\ R(1, 2, 2; 5) &: \hat{u}_{14}^5 \hat{u}_{22}^4 = \hat{u}_{23}^5 \hat{u}_{12}^3 \end{aligned} \quad (5.1.8)$$

and one defect-1 equation:

$$R(1, 1, 2; 5) : \hat{u}_{13}^5 \hat{u}_{12}^3 + \hat{u}_{14}^5 \hat{u}_{12}^4 = \hat{u}_{11}^2 \hat{u}_{22}^5 + \hat{u}_{23}^5 \hat{u}_{11}^3 \quad (5.1.9)$$

We observe that the toric equations (5.1.8) describe the vanishing of the 3 maximal minors of a 2×3 matrix. This is an irreducible toric variety, thus we can again argue that it coincides with $\widehat{\mathcal{T}}_5$. Fortunately, this variety is a special case of the A_1 -singularity, this time with $n = 2$ and $k = 3$. Substituting the appropriate weights into (3.2.2), we obtain:

$$\begin{aligned} \text{eP}[\widehat{\mathcal{T}}_5, \widehat{\mathcal{N}}_d^0] &= \\ &= \frac{(z_1 + z_2 - z_3)(2z_1 - z_2)(z_1 + z_4 - z_5) - (2z_2 - z_4)(z_1 + z_3 - z_4)(z_2 + z_3 - z_5)}{z_1 + z_4 - z_2 - z_3} = \\ &= 2z_1^2 + 3z_1z_2 - 2z_1z_5 + 2z_2z_3 - z_2z_4 - z_2z_5 - z_3z_4 + z_4z_5. \end{aligned} \quad (5.1.10)$$

Next, inspecting (5.1.9), it is not difficult to convince ourselves that the map $\text{pr}_0 : \widehat{\mathcal{O}}_5 \rightarrow \widehat{\mathcal{T}}_5$ is the projection of a vector bundle onto its base; the fibers of this vector bundle are hyperplanes in the 7-dimensional complement of $\widehat{\mathcal{N}}_5^0$ in $\widehat{\mathcal{N}}_5$. This implies that $\text{eP}[\widehat{\mathcal{O}}_5, \widehat{\mathcal{N}}_5]$ is the product of $\text{eP}[\widehat{\mathcal{T}}_5, \widehat{\mathcal{N}}_5^0]$ and the weight of the relation $R(1, 1, 2; 5)$. The latter equals $2z_1 + z_2 - z_5$, hence the final result is

$$\widehat{Q}_5(z_1, z_2, z_3, z_4, z_5) = (2z_1 + z_2 - z_5)(2z_1^2 + 3z_1z_2 - 2z_1z_5 + 2z_2z_3 - z_2z_4 - z_2z_5 - z_3z_4 + z_4z_5).$$

d=6

Now \widehat{Q}_6 is a degree-7 polynomial in 6 variables, and one needs the help of a computer algebra program to do the calculations. One obtains this way the equivariant Poincaré dual of the variety M_6 defined by the basic equations rather quickly, however, it turns out that this variety is *not* irreducible, and $\widehat{\mathcal{O}}_6$ is only one of its two components. The other component is a linear subspace, hence we obtain \widehat{Q}_6 by omitting the appropriate term from $\text{eP}[M_6]$. The details of this computation will be published elsewhere.

5.2 Computing \widehat{Q}_d in general

In this section we compute a simple relation between $\text{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_d^0]$ and $\text{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$. Recall, that $\text{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_d^0]$ is the multidegree of a toric variety; computation of this is a well-known and investigated problem in commutative algebraic geometry. The multidegree can be read off the polytope of the toric variety several different ways. (see [35] for details) However, our main formula (4.4.19) contains $\text{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$, which is the multidegree of a Borel orbit. Multidegrees of Borel orbits is a harder task, and almost nothing is known in general.

Since $\widehat{\mathcal{T}} = \overline{T_d \widehat{\mathcal{E}}_{\text{ref}}}$ and $\widehat{\mathcal{O}}_d = \overline{B_d \widehat{\mathcal{E}}_{\text{ref}}}$, we may think of $\widehat{\mathcal{T}}$ as the toric 'head' of $\widehat{\mathcal{O}}$: a toric subvariety of $\widehat{\mathcal{O}}$.

$\widehat{\mathcal{O}}$ is a homogeneous polynomial in d variables, which correspond to a basis of the d -dimensional weight lattice. After a proper linear coordinate change we are going to distinguish the first coordinate, and view $\text{eP}[\widehat{\mathcal{O}}, \widehat{\mathcal{N}}_d]$ as a polynomial in this coordinate, whose coefficients are homogeneous polynomials in the remaining $d - 1$ coordinates. It turns out that the leading coefficient (whose degree is minimal) is $\text{eP}[\widehat{\mathcal{T}}, \widehat{\mathcal{N}}_{d-1}^0]$, and this is the main result of this section.

5.2.1 The degree of \widehat{Q}_d

We use the same terms and notations as in the previous section. Recall the structure of the Borel orbit $B_d \widehat{\mathcal{E}}_{\text{ref}}$ and its closure, $\widehat{\mathcal{O}} = \overline{B_d \widehat{\mathcal{E}}_{\text{ref}}}$ in $\widehat{\mathcal{N}}_d$. $\widehat{\mathcal{N}}_d$ has a basis q_{mr}^l , indexed by the set of indices $\{m + r \leq l \leq d\}$.

We denoted by $\widehat{\mathcal{N}}_d^0$ the *minimal* or defect-zero part of $\widehat{\mathcal{N}}_d$ spanned by the vectors $\{q_{mr}^l; m + r = l \leq d\}$, and $\text{pr}_0 : \widehat{\mathcal{N}}_d \rightarrow \widehat{\mathcal{N}}_d^0$ is the natural projection; note that

$\hat{\varepsilon}_{\text{ref}} \in \hat{\mathcal{N}}_d^0$. Consequently, we denote by $\hat{\mathcal{N}}_d^>$ the positive-defect part of $\hat{\mathcal{N}}_d$ spanned by the vectors $\{q_{mr}^l; m+r < l \leq d\}$. Recall that $B_d = T_d U_d$, where $U_d \subset B_d$ is the subgroup of unipotent matrices, and $\hat{\mathcal{T}} = \overline{T_d \hat{\varepsilon}_{\text{ref}}} \subset \hat{\mathcal{N}}_d^0$. The projection pr_0 restricted to the orbit $B_d \hat{\varepsilon}_{\text{ref}}$ establishes a fibration over the toric orbit $T_d \hat{\varepsilon}_{\text{ref}}$. This map extends to a map between the closures $\hat{\mathcal{O}} \rightarrow \hat{\mathcal{T}}$.

Lemma 5.1.1 implies, in particular, that the codimension of $B_d \hat{\varepsilon}_{\text{ref}}$ is the sum of the codimensions of $\hat{\mathcal{T}}$ in $\hat{\mathcal{N}}_d^0$ and the codimension in the fiberwise directions. We use the shortened notation $\Gamma_d = \text{eP}[\hat{\mathcal{T}}, \hat{\mathcal{N}}_d^0]$ and recall that $\hat{Q}_d = \text{eP}[\hat{\mathcal{O}}, \hat{\mathcal{N}}_d]$. We collected the appropriate numeric values in the following table again, as in the previous section:

d	$\dim \hat{\mathcal{O}} = \binom{d}{2}$	$\dim \hat{\mathcal{N}}_d$	$\deg \hat{Q}_d = \text{codim}(\hat{\mathcal{O}})$	$\dim(\hat{\mathcal{T}}) = d - 1$	$\dim \hat{\mathcal{N}}_d^0$	$\text{codim}(\hat{\mathcal{T}})$
1	0	0	0	0	0	0
2	1	1	0	1	1	0
3	3	3	0	2	2	0
4	6	7	1	3	4	1
5	10	13	3	4	6	2
6	15	22	7	5	9	4

Let us compute the degree of the homogeneous polynomial \hat{Q}_d : this is the codimension of the orbit $B_d \hat{\varepsilon}_{\text{ref}}$, or that of its closure $\hat{\mathcal{O}}_d$, in $\hat{\mathcal{N}}_d$.

Let t_l denote the number of defect-zero vectors q_{mr}^l with $m+r = l \leq d$.

Lemma 5.2.1. $t_1 = 0$, and

1. $\dim \hat{\mathcal{N}}_d^0 = t_2 + t_3 + \dots + t_d$
2. $\dim \hat{\mathcal{N}}_d = t_d + 2t_{d-1} + 3t_{d-2} + \dots + (d-1)t_2$
3. $\dim(\hat{\mathcal{T}}) = d - 1$
4. $\dim(\hat{\mathcal{O}}) = \dim(B_d) - \dim(H_d) = \binom{d+1}{2} - d = \binom{d}{2}$.

Proof. The first part is obvious. The basis of $\hat{\mathcal{N}}_d$ described above is the union of the basis of $\hat{\mathcal{N}}_d^0, \hat{\mathcal{N}}_{d-1}^0, \dots, \hat{\mathcal{N}}_2^0$. This implies the second equation. The last two are straightforward. \square

Since $t_{2k} = t_{2k+1} = k$, the exact values are the following:

- $\dim \hat{\mathcal{N}}_d = \begin{cases} \frac{(4k-1)k(k+1)}{6} & \text{if } d = 2k \\ \frac{(4k+5)k(k+1)}{6} & \text{if } d = 2k + 1 \end{cases}$
- $\deg \hat{Q}_d = \dim Y_2 - \binom{d}{2} = \begin{cases} \frac{4k^3 - 9k^2 + 5k}{6} & \text{if } d = 2k \\ \frac{4k^3 - 3k^2 - k}{6} & \text{if } d = 2k + 1. \end{cases}$

- $\deg T_d = \dim Y_{2,0} - (d-1) = \begin{cases} (k-1)^2 & \text{if } d = 2k \\ k(k-1) & \text{if } d = 2k+1 \end{cases}$

As a consequence we have the following relations

Proposition 5.2.2. 1.

$$\deg \widehat{Q}_d - \deg T_d = \deg \widehat{Q}_{d-1} \quad (5.2.1)$$

2.

$$\deg \widehat{Q}_d - \deg T_d = \dim \widehat{\mathcal{N}}_{d-3} = \dim \widehat{\mathcal{N}}_{d-2}^> \quad (5.2.2)$$

Now, (5.2.1) says that the codimension of the generic fiber in Lemma 5.1.1 equal to the codimension of $\widehat{\mathcal{O}}_{d-1}$ in $\widehat{\mathcal{N}}_{d-1}$. This can give the false impression that $\widehat{Q}_d = T_d \cdot \widehat{Q}_{d-1}$, but it is true only for $d \leq 5$.

The RHS of 5.2.2 is the number of vectors q_{mr}^l with $m+r < l \leq d-2$. So this is the number of vectors with positive defect on level $l \leq d-2$.

As before, \hat{u}_{mr}^l denotes the dual coordinate of q_{mr}^l . We have $\widehat{\mathcal{O}}_d = \text{Spec} \mathbb{C}[\hat{u}_{mr}^l; m+r \leq l \leq d] / \mathcal{I}_d$ for some ideal $\mathcal{I}_d \triangleleft \mathbb{C}[\hat{u}_{mr}^l; m+r \leq l \leq d]$, and $\widehat{\mathcal{T}}_d = \text{Spec} \mathbb{C}[\hat{u}_{mr}^l; m+r = l \leq d] / \mathcal{J}_d$ for some ideal $\mathcal{J}_d \triangleleft \mathbb{C}[\hat{u}_{mr}^l; m+r = l \leq d]$. Note that $\mathcal{J}_d = \mathcal{I}_d \cap \mathbb{C}[\hat{u}_{mr}^l; m+r = l \leq d]$.

5.2.2 Primary decomposition of monomial ideals

This subsection is a concise summary of the material we need in commutative algebra related to Groebner degeneration and primary decomposition. All this can be found in [14], Chapter 15.

Definition 5.2.3. A monomial order on the polynomial ring $\mathbb{C}[x_1, \dots, x_s]$ is a total order \succ on the monomials, such that if M_1, M_2 and $1 \neq N$ are monomials then $M_1 > M_2$ implies $NM_1 > NM_2 > M_2$.

For example, the lexicographic monomial order w.r.t any order on the variables x_1, \dots, x_s is a monomial order. Let \succ be a monomial order on $\mathbb{C}[x_1, \dots, x_s]$, and $I \triangleleft \mathbb{C}[x_1, \dots, x_s]$ an ideal. For any $p \in I$ denote $in_\succ(p)$ the greatest term of p with respect of the order \succ .

Definition 5.2.4. The ideal $in_\succ(I)$ of $\mathbb{C}[x_1, \dots, x_s]$ generated by the initial terms $in_\succ(p)$ for all $p \in I$ is called the initial monomial ideal of I w.r.t \succ .

The scheme $A = \text{Spec}(\mathbb{C}[x_1, \dots, x_s]/I)$ has a flat deformation, whose zero-fiber is $\text{Groeb}(A) = \text{Spec}(\mathbb{C}[x_1, \dots, x_s]/in_\succ(I))$. This is called Groebner degeneration, and by the axioms of section 2.2.1, $\text{Groeb}(A)$ has the same multidegree as A .

Let

$$in_\succ(I) = \bigcap_{\mathbf{a} \in A} P_{\mathbf{a}} \quad (5.2.3)$$

be the primary decomposition of the monomial ideal $in_{\succ}(I)$. The primary ideals in the decomposition (5.2.6) correspond to the 'blocks' of A , i.e the – possibly nonreduced, and different dimensional – components, whose 'union' is A . (see [13] sect II.3.3) The maximal dimensional components $\text{Spec}(\mathbb{C}[x_1, \dots, x_s]/P_{\mathbf{a}})$ correspond to the maximal dimensional associated primes $P_{\mathbf{a}}$. These components and their multiplicities are well-defined in the decomposition. By the first axiom in sect. 2.2.1 the multidegree $\text{mdeg}[A, \mathbb{C}^d]$ depends only on the maximal dimensional components of A . Here $\mathbb{C}^d = \text{Spec}\mathbb{C}[x_1, \dots, x_s]$ is the affine d -space. We state the following proposition for $in_{\succ}I$, but it is true for any ideal generated by monomials.

Proposition 5.2.5. 1. *The maximal primary ideals in the decomposition (5.2.6) has the form*

$$P = \langle x_{i_1}^{r_1}, x_{i_2}^{r_2}, \dots, x_{i_c}^{r_c} \rangle \triangleleft \mathbb{C}[x_1, \dots, x_s] \quad (5.2.4)$$

with $1 \leq x_{i_1} < \dots < x_{i_c} \leq s$ and $r_1, \dots, r_c \in \mathbb{Z}^+$. Here c is the codimension of A , i.e the codimension of its maximal dimensional components.

2. *The maximal dimensional component corresponding to (5.2.4) is*

$$A_P = \text{Spec}(\mathbb{C}[x_1, \dots, x_s]/P)$$

. This is a linear subspace of \mathbb{C}^s , with multiplicity $r_1 r_2 \dots r_s$.

3. *If the action on \mathbb{C}^s is diagonal with respect to the coordinates x_1, \dots, x_s and weights w_1, \dots, w_s ,*

$$\text{mdeg}[A_P, \mathbb{C}^s] = r_1 r_2 \dots r_s \cdot w_1 w_2 \dots w_s. \quad (5.2.5)$$

Proof. The first two part is straightforward from the definitions, and the third follows from the third axiom of sect. 2.2.1. \square

Although the following observation is trivial from the definition, we formulate as a separate proposition, and we use it continuously in section 5.2.3 and 5.2.4.

Proposition 5.2.6. *Suppose that the monomial $in_{\succ}(p)$ is divisible by any of $x_{i_1}^{r_1}, \dots, x_{i_c}^{r_c}$ for all $p \in I$, but this is not true for bigger exponents r_i . Then (5.2.4) is a maximal primary component in the primary decomposition of I .*

5.2.3 The main theorem

Definition 5.2.7. Let $>_1, >_2, \dots$ be partial orders of the variables x_1, \dots, x_s . The lexicographic product of $>_1, >_2, \dots$ is a partial order \succ , where $x_i \succ x_j$ if $x_i >_k x_j$ for the first k such that x_i and x_j are compatible.

We use the lexicographic product of 4 partial orders on the coordinates \hat{u}_{mr}^l for our Groebner degeneration of $\hat{\mathcal{O}}$. We use the following definition throughout this chapter:

Definition 5.2.8. We call \hat{u}_{mr}^l and the corresponding u_{mr}^l *good coordinate*, if $m+r < l \leq d-2$.

The partial orders $>_1, >_2, >_3$ and $>_4$ are defined in the following way

Definition 5.2.9. 1.

$$\hat{u}_{mr}^l >_1 \hat{u}_{\bar{m}\bar{r}}^{\bar{l}} \iff \hat{u}_{mr}^l \text{ is good, but } \hat{u}_{\bar{m}\bar{r}}^{\bar{l}} \text{ is not good coordinate}$$

2.

$$\hat{u}_{mr}^l >_2 \hat{u}_{\bar{m}\bar{r}}^{\bar{l}} \iff l - (m+r) < \bar{l} - (\bar{m} + \bar{r})$$

3.

$$\hat{u}_{mr}^l >_3 \hat{u}_{\bar{m}\bar{r}}^{\bar{l}} \iff l < \bar{l}$$

4.

$$\hat{u}_{mr}^l >_4 \hat{u}_{\bar{m}\bar{r}}^{\bar{l}} \iff m < \bar{m}$$

Note that by definition $m \leq r$ and $\bar{m} \leq \bar{r}$.

5. Let \succ denote the lexicographic product of $>_1, >_2, >_3$ and $>_4$.

Proposition 5.2.10. \succ is a total order of the coordinates \hat{u}^\bullet , i.e any two coordinates are compatible.

Definition 5.2.11. Let $>$ be an order of the variables $x_1, \dots, x_s, x_1 > x_2 > \dots > x_s$. The generated lexicographic monomial order on $\mathbb{C}[x_1, \dots, x_s]$ is defined as follows: $x_1^{a_1} \dots x_s^{a_s} > x_1^{b_1} \dots x_s^{b_s}$ iff $a_i > b_i$ for the first index with $a_i \neq b_i$.

We denote with the same \succ the generated lexicographic monomial order on $\mathbb{C}[\hat{u}^\bullet]$. Let \mathcal{IN}_d denote the initial monomial ideal of the ideal \mathcal{I}_d and \mathcal{JN}_d the initial monomial ideal of \mathcal{J}_d , with respect to \succ . Note that $\mathcal{JN}_d = \mathcal{IN}_d \cap \mathbb{C}[u_{mr}^l; m+r = l \leq d]$.

\mathcal{IN}_d is a flat deformation of \mathcal{I}_d hence $\text{Spec}(\mathcal{IN}_d)$ has the same multidegree as $\text{Spec}(\mathcal{I}_d)$, and the same true for \mathcal{JN}_d

To state the main result of this section we need some extra notation.

Let $\mathcal{IN}_d = \bigcap_{a \in A} \mathbf{P}_a$ be the primary decomposition of the initial ideal. Here \mathbf{P}_a is a monomial primary ideal, so by Prop 5.2.5 it has the form

$$\mathbf{P}_a = \langle \mathbf{u}_1^{s_1}, \dots, \mathbf{u}_\alpha^{s_\alpha}, \mathbf{u}_{\alpha+1}^{s_{\alpha+1}}, \dots, \mathbf{u}_{\alpha+\beta}^{s_{\alpha+\beta}} \rangle \quad (5.2.6)$$

where $\mathbf{u}_i = \hat{u}_{m_i r_i}^{l_i}$, and $s_i \in \mathbb{Z}^+$ for $i = 1, \dots, \alpha + \beta$. The first α coordinates have positive defect and the last β coordinates are the zero-defect coordinates, i.e

$$m_1 + r_1 - l_1 > 0, m_2 + r_2 - l_2 > 0, \dots, m_\alpha + r_\alpha - l_\alpha > 0 \quad (5.2.7)$$

$$m_{\alpha+1} + r_{\alpha+1} - l_{\alpha+1} = \dots = m_{\alpha+\beta} + r_{\alpha+\beta} - l_{\alpha+\beta} = 0 \quad (5.2.8)$$

By Prop. 5.2.5, $\alpha + \beta \geq \deg \widehat{Q}_d$, with equality for the maximal dimensional components of $\widehat{\mathcal{O}}$. The lower dimensional components are redundant when computing multidegree, so we restrict our attention to the $\alpha + \beta = \deg \widehat{Q}_d$ case. By Prop 5.2.5 $\beta \geq \deg T_d$, so (5.2.2) implies $\alpha \leq \dim \widehat{\mathcal{N}}_{d-2}^0$.

Definition 5.2.12. We call \mathbf{P}_a *minimal*, if it has minimal number of zero-defect generators, that is $\beta = \deg T_d = \text{codim}(\widehat{T} \subset \widehat{\mathcal{N}}_d^0)$ and $\alpha = \dim \widehat{\mathcal{N}}_{d-2}^0$.

We suppose that \mathbf{P}_a is minimal for all $a \in A$.

Theorem 5.2.13. 1. For any $a \in A$

$$\langle \mathbf{u}_{\alpha+1}, \dots, \mathbf{u}_{\alpha+\beta} \rangle \triangleleft \mathbb{C}[\hat{u}_{mr}^l; m+r=l \leq d]$$

is an element of the primary decomposition of \mathcal{JN}_d .

2. For any $a \in A$ the set $\{\mathbf{u}_1, \dots, \mathbf{u}_\alpha\}$ equals to the set $\{\hat{u}_{mr}^l; m+r < l \leq d-2\}$.

3. The reverse is also true. Let \mathbf{P} be a primary ideal in the primary decomposition of \mathcal{JN}_d . If this corresponds to maximal dimensional component, then

$$\langle \hat{u}_{mr}^l : m+r < l \leq d-2, \mathbf{P} \rangle \triangleleft \mathbb{C}[\hat{u}_{mr}^l; m+r \leq l \leq d]$$

is a primary ideal in the primary decomposition of \mathcal{IN}_d .

Recall that the weights z_1, \dots, z_d form a basis of the d -dimensional weight space, and the weight of \hat{u}_{mr}^l under the torus action is $z_m + z_r - z_l$. The defect is $l - (m + r)$. Let us choose another basis measuring the defect in a more appropriate way. Introduce

$$w_1 = z_1, w_2 = 2z_1 - z_2, w_3 = z_1 + z_2 - z_3, \dots, w_d = z_1 + z_{d-1} - z_d \quad (5.2.9)$$

The inverse transformation is

$$z_1 = w_1, z_2 = 2w_1 - w_2, \dots, z_d = dw_1 - w_2 - w_3 - \dots - w_d \quad (5.2.10)$$

The weight of \hat{u}_{mr}^l in the new coordinates is

$$(m+r-l)w_1 - w_2 - \dots - w_m + w_{r+1} + \dots w_l. \quad (5.2.11)$$

Note that the coefficient of w_1 is the opposite of the defect.

By Proposition 5.2.5 $\widehat{Q}_d(w_1, \dots, w_d)$ is the sum of terms, each corresponding to some $a \in A$, namely, this is the product of the weights of the generators of \mathbf{P}_a . There are exactly α positive-defect coordinates among the generators, the degree of $\widehat{Q}(w_1, \dots, w_d)$ in w_1 is α . The top coefficient is a degree- β homogeneous polynomial in w_2, \dots, w_d , and by the third part of Theorem 5.2.13,

Corollary 5.2.14.

$$\text{coeff}_{w_1^{\widehat{Q}}_d}(w_1, \dots, w_d) = C_d \cdot T_d$$

with the constant

$$C_d = (-1)^{t_{d-3}+t_{d-4}+\dots+t_2} (-2)^{t_{d-4}+\dots+t_2} \dots (-d+4)^{t_2} = \begin{cases} (-1)^{(k-2)(k-1)} (-2)^{(k-2)(k-2)} \dots (-2k+4)^1 & \text{if } d = 2k \\ (-1)^{(k-1)(k-1)} (-2)^{(k-2)(k-1)} \dots (-2k+3)^1 & \text{if } d = 2k+1 \end{cases} \quad (5.2.12)$$

The theorem implies something stronger. Let $\widehat{Q}_d^{\text{top}}$ denote the sum of the terms in \widehat{Q}_d corresponding to the minimal primary ideals. Then

Corollary 5.2.15.

$$\widehat{Q}_d^{\text{top}} = T_d \cdot \prod_{m+r < l \leq d-2} (z_m + z_r - z_l) \quad (5.2.13)$$

The proof of Theorem 5.2.13 fills out the remaining part of this section.

5.2.4 Proof of Theorem 5.2.13

Recall that the elements of \mathcal{I}_d are local equivalents of the relations on the orbit closure \mathcal{O} in $\widetilde{\mathcal{E}}$ near the distinguished fixed point $\widetilde{\varepsilon}_{\text{dist}}$. At this point the coordinates u_τ^l of $\widetilde{\mathcal{E}} = \mathcal{E}/B_R$ can be eliminated for $|\tau| > 2$, in other words the Zariski tangent space to \mathcal{O} at $\widetilde{\varepsilon}_{\text{dist}}$ is contained in the linear subspace of $T_{\widetilde{\varepsilon}_{\text{dist}}}\widetilde{\mathcal{E}}$ spanned by the tangent vectors indexed by the length-2 partitions, which was denoted by $\widehat{\mathcal{N}}_d$. The projection of $T_{\widetilde{\varepsilon}_{\text{dist}}}\mathcal{O}$ onto $\widehat{\mathcal{N}}_d$ is $\widehat{\mathcal{O}}_d$, and \widehat{Q}_d is the multidegree of $\widehat{\mathcal{O}}$ in $\widehat{\mathcal{N}}_d$.

Recall that the space $\mathbb{C}[u^\bullet]$ of polynomials on \mathcal{E} forms a left-right representation of the group $B_L \times B_R$; the variables are u_τ^l , $1 \leq l \leq d$, $\sum(\tau) \leq l$. In particular, it has two multigradings inherited from the T_R and T_L actions: $\text{deg}_R(u_\tau^l)$ is the l th basis vector in \mathbb{Z}^d , while $\text{deg}_L(u_\tau^l)$ is the vector of multiplicities $(\text{mult}(i, \tau), i = 1, \dots, d)$. A combination of these gradings will be particularly important for us: $\text{defect}(u_\tau^l) = l - \text{sum}(\tau)$; this induces a $\mathbb{Z}^{\geq 0}$ -grading on $\mathbb{C}[u^\bullet]$. Denote the nilpotent Lie algebras of strictly upper-triangular matrices corresponding to B_R and B_L by \mathfrak{n}_R and \mathfrak{n}_L , respectively. These Lie algebras are generated by the simple root vectors

$$\Delta_L = \{E_{l,l+1}^L; l = 1, \dots, d-1\}, \quad \Delta_R = \{E_{l,l+1}^R; l = 1, \dots, d-1\}.$$

The action of these root vectors on the coordinates of \mathcal{E} is the following:

$$\begin{cases} \mathfrak{n}_R u_\tau^l = u_\tau^l \mathfrak{n}_L = 0, & \text{if } \text{sum}(\tau) = l, \\ E_{m,m+1}^R u_\tau^l = \delta_{l,m+1} u_\tau^{l-1}, \quad u_\tau^l E_{m,m+1}^L = \text{mult}(m, \tau) u_{\tau-m \cup m+1}^l, & \text{if } \text{sum}(\tau) < l. \end{cases} \quad (5.2.14)$$

where $\delta_{a,b}$ is the Kronecker delta.

A relation Z for \mathcal{O} in \mathcal{E} is an element of the ideal $I_{\mathcal{O}} < \mathbb{C}[u^\bullet]$ characterized by the properties (see Prop. 4.4.7)

Proposition 5.2.16. $Z \in I_{\mathcal{O}}$ iff it satisfies

1. Z is of pure $T_R \times T_L$ weight,
2. $\mathbf{n}_R Z = 0$
3. $[Z\mathbf{n}_L^N](\varepsilon_{\text{ref}}) = 0$ for all $N = 1, \dots$

Definition 5.2.17. The *defect* of Z is the defect of its monomials. Since Z is a pure $T_R \times T_L$ weight, this is well-defined. The defect of a monomial is the sum of the defects of its terms. We call the zero-defect relations *toric* relations.

Recall that locally, near $\tilde{\varepsilon}_{\text{dist}}$, Z has a particular simple form.

Lemma 5.2.18. Let $Z \in I_{\mathcal{O}}$ be a global relation in $\mathbb{C}[u^\bullet]$. Then the corresponding local relation \tilde{Z} near the fixed point $\tilde{\varepsilon}_{\text{dist}}$ in terms of the coordinates \hat{u}_{mr}^l ($m+r \leq l$) and \hat{u}_m^l ($m \leq l$) is obtained by

- setting $u_{l,l}$ to 1, for $l = 1, \dots, d$,
- setting $u_{m,l}$ to 0, for $1 \leq m < l \leq d$,
- replacing the remaining variables u_τ^l by \hat{u}_τ^l .

The elements of \mathcal{I}_d are the local forms of elements in $I_{\mathcal{O}}$ near $\tilde{\varepsilon}_{\text{dist}}$. So each $\tilde{Z} \in \mathcal{I}_d$ comes from a global form $Z \in I_{\mathcal{O}}$, which is a polynomial in $\mathbb{C}[u_{mr}^l : m+r \leq l, u_m^l : m \leq l]$ satisfying the properties of Prop. 5.2.16.

The first part of Theorem 5.2.13 follows from the fact that \mathcal{JN}_d consist of the toric elements of \mathcal{IN}_d . Consequently, any $M \in \mathcal{JN}_d$ is a toric element of \mathcal{IN}_d , so it is divisible by $\mathbf{u}_i^{s_i}$ for some $i > \alpha$.

By Proposition 5.2.6, the second half of Theorem 5.2.13 follows from the following

Lemma 5.2.19. Let $\tilde{Z} \in \mathcal{I}_d$ be a local relation with defect > 0 . Then the initial monomial with respect to \succ contains at least one good coordinate as a factor, whose exponent is 1.

Proof of Lemma 5.2.19

Let $Z \in I_{\mathcal{O}}$ be a global form of \tilde{Z} .

To simplify our life, if $\pi_2 = \{m, l\}$ is a pair with $m \leq l \leq d$ and $\pi_3 = \{m, r, l\}$ is a triple with $m+r \leq l \leq d$ then we also use the notations $u_{\pi_2} = u_m^l$ and $u_{\pi_3} = u_{mr}^l$ for the coordinates on $\tilde{\mathcal{E}}$. We refer a coordinate u_{π_2} (resp. u_{π_3}) as two-index (resp. three-index) coordinate.

Let $M_{\mathbf{a}}$ ($\mathbf{a} \in A(Z)$) be the monomials of Z , i.e

$$Z = \sum_{\mathbf{a} \in A(Z)} C_{\mathbf{a}} M_{\mathbf{a}} = \sum_{\mathbf{a} \in A(Z)} C_{\mathbf{a}} \prod_{\pi_2 \in \Pi_2^{\mathbf{a}}} \prod_{\pi_3 \in \Pi_3^{\mathbf{a}}} u_{\pi_2} u_{\pi_3} \quad (5.2.15)$$

When M is a monomial term of Z we also use Π_2^M and Π_3^M for the two- and three-index coordinates of M . To avoid the exponents, we think of these as multisets, i.e elements of Π_2^M can be equal, and the same for Π_3^M .

Since Z is $T_L \times T_R$ -homogeneous, $\Pi_2^{\mathbf{a}}$ has the same number of elements for all $\mathbf{a} \in A$, and the same is true for $\Pi_3^{\mathbf{a}}$. This is obvious from Comparing the T_L weights and T_R weights of two monomials.

By Lemma 5.2.18 \tilde{Z} is the sum of those $C_{\mathbf{a}}M_{\mathbf{a}}$'s where $\Pi_2^{\mathbf{a}}$ does not contain a positive-defect coordinate, i.e (m, l) with $m < l$. In particular, if $\tilde{Z} \neq 0$ then there is no positive-defect two-index coordinate in some $M_{\mathbf{a}}$. Using the definition of \prec Lemma 5.2.19 follows from

Proposition 5.2.20. *Let Z be a global form of \tilde{Z} . Then there is $\mathbf{a} \in A$ such that $M_{\mathbf{a}}$ contains at least one good coordinate but $\Pi_2^{\mathbf{a}}$ does not contain positive-defect coordinate, i.e u_m^l with $m < l$.*

We prove by induction on the defect of Z . The $\text{def}(Z) = 1$ case is Lemma 5.2.25 below. Now suppose that $\text{def}(Z) \geq 2$, and the Lemma is true for relations with smaller defect.

Definition 5.2.21. We say that $Z \in I_{\mathcal{O}}$ is a *primitive* element of the ideal if there is no $\tilde{Z} \in I_{\mathcal{O}}$ and $p \in \mathbb{C}[u^{\bullet}]$ such that

$$Z = p\tilde{Z}$$

Clearly, it is enough to prove Prop. 5.2.20 for primitive elements, and we suppose that Z is such an element of $I_{\mathcal{O}}$.

Define

$$m_0(Z) = \min \{m : \exists \mathbf{a} \in A(Z), 1 \leq m \leq r \leq d \text{ such that } m + r < l, (m, r, l) \in \Pi_3^{\mathbf{a}}\} \quad (5.2.16)$$

Note that the definition is meaningless if the defined set is empty. However, if there is no positive-defect coordinate in $\Pi_3^{\mathbf{a}}$ for all $\mathbf{a} \in A(Z)$, then $\Pi_2^{\mathbf{a}}$ contains at least one positive-defect coordinate for all \mathbf{a} , and therefore $\tilde{Z} = 0$ by Lemma 5.2.16, so (5.2.16) is non-empty.

Definition 5.2.22. We say that the index m (and r) has defect k in the variable u_{mr}^l (resp. u_m^l) if $l - (m + r) = k$ (resp. $l - m = k$). The notation is $\text{def}(m, \pi_3)$ ($\text{def}(m, \pi_2)$).

Take $Z^* = ZE_{m_0, m_0+1}^L$. Since $\text{def}Z^* = \text{def}Z - 1$, by the induction hypothesis Z^* has a monomial term M^* without positive-defect two-index coordinate u_{π_2} :

$$M^* = u_{m_1 r_1}^{l_1} u_{m_2 r_2}^{l_2} \dots u_{m_s r_s}^{l_s} u_{l_{s+1}}^{l_{s+1}} \dots u_{l_t}^{l_t} \quad (5.2.17)$$

where one of the first s terms is a good coordinate, say $u_{m_1 r_1}^{l_1}$.

This coordinate comes from a term M of Z , by changing the lower index m_0 to $m_0 + 1$ in one coordinate of M where m_0 has positive defect. We call this *shifting* coordinate.

Case 1) *The shifting coordinate is one of the first s .*

Then $u_{m_1 r_1}^{l_1}$ is a good coordinate in M and the two-index coordinates have defect 0, since they coincide with the two-index coordinates of M^* . So M satisfies Prop. 5.2.20

Case 2) *The shifting coordinate is among the two-index coordinates,*

say the $s + 1$ th. That is, $m_0 + 1 = l_{s+1}$ and M contains $u_{m_0}^{m_0+1}$, but all the other two-index coordinates have zero defect, and $\Pi_3^M = \Pi_3^{M^*}$.

Suppose moreover, that $m_0(Z) = \max_{Z' \in I_{\mathcal{O}}} m_0(Z')$. We prove the following

Proposition 5.2.23. *Z must have a monomial N , which has a coordinate u_{mr}^l with $m \geq 1, r \geq 1, m + r = m_0, l = m_0 + 1$.*

Since m and r have defect 1 in this coordinate and one of them is less than m_0 , this contradicts to the minimality of m_0 .

Proof of Prop. 5.2.23

Suppose that $u_{m(m_0-m)}^{m_0+1}$ does not occur in Z with $m \geq 1$ and $m_0 - m \geq 1$. Then by the minimality of m_0 the possible positive-defect coordinate on the $m_0 + 1$ th level is $u_{m_0}^{m_0+1}$. Take $Z^{\text{shift}} = E_{m_0, m_0+1}^R M$. Then $Z^{\text{shift}} \in I_{\mathcal{O}}$ by Prop. 5.2.16. Since $E_{m_0, m_0+1}^R u_{m_0}^{m_0+1} = u_{m_0}^{m_0}$, any monomial M^{shift} of Z^{shift} must contain the coordinate $u_{m_0}^{m_0}$ which comes from $u_{m_0}^{m_0+1}$ in the corresponding monomial of Z . So

$$Z^{\text{shift}} = u_{m_0}^{m_0} \cdot \check{Z}. \quad (5.2.18)$$

Since $Z^{\text{shift}} \in i_{\mathcal{O}}$, and $u_{m_0}^{m_0}$ is a zero-defect coordinate, \check{Z} must also satisfy the properties in Lemma 5.2.16, so $\check{Z} \in I_{\mathcal{O}}$. That is

$$Z^{\text{rem}} = Z - u_{m_0}^{m_0+1} \cdot \check{Z} \in I_{\mathcal{O}}. \quad (5.2.19)$$

If $Z^{\text{rem}} = 0$ then Z is not primitive; contradiction. Otherwise, either the corresponding minimum 5.2.16 for Z^{rem} is greater than m_0 , which is again a contradiction, or Case 1) holds for Z^{rem} . In the latter case Z^{rem} contains a searched monomial, which has a good coordinate but no double index with positive defect, and this is also a monomial of Z . Prop. 5.2.23 and Lemma 5.2.20 is proved.

Note that we proved the following more general

Proposition 5.2.24. *If the primitive Z contains a monomial with a term u_{l-1}^l but u_m^l and u_{mr}^l do not occur when $m < l - 1, m + r < l - 1$ then some u_{mr}^l with $m + r = l - 1$ must occur in Z*

To finish the proof of the second half of Theorem 5.2.13 only the $\text{def}(Z) = 1$ case is a left:

Proposition 5.2.25. *Let $Z \in I_{\mathcal{O}}$ be a relation, $\text{def}Z = 1$. Then there is $\mathbf{a} \in A$ such that $M_{\mathbf{a}}$ contains at least one good coordinate but $\Pi_2^{\mathbf{a}}$ does not contain positive-defect coordinate.*

Proof of Prop. 5.2.25

Each monomial of Z has exactly one coordinate of defect 1. So we have to prove that one of them is a good coordinate. We prove a slightly weaker fact, namely: Z contains one of $\{u_{mr}^l : m+r \leq d-2, l-(m+r) = 1; u_m^l : m \leq d-2, l-m = 1\}$. If Z contains a 1-defect two-index coordinate, then it also contains a 1-defect three-index coordinate on the same level by Prop. 5.2.24, so this weaker result is sufficient.

From now on $d = 2k$. The odd case can be handled similarly. Suppose that there is no 1-defect coordinate on level $\leq d-2$ in Z , i.e the occurring 1-defect coordinates for $d = 2k$ are

$$u_{2k-2}^{2k-1}, u_{1(2k-3)}^{2k-1}, u_{2(2k-4)}^{2k-1}, \dots, u_{(k-1)(k-1)}^{2k-1} \quad (5.2.20)$$

on level $2k-1$ and

$$u_{2k-1}^{2k}, u_{1(2k-2)}^{2k}, u_{2(2k-3)}^{2k}, \dots, u_{(k-1)k}^{2k} \quad (5.2.21)$$

on level $2k$. We denote by $A_{m,r}$ (resp. $B_{m,r}$) the sum of the coefficients of the monomials which have their unique 1-defect coordinate on the $2k$ th (resp. $2k-1$ th) level. We use A_m and B_m when the unique 1-defect coordinate is a two-index one.

By Prop. 5.2.16 $ZE_{i,i+1}^L \in I_{\mathcal{O}}$ is a toric relation ($i = 1, 2, \dots, d-1$), and $ZE_{i,i+1}^L(p_{\text{ref}}) = 0$. This can be rewritten using $A_{m,r}$ and $B_{m,r}$ as following:

$$\begin{aligned} B_{i,2k-2-i} + A_{i,2k-1-i} &= 0 \text{ for } i = 1, \dots, k-2, k, \dots, 2k-3 \\ 2B_{k-1,k-1} + A_{k-1,k} &= 0 \text{ for } i = k-1 \\ B_{2k-2} + A_{1,2k-2} &= 0 \text{ for } i = 2k-2 \\ A_{2k-1} &= 0 \text{ for } i = 2k-1 \end{aligned}$$

This gives us the following:

$$\begin{aligned} B_{2k-2} &= -A_{1,2k-2} = -B_{1,2k-3} = -A_{2,2k-3} = \\ &= -B_{2,2k-4} = -A_{3,2k-4} = \dots = -A_{k-1,k} = -2B_{k-1,k-1} \end{aligned} \quad (5.2.22)$$

On the other hand $E_{d-2,d-1}^R Z(p_{\text{ref}}) = 0$, in coordinates

$$B_{2k-2} + B_{1,2k-3} + B_{2,2k-4} + \dots + B_{k-1,k-1} = 0 \quad (5.2.23)$$

and $E_{d-1,d}^R Z(p_{\text{ref}}) = 0$, i.e

$$A_{2k-1} + A_{1,2k-2} + A_{2,2k-3} + \dots + A_{k-1,k} = 0 \quad (5.2.24)$$

Adding equations of 5.2.22 we get

$$B_{2k-2} + A_{1,2k-2} + B_{1,2k-3} + A_{2,2k-3} + \dots + A_{k-1,k} + 2B_{k-1,k-1} = 0 \quad (5.2.25)$$

The LHS of 5.2.23+5.2.24 is almost the same, the difference is $0 = A_{2k-1} - B_{k-1,k-1} = B_{k-1,k-1}$. From 5.2.22 then all coefficients are 0, which is obvious contradiction. So Prop. 5.2.25 is proved.

Now we prove the first half of Theorem 5.2.13. Recall that we call the elements of

$$\{\hat{u}_{mr}^l : m + r < l \leq d - 2\}$$

good coordinates. The theorem comes from the following two lemmas

Lemma 5.2.26. *Let \hat{u}_{mr}^l be a good coordinate. The initial ideal \mathcal{IN}_d contains a monomial M with the properties:*

1. *Its unique positive defect term is \hat{u}_{mr}^l with exponent 1*
2. *The remaining toric coordinates are different (i.e their exponent is 1.)*
3. *There is $\tilde{Z} \in \mathcal{I}_d$ such that M is the initial monomial of \tilde{Z} and \hat{u}_{mr}^l does not occur in other monomials of \tilde{Z} .*

Lemma 5.2.27. *Suppose $\tilde{M} \in \mathcal{IN}_d$ is a monomial satisfying Lemma (5.2.26) (1),(2),(3). Then \hat{u}_{mr}^l must occur in every minimal element of the primary decomposition of \mathcal{IN}_d .*

Proof of Lemma (5.2.27) Suppose

$$\tilde{M} = \hat{u}_{mr}^l \prod_{\pi_3 \in \Pi_3} \hat{u}_{\pi_3} \in \mathcal{IN}_d \quad (5.2.26)$$

where $l - (m + r) > 0$ but the other terms are toric coordinates. Note that there are no two-index coordinates by Lemma 5.2.18. Moreover, the second property of Lemma 5.2.26 says that the elements of Π_3 are different in (5.2.26).

Suppose \hat{u}_{mr}^l is not contained in \mathbf{P} , where \mathbf{P} is a minimal primary ideal in the primary decomposition. Let $\tilde{Z} \in \mathcal{I}_d$ be an element with initial monomial \tilde{M} satisfying (3) of Lemma (5.2.26) and $Z \in I_{\mathcal{O}}$ a global form of \tilde{Z} . Note that \tilde{M} comes from a monomial M of Z applying Lemma 5.2.18.

$$M = u_{mr}^l \prod_{\pi_3 \in \Pi_3} u_{\pi_3} \prod_{\pi_2 \in \Pi_2} u_{\pi_2}, \quad (5.2.27)$$

where Π_3 is the same set as in (5.2.26), and Π_2 contains only toric two-indices. Z may have some additional monomials which must have a positive-defect double index coordinate, and therefore vanish after using Lemma 5.2.18.

Since $\tilde{M} \in \mathcal{IN}_d$ but $\hat{u}_{mr}^l \notin \mathbf{P}$, there is a $\tau_3 \in \Pi_3$ such that $\hat{u}_{\tau_3} \in \mathbf{P}$, that is, the set

$$\Pi_{\mathbf{P}} = \Pi_3 \cap \mathbf{P} \quad (5.2.28)$$

is nonempty. The elements of \mathcal{J}_d have the form $\tilde{M}_1 - \tilde{M}_2$ (every toric ideal is generated by polynomials with two monomials) Define the following sets of monomials for $\tau \in \Pi_{\mathbf{P}}$

$$\mathbf{Mon}_{\tau}^1 = \left\{ \tilde{M}_1 : \tilde{M}_1 - \tilde{M}_2 \in \mathcal{J}_d, \hat{u}_{\tau} \in \tilde{M}_1 \right\} \quad (5.2.29)$$

$$\mathbf{Mon}_{\tau}^2 = \left\{ \tilde{M}_2 : \tilde{M}_1 - \tilde{M}_2 \in \mathcal{J}_d, \hat{u}_{\tau} \in \tilde{M}_1 \right\} \quad (5.2.30)$$

If both \tilde{M}_1 and \tilde{M}_2 are divisible by \hat{u}_{τ} , we can simplify the toric relation $\tilde{M}_1 - \tilde{M}_2$, so we suppose that $\mathbf{Mon}_{\tau}^1 \cap \mathbf{Mon}_{\tau}^2 = \emptyset$.

Let $\tilde{M} \in \mathcal{IN}_d$ be of the form (5.2.26), and choose $\tilde{M}_{\tau}^2 \in \mathbf{Mon}_{\tau}^2$ for all $\tau \in \Pi_{\mathbf{P}}$. Let $\tilde{M}_{\tau}^1 \in \mathbf{Mon}_{\tau}^1$ the pair of \tilde{M}_{τ}^2 , i.e $\tilde{M}_{\tau}^1 - \tilde{M}_{\tau}^2 \in \mathcal{I}_d$. The following small observation is the crucial step in the proof.

Proposition 5.2.28.

$$\tilde{M} = \hat{u}_{mr}^l \prod_{\pi_3 \in \Pi_3 \setminus \Pi_{\mathbf{P}}} \hat{u}_{\pi_3} \prod_{\tau \in \Pi_{\mathbf{P}}} \tilde{M}_{\tau}^2 \quad (5.2.31)$$

is an element of \mathcal{IN}_d .

Proof of Prop. (5.2.28)

The initial monomial of

$$\check{Z} = \tilde{Z} \prod_{\tau \in \Pi_{\mathbf{P}}} \frac{\tilde{M}_{\tau}^1}{\hat{u}_{\tau}} \quad (5.2.32)$$

is

$$\text{In}(\check{Z}) = \tilde{M} \prod_{\tau \in \Pi_{\mathbf{P}}} \frac{\tilde{M}_{\tau}^1}{\hat{u}_{\tau}} = \hat{u}_{mr}^l \prod_{\pi_3 \in \Pi_3 \setminus \Pi_{\mathbf{P}}} \hat{u}_{\pi_3} \prod_{\tau \in \Pi_{\mathbf{P}}} \tilde{M}_{\tau}^1 \quad (5.2.33)$$

Let $\check{Z}^{\text{repl}} \in I_{\mathcal{O}}$ be the modified polynomial arising from \check{Z} by replacing $\text{In}(\check{Z})$ by \tilde{M} . (Since $\text{In}(\check{Z}) - \tilde{M} = 0$ a toric relation, \check{Z}^{repl} is also relation.) We claim that the initial monomial of \check{Z}^{repl} is \tilde{M} . This is guaranteed by the term \hat{u}_{mr}^l in \tilde{M} , namely \check{Z}^{repl} has the following properties:

1. \hat{u}_{mr}^l is a term in \tilde{M}
2. \hat{u}_{mr}^l is not a term in any other monomial of \check{Z}^{repl}
3. \hat{u}_{mr}^l is the greatest element w.r.t \succ among all the coordinates of \check{Z}^{repl} .

Only (2) and (3) need some explanation. But the same is true for \tilde{Z} by the prescribed properties of Lemma 5.2.26, and multiplying by toric coordinates and substituting \mathbf{Mon}^1 by \mathbf{Mon}^2 does not bother the positive-definite coordinates, so the same is true for \check{Z} and \check{Z}^{repl} . So Prop. (5.2.28) is proved.

\mathbf{P} have the form

$$\mathbf{P} = \langle \mathbf{u}_1^{s_1}, \dots, \mathbf{u}_{\alpha}^{s_{\alpha}}, \mathbf{u}_{\alpha+1}^{s_{\alpha+1}}, \dots, \mathbf{u}_{\alpha+\beta}^{s_{\alpha+\beta}} \rangle \quad (5.2.34)$$

where $\mathbf{u}_i = \hat{u}_{m_i r_i}^{l_i}$, and $s_i \in \mathbb{Z}^+$ for $i = 1, \dots, \alpha + \beta$. As in 5.2.6, the first α coordinates have positive defect and the last β coordinates are the zero-defect coordinates.

Take a look at $\tilde{M} \in \mathcal{IN}_d$. Since $\hat{u}_{mr}^l \notin \mathbf{P}$, $\prod_{\pi_3 \in \Pi_3 \setminus \Pi_{\mathbf{P}}} \hat{u}_{\pi_3} \notin \mathbf{P}$, there must be $\tau \in \Pi_{\mathbf{P}}$ and $\alpha + 1 \leq i \leq \alpha + \beta$ such that $\mathbf{u}_i | \tilde{M}_{\tau}^2$.

This implies

Corollary 5.2.29. *There must be a $\tau_0 \in \Pi_{\mathbf{P}}$, such that for any $\tilde{M}_{\tau_0}^2 \in \mathbf{Mon}_{\tau_0}$ there is $\alpha + 1 \leq i \leq \alpha + \beta$ with $\mathbf{u}_i | \tilde{M}_{\tau_0}^2$.*

For simplicity, suppose $\mathbf{u}_{\tau_0} = \mathbf{u}_{\alpha+1}$. (recall that \mathbf{u}_{tau_0} is a toric coordinate)
Introduce

$$\mathcal{JN}_{\tau_0} = \{M \in \mathcal{JN}_d : \mathbf{u}_{\tau_0} \nmid M\} \quad (5.2.35)$$

Corollary 5.2.29 implies

Proposition 5.2.30. *The ideal*

$$\langle \mathbf{u}_{\alpha+2}, \dots, \mathbf{u}_{\alpha+\beta} \rangle \quad (5.2.36)$$

is an associated prime in the primary decomposition of the ideal generated by $\mathbf{Mon}_{\tau_2}^2$ and \mathcal{JN}_{τ_0} .

We define a new order \succ_0 on the toric coordinates, namely

$$\hat{u}_{mr}^l \succ_0 \hat{u}_{m\bar{r}}^{\bar{l}}$$

if

$$\hat{u}_{mr}^l = \mathbf{u}_{\alpha+1} \neq \hat{u}_{m\bar{r}}^{\bar{l}}$$

or neither of them is $\mathbf{u}_{\alpha+1}$, but $\hat{u}_{mr}^l \succ \hat{u}_{m\bar{r}}^{\bar{l}}$

Let \succ_{wt} be the lexicographic monomial order corresponding to \succ_0 on the coordinates. Note that \succ_{wt} is a weighted lexicographic monomial order, where $\mathbf{u}_{\alpha+1}$ has weight ε , the other coordinates have weight 1 with $\varepsilon \ll 1$.

The following observation is straightforward from the definition:

Proposition 5.2.31. *Let \mathcal{JN}_{wt} be the initial monomial ideal of \mathcal{J}_d with respect to \succ_{wt} . Then $\mathcal{JN}_{wt} = \langle \mathbf{Mon}_{\tau_0}^2 \cup \mathcal{JN}_{\tau_0} \rangle$*

By Proposition 5.2.30 and 5.2.31, $\text{Spec} \mathcal{JN}_{wt}$ has a component of codimension $\beta - 1$, but the maximal dimensional components of $\text{Spec} \mathcal{JN}_d$ have codimension β by the minimality of \mathbf{P} , which is impossible, since both \mathcal{JN}_d and \mathcal{JN}_{wt} are initial ideals of \mathcal{J}_d . Lemma 5.2.27 is proved.

Proof of Lemma 5.2.26

Briefly, we need to find relations in $\mathcal{I}_d \triangleleft \mathbb{C}[\hat{u}^\bullet]$ with initial monomial ideal satisfying the requirements of the lemma. The miraculous thing is that we can find such relations in the subideal $\mathcal{B}_d \subset \mathcal{I}_d$ generated by the basic equations! Observe that

in this way we use only \mathcal{B}_d and the toric elements of \mathcal{I}_d to prove the first half of Theorem 5.2.13.

The good coordinates $\hat{u}_{1,r}^l$ ($r+1 < l \leq d-2$) are easy to handle: the initial term of the basic equation $B(1, r, 2; l+2)$ satisfies (1) and (2), and $B(1, r, 2; l+2)$ itself satisfies (3) of Lemma 5.2.26. Namely, the initial monomial of

$$B(1, r, 2; l+2) = (\hat{u}_{1,r}^l \hat{u}_{2,l}^{l+2} + \hat{u}_{1,r}^{l-1} \hat{u}_{2,l-1}^{l+2} + \dots + \hat{u}_{1,r}^{r+1} \hat{u}_{2,r+1}^{l+2}) - (\hat{u}_{2,r}^{l+1} \hat{u}_{1,l+1}^{l+2} + \hat{u}_{2,r}^l \hat{u}_{1,l}^{l+2} + \dots + \hat{u}_{2,r}^{r+2} \hat{u}_{1,r+2}^{l+2}) \quad (5.2.37)$$

is

$$\hat{u}_{1,r}^l \hat{u}_{2,l}^{l+2} \quad (5.2.38)$$

since $\hat{u}_{1,r}^l$ and $\hat{u}_{2,r}^{l+1}$ have the same maximal defect among the coordinates of this basic relation (hence they are maximal with respect to $>_2$) but

$$\hat{u}_{1,r}^l >_3 \hat{u}_{2,r}^{l+1}. \quad (5.2.39)$$

The other properties of Lemma 5.2.26 are straightforward.

The remaining good coordinates are the initial monomials of a linear combination of 2 basic relations. For the good coordinate $\hat{u}_{m,r}^l$, $m > 1, m+r < l \leq d-2$ these basic equations are $B(m, r, 1; l+1)$ and $B(m+1, r, 1; l+2)$:

$$B(m, r, 1; l+1) = (\hat{u}_{m,r}^l \hat{u}_{1,l}^{l+1} + \hat{u}_{m,r}^{l-1} \hat{u}_{1,l-1}^{l+1} + \dots + \hat{u}_{m,r}^{m+r} \hat{u}_{1,m+r}^{l+1}) - (\hat{u}_{1,r}^{l-m+1} \hat{u}_{m,l-m+1}^{l+1} + \hat{u}_{1,r}^{l-m} \hat{u}_{m,l-m}^{l+1} + \dots + \hat{u}_{1,r}^{r+1} \hat{u}_{m,r+1}^{l+1}) = 0 \quad (5.2.40)$$

and

$$B(m+1, r, 1; l+2) = (\hat{u}_{m+1,r}^{l+1} \hat{u}_{1,l+1}^{l+2} + \hat{u}_{m+1,r}^l \hat{u}_{1,l}^{l+2} + \dots + \hat{u}_{m+1,r}^{m+r+1} \hat{u}_{1,m+r+1}^{l+2}) - (\hat{u}_{1,r}^{l-m+1} \hat{u}_{m+1,l-m+1}^{l+2} + \hat{u}_{1,r}^{l-m} \hat{u}_{m+1,l-m}^{l+2} + \dots + \hat{u}_{1,r}^{r+1} \hat{u}_{m+1,r+1}^{l+2}) \quad (5.2.41)$$

The maximum of the occurring defects in (5.2.40) and (5.2.41) is $l-m-r$, and only

$$\hat{u}_{m,r}^l, \hat{u}_{1,m+r}^{l+1}, \hat{u}_{1,r}^{l-m+1}, \hat{u}_{m,r+1}^{l+1}$$

in (5.2.40) and

$$\hat{u}_{m+1,r}^{l+1}, \hat{u}_{1,m+r+1}^{l+2}, \hat{u}_{1,r}^{l-m+1}, \hat{u}_{m+1,r+1}^{l+2}$$

in (5.2.41) have this maximal defect. The good coordinates among these are $\hat{u}_{m,r}^l, \hat{u}_{1,r}^{l-m+1}$ if $l = d-2$ and if $l < d-2$ there are even more.

For a partial order $>$ we use $a \geq b$ when $a > b$ or a is not comparable with b . Let $a >_{1,2,3} b$ stand for $a \geq_1 b, a \geq_2 b, a >_3 b$. If $m < r+1$ the coordinates above have the following order:

$$\hat{u}_{1,r}^{l-m+1} \geq_{1,2,3} \hat{u}_{m,r}^l \geq_{1,2,3} \hat{u}_{1,m+r}^{l+1} \geq_{1,2,3,4} \hat{u}_{m,r+1}^{l+1} \geq_{1,2,3,4} \hat{u}_{m+1,r}^{l+1} \geq_{1,2,3} \hat{u}_{1,m+r+1}^{l+2} \geq_{1,2,3,4} \hat{u}_{m+1,r+1}^{l+2} \quad (5.2.42)$$

(If $m = r + 1$ $\hat{u}_{m,r+1}^{l+1}$ and $\hat{u}_{m+1,r}^{l+1}$ change order.) But we can get rid of $\hat{u}_{1,r}^{l-m+1}$ easily: multiplying (5.2.40) and (5.2.41) by the toric $\hat{u}_{m+1,l-m+1}^{l+2}, \hat{u}_{m,l-m+1}^{l+1}$, respectively, in the difference $v_{1,r}^{l-m+1}$ cancels, and we get

$$\hat{u}_{m,r}^l \hat{u}_{1,l}^{l+1} \hat{u}_{m+1,l-m+1}^{l+2} + \dots = 0$$

which is a relation in \mathcal{B} which satisfies (1),(2),(3) of Lemma 5.2.26.

5.3 An application: the positivity of Thom polynomials

It is conjectured in [43, Conjecture 5.5] that all coefficients of the Thom polynomials $\text{Tp}_d^{n \rightarrow k}$ expressed in terms of the relative Chern classes are nonnegative. Rimányi also proves that this property is special to the A_d -singularities. In this final paragraph, we would like to show that our formalism is well-suited to approach this problem. We will also formulate a more general positivity conjecture, which will imply this statement.

We start with a comment about the sign $(-1)^d$ in our main formula (4.4.19). Recall from (4.3.5) in §4.3.2 that, according to our convention, the iterated residue at infinity may be obtained by expanding the denominators in terms of z_i/z_j with $i < j$ and then *multiplying the result by* $(-1)^d$. This sign appears because of the change of orientation of the residue cycle when passing to the point at infinity. This means that if we compute (4.4.19) via expanding the denominators, then the sign in the formula cancels.

Now we are ready to formulate our positivity conjecture.

Conjecture: *Expanding the rational function*

$$\frac{\prod_{m < l} (z_m - z_l) \widehat{Q}_d(z_1, \dots, z_d)}{\prod_{l=1}^d \prod_{m=1}^{l-1} \prod_{r=1}^{\min(m, l-m)} (z_m + z_r - z_l)}$$

in the domain $|z_1| \ll \dots \ll |z_d|$, one obtains a Laurent series with nonnegative coefficients.

This statement clearly implies the non-negativity of the coefficients of the Thom polynomial.

At the moment we do not know how to prove this conjecture in general. However, we observe that the expansion of a fraction of the form $(1 - f)/(1 - (f + g))$ with f and g small has positive coefficients. Indeed, this follows from the identity

$$\frac{1 - f}{1 - f - g} = 1 + \frac{g}{1 - f - g}.$$

Now, introducing the variables $a = z_1/z_2$ and $b = z_2/z_3$, we can rewrite the above fraction in the $d = 3$ case as follows:

$$\frac{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)}{(2z_1 - z_2)(z_1 + z_2 - z_3)(2z_1 - z_3)} = \frac{1 - a}{1 - 2a} \cdot \frac{1 - ab}{1 - 2ab} \cdot \frac{1 - b}{1 - b - ab}.$$

Applying the above identity to the right hand side of this formula immediately implies our conjecture for $d = 3$. We offer to the reader the rather amusing exercise of proving the same statement for $d = 4$.

Chapter 6

List of Notations and Bibliography

- $\mathcal{J}(n)$: algebra of germs $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$; see §2.1.1.
- $-\mathcal{J}_d(n, k)$: d -jets of maps $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$, see §2.1.1.
- $\mathcal{J}_d(n)$: short notation for $\mathcal{J}_d(n, n)$, see §2.1.1.
- Lin: linear part of a germ or jet, see §2.1.1.
- $\text{Diff}_d(n) = \{\Delta \in \mathcal{J}_d(n, n); \text{Lin}(\Delta) \text{ invertible}\}$, i.e the group of diffeomorphism d -jets of \mathbb{C}^n at the origin, see §2.1.1.
- $-A_\Psi$: the nilpotent algebra of the map germ Ψ , see §2.1.1.
- A_d : the nilpotent algebra $t\mathbb{C}[t]/t^{d^t}$, see §2.1.2.
- $-\Theta_A, \theta_A[n, k]$: set of jets with nilpotent algebra A , see (2.1.2).
- $\Theta_d, \theta_d[n, k]$ short notation for Θ_{A_d} , see §2.1.2.
- $\mathcal{K}, \mathcal{K}_d(n, k)$: the contact group, see (2.1.3).
- $eP[\Sigma, W]$: equivariant Poincaré dual class of $\Sigma \subset W$, see the axiomatic definition in §2.2.1.
- $-\text{Euler}^T(W)$: the equivariant Euler class of a vector space W with respect to the action of the torus T , see §2.2.1.
- $\text{emult}_p[M, Z]$: equivariant multiplicity of M in Z at $p \in M$, see (2.2.10) in §2.2.3.
- $\text{Thom}(W)$: equivariant Thom class of W , see (2.2.13) in §2.2.3.
- $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n), \boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$: weights of T^n, T^k , the maximal tori of GL_n, GL_k , respectively, see (2.2.15) in §2.2.5.
- $-\text{Tp}_A^{n \rightarrow k}, \text{Tp}_A^{n \rightarrow k}(\boldsymbol{\lambda}, \boldsymbol{\theta})$: the Thom polynomial of Θ_A for a nilpotent algebra A , see (2.2.18) in §2.2.5.
- $\text{Tp}_d^{n \rightarrow k}$: the Thom polynomial in case $A = A_d$.

- $|\pi|, \text{sum}(\pi), \text{max}(\pi), \text{perm}(\pi)$: the length, the sum, the maximal element and the number of different permutations of the partition π , respectively, see the definitions before Lemma 4.1.2.
- $\mathbf{v}_\pi = \prod_{j=1}^l v_{i_j}, \Psi(\mathbf{v}_\pi) = \Psi(v_{i_1}, \dots, v_{i_l})$, see (4.1.5).
- $-\mathcal{J}_d^{\text{reg}}(1, n)$: set of regular curves, i.e with nonvanishing linear part, see (4.1.1) in §4.1.
 $-\Psi = (\Psi^1, \dots, \Psi^d) = (A, B, C, \dots)$: element of $\mathcal{J}_d(n, k)$, see (4.1.4) in §4.1.
- $\gamma \in \mathcal{J}_d(1, n)$: γ always stand for a test curve, see (4.1.3) in §4.1.
- $-\text{Sol}_\gamma \subset \mathcal{J}_d(n, k)$: the linear subspace of solutions in $\mathcal{J}_d(n, k)$ corresponding to the curve γ , see the definition after (4.1.7) in §4.1.
 $-\text{Sol}_\varepsilon$: the linear subspace of solutions in $\mathcal{J}_d(n, k)$ corresponding $\varepsilon \in \mathcal{F}_d(n)$, see the Definition 4.2.4
 $-\text{Sol}_{\tilde{\varepsilon}}$: the linear subspace of solutions in $\mathcal{J}_d(n, k)$ corresponding to $\tilde{\varepsilon} \in \tilde{\mathcal{F}}_d(n)$, see Definition 4.2.4.
 $-\text{Sol}_{\tilde{\mathcal{F}}}$: the vector bundle with fibers $\text{Sol}_{\tilde{\varepsilon}}$ and base $\tilde{\mathcal{F}}_d(n)$, see Definition 4.2.4.
- $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$: the filtration preserving maps with respect to the filtrations (4.2.5) and (4.2.4), see (4.2.6) in §4.1.
- $-\mathcal{F}_d(n)$: the subspace of nondegenerate systems in $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$, see (4.2.7) in §4.1.
 $-\tilde{\mathcal{F}}_d(n)$: this is $\mathcal{F}_d(n)/B_R$: see Lemma 4.2.2 in §4.1.
- ε is always an element of $\mathcal{F}_d(n)$, its image under the projection is $\tilde{\varepsilon} \in \tilde{\mathcal{F}}_d(n)$, see Definition 4.2.4.
- V : The bundle over $\tilde{\mathcal{F}}_d(n)$ associated to the standard representation of B_R on \mathbb{C}^d , see Lemma 4.2.5.
- $\mathcal{F}_d^{\text{reg}}(n)$: the regular part of $\mathcal{F}_d(n)$, see (4.2.8).
- $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$: the regular part of $\text{Hom}(\mathbb{C}^d, \mathbb{C}^n)$, see (4.2.18).
- $-\text{Flag}_d(\mathbb{C}^n) = \text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)/B_d$ the smooth variety of full flags of d-dimensional subspaces of \mathbb{C}^n , see Lemma 4.2.10.
 $-\gamma_{\text{ref}}, \mathbf{f}_{\text{ref}}, \pi_{\text{Fl}}$ denotes the reference sequence in $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$, the corresponding flag in $\text{Flag}_d(\mathbb{C}^n)$, and the projection from $\text{Hom}^{\text{reg}}(\mathbb{C}_L^d, \mathbb{C}^n)$ to $\text{Flag}_d(\mathbb{C}^n)$, respectively, see Definition 4.2.11.
- $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$: the space of filtration preserving maps, with filtrations introduced in (4.2.24) and (4.2.5), a subspace of $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Sym}_d^\bullet \mathbb{C}^n)$.

- $-\mathcal{E}$: the nondegenerate part of $\text{Hom}^\Delta(\mathbb{C}_R^d, \text{Ym}^\bullet \mathbb{C}_L^d)$, see (4.2.26)
- $-\tilde{\mathcal{E}}$: the factor \mathcal{E}/B_R , see Proposition 4.2.13.
- $-\varepsilon_{\text{ref}}$: the reference system $\psi(\gamma_{\text{ref}})$ in \mathcal{E} , see (4.2.19).
- $-\tilde{\varepsilon}_{\text{ref}}$: the corresponding point in $\tilde{\mathcal{E}}$, i.e $\tilde{\varepsilon}_{\text{ref}} = \tilde{\text{pr}}(\varepsilon_{\text{ref}})$ where $\tilde{\text{pr}} : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ is the projection, see (4.3.10).
- $u_\pi^l, \hat{u}_{\tau|\pi}^l$: the coordinates on \mathcal{E} and \mathcal{N}_π defined in (4.2.23) and (4.3.13), respectively.
- $\mathcal{O} = \overline{B_L \tilde{\varepsilon}_{\text{ref}}} \subset \tilde{\mathcal{E}}$, the closure of the Borel orbit, see (4.3.1).
- $-\mathbf{\Pi}_d$: set of admissible sequences, see Definition 4.3.6.
- $-\mathbf{\Pi}_{\mathcal{O}}$: the set of admissible sequences corresponding to fixed points in \mathcal{O} , see (4.3.25).
- $-\pi_{\text{dist}}$: distinguished sequence of partitions, see (4.4.1).
- $-\tilde{\varepsilon}_{\text{dist}}$: the corresponding distinguished fixed point in $\tilde{\mathcal{E}}$.
- $-\mathcal{N}_{\text{dist}}$: the chart corresponding to the distinguished sequence of partitions.
- $-\mathcal{O}_{\text{dist}} = \mathcal{O} \cap \mathcal{N}_{\text{dist}} \subset \mathcal{N}_{\text{dist}}$.
- $-\hat{\mathcal{N}}_d \subset \mathcal{N}_{\text{dist}}$ is the linear subspace generated by partitions not longer than 2, see (4.4.16).
- $-\hat{\mathcal{O}}_d = \hat{\text{pr}}(\mathcal{O}_{\text{dist}}) \subset \hat{\mathcal{N}}_d$
- $-R_{\tilde{\mathcal{E}}}$: the coordinate ring of $\tilde{\mathcal{E}}$, see Definition 4.4.6.
- $-I_{\mathcal{O}} \triangleleft R_{\tilde{\mathcal{E}}}$: the ideal of the subvariety $\mathcal{O} \subset \tilde{\mathcal{E}}$, see Definition 4.4.6.
- $\deg(p(\mathbf{z}); S), \text{coeff}(L, z_l), \text{lead}(q(\mathbf{z}); m)$ denotes the degree, some coefficient and leading coefficient of some polynomials defined in (4.4.2).

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