Asymptotic behavior of random graphs evolving in time
SYNOPSIS

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Contents

1 Introduction

In my PhD thesis I investigate the time evolution of certain graph-valued Markov chains: the vertex set and edge set of the graph changes over time: with probabilities that depend on the present structure of the graph we add/delete edges/vertices from the graph. If we consider a sequence of Markov chains with a sequence of initial graphs in which the number of vertices n goes to infinity, but assume that some family of statistics of the initial graphs converge as $n \to \infty$, then with the appropriate scaling of time we are able to translate the microscopic transition rules of the Markov chain into differential equations governing the time evolution of the limiting values of the family of statistics that we consider. By the analysis of the solutions of these differential equations we are able to describe the large-scale evolution and properties of random graphs.

We consider two different families of models in the thesis:

The **mean field forest fire model** and the **mean field frozen percolation model** are two closely related random graph models: in both cases we modify the dynamical Erdős-Rényi random graph model so that large connected components are destroyed, which creates a competition between coagulation and fragmentation. The most important property of these models is that they exhibit *self-organized criticality*, which can be proved by the analysis of the corresponding modifications of the *Smoluchowski coagulation equations*.

The **edge reconnecting model** is a dense multigraph evolving in discrete timesteps: at each step we reconnect one endpoint of a uniformly chosen edge according to the rules of linear preferential attachment. We investigate the model using the notion of *dense graph limits* and give a full description of the time evolution of the limit objects. The number of parallel edges and the degrees evolve on different timescales and because of this the model exhibits *aging*.

The thesis is divided into three chapters in which we provide the analysis of these three random graph models:

The topic of Chapter 2 is the mean field forest fire model and is based on [36].

The topic of Chapter 3 is the mean field frozen percolation and is based on [33].

The topic of Chapter 4 is the edge reconnecting model and is based on [34], joint work with László Szakács.

The rest of this document is also divided into three sections. We begin each section with a short résumé of the context, description and relevance of these three models, and after the necessary definitions we precisely state the results of the PhD thesis.

2 Mean field forest fire model

2.1 Context

The dynamical Bernoulli bond percolation model is a random graph evolving in continuous time according to the following Markovian dynamics:

Fix an infinite homogenous graph (e.g. the lattice $\mathbb{Z}^d$, $d \geq 2$). The edges of the graph can be either "open" or "closed". We start the process from the state where all edges are closed. Edges are independently switched from closed to open with rate 1. As the time parameter t increases, the model undergoes *phase transition*: there is a particular value $t_c \in (0, +\infty)$ (the critical time) such that if we define

$$v_k(t) := \mathbf{P}\big(\text{ the size of the connected open component of the origin is } k \text{ at time } t \, \big)$$

then

- for $t < t_c$ the model is *subcritical*: $v_k(t)$ decays exponentially in k.
- for $t > t_c$ the model is *supercritical*: $\sum_{k=1}^{\infty} v_k(t) = 1 - \theta(t)$ where $\theta(t) > 0$ is the probability that the origin is contained in an infinite component. $v_k(t)$ decays exponentially in k.
- for $t = t_c$ the model is *critical*: $\theta(t_c) = 0$ and $v_k(t)$ decays polynomially in k.

The mean field version of dynamical bond percolation is the dynamical Erdős-Rényi random graph model: the edges of the complete graph on n vertices are turned from closed to open with rate $\frac{1}{n}$. If we define

$$v_{n,k}(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[\text{ the size of the connected open component of vertex } i \text{ is } k \text{ at time } t \,] \tag{1}$$

then $v_{k,n}(t) \to v_k(t)$ in probability as $n \to \infty$ where $(v_k(t))_{k=1}^{\infty}$ is the solution of the *Smoluchowski coagulation equations* with multiplicative kernel

$$\dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k v_k(t), \qquad k \geq 1 \tag{2}$$

with initial condition $v_k(0) = \mathbb{1}[k = 1]$.

With $t_c = 1$ the phase transition can be described exactly the same way as before (the only difference is that $\theta(t)$ is the density of the *giant component*). The decay rate of component size densities at the critical time is $v_k(t_c) \asymp k^{-3/2}$ in the Erdős-Rényi model.

The critical forest fire model on the lattice $\mathbb{Z}^d$ might be informally defined in the following way: edges are independently switched from closed to open with rate 1, but if an infinite open component appears, we switch its edges to closed instantaneously. It is conjectured by physicists (see [19]) that the forest fire model exhibits self-organized criticality (S.O.C.): for all $t \geq t_c$ the graph is critical: $\theta(t_c) = 0$ and $v_k(t)$ decays polynomially in k.

There are few mathematical results describing forest fire models on $\mathbb{Z}^d$: in fact a rigorously defined graph-valued stochastic process satisfying the informal definition of the critical forest fire model has not been constructed yet. In [20] and [21] M. Dürre proves the existence and uniqueness of the subcritical forest fire model satisfying the following informal definition: edges are independently switched from closed to open with rate 1 and all the edges of open connected components of size k are switched to closed with rate λk (a lightning strikes the component with a rate proportional to its size). Heuristically $\lambda \to 0$ yields the critical forest fire model. We modify the dynamical Erdős-Rényi model in a similar fashion to obtain the mean field forest fire model:

2.2 The model

We start with a graph on n vertices and between each pair of unconnected vertices an edge appears with rate $1/n$, moreover each vertex is exposed to a Poisson process of lightnings with rate $\lambda(n)$. If a lightning strikes a vertex, then fire spreads along the edges instantaneously and burns them: connected components of size k burn down with rate $\lambda(n) \cdot k$ and the component is replaced with k isolated vertices. The total number of vertices remains n.

We investigate the model in the *critical regime* $\frac{1}{n} \ll \lambda(n) \ll 1$ as $n \to \infty$. Since $\lambda(n) \ll 1$, the fire doesn't do much harm to small components, but by $\frac{1}{n} \ll \lambda(n)$ giant components of size comparable to n burn immediately.

In order to formulate our problem first have to introduce the proper spaces on which our processes are defined.

We denote

$$\mathcal{V} :=\big\{\mathbf{v} = \big(v_k\big)_{k\in\mathbb{N}} \,:\, v_k \geq 0, \quad \sum_{k\in\mathbb{N}} v_k \leq 1\big\}, \quad \theta(\mathbf{v}) := 1 - \sum_{k\in\mathbb{N}} v_k, \tag{3}$$

$$\mathcal{V}_1 :=\big\{\mathbf{v} \in \mathcal{V} \,:\, \theta(\mathbf{v}) = 0\big\}. \tag{4}$$

We endow $\mathcal{V}$ with the (weak) topology of component-wise convergence. We may interpret θ as the density of the giant component.

A map $[0, \infty) \ni t \mapsto \mathbf{v}(t) \in \mathcal{V}$ which is component-wise of bounded variation on compact intervals of time and continuous from the left in $[0, \infty)$, will be called *a forest fire evolution (FFE)*. If $\mathbf{v}(t) \in \mathcal{V}_1$ for all $t \in [0, \infty)$ we call the FFE *conservative*. Denote the space of FFE-s and conservative FFE-s by $\mathcal{E}$, respectively, $\mathcal{E}_1$. The space $\mathcal{E}$ is endowed with the topology of component-wise weak convergence of the signed measures corresponding to the functions $v_k(\cdot)$ on compact intervals of time. This topology is metrizable and the space $\mathcal{E}$ endowed with this topology is complete and separable.

Define $v_{n,k}(t)$ by (1) and let $\mathbf{v}_n(t) := \big(v_{n,k}(t)\big)_{k\in\mathbb{N}}$. Clearly, the random trajectory $t \mapsto \mathbf{v}_n(t)$ is a (conservative) FFE. We investigate the asymptotics of this process, as $n \to \infty$.

2.3 Theorems

If $\frac{1}{n} \ll \lambda(n) \ll 1$ then

$$\mathbf{v}_n(\cdot) \xrightarrow{\mathbf{P}} \mathbf{v}(\cdot) = \big(v_k(\cdot)\big)_{k\in\mathbb{N}} \quad \text{as} \quad n \to \infty$$

holds, where the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of *constrained ODE-s*

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t), \qquad k \geq 2, \tag{5}$$

$$\sum_{k\in\mathbb{N}} v_k(t) = 1. \tag{6}$$

Mind the difference between the system (2) at one hand and the constrained system (5)+(6) at the other: the first equation from (2) is replaced by the global constraint (6). A first consequence is that the ODE-s in (5) cannot be solved for $k = 1, 2, \dots$, one-by-one, in turn as opposed to (2): the system of ODE-s is *genuinely infinite*. The first main theorem of Chapter 2 states that for a certain class of initial states the system of constrained differential equations (5)+(6) is well-posed:

Theorem 2.1. *If the initial condition* $\mathbf{v}(0) \in \mathcal{V}_1$ *is such that* $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$ *and the gelation time* T_{gel} *is defined by*

$$T_{gel} := \Big(\sum_{k=1}^{\infty} k \cdot v_k(0)\Big)^{-1} \tag{7}$$

then the critical forest fire equations (5)+(6) *have a unique solution with the following properties:*

1. *For* $t \leq T_{gel}$ *the solution coincides with that of* (2) *with the same initial condition.*
2. *For* $t \geq T_{gel}$ *there exists a positive, locally Lipschitz-continuous function* φ *such that*

   $$\dot{v}_1(t) = -v_1(t) + \varphi(t) \tag{8}$$

   and

   $$\sum_{l=k}^{\infty} v_l(t) \sim \sqrt{\frac{2\varphi(t)}{\pi}}k^{-1/2}, \qquad \textit{as} \qquad k \to \infty. \tag{9}$$

Up to T_{gel} the solutions of (2), respectively, of (5)+(6) coincide because for $t \in [0, T_{\text{gel}}]$ the solution of (2) satisfies (6). But dramatic differences arise beyond this critical time: (9) shows that in this regime the random graph dynamics exhibits indeed *self-organized critical behavior*: beyond the critical time T_{gel} it stays critical for ever.

Theorem 2.2. *Let $\mathbb{P}_n$ denote the law of the random FFE of the forest fire Markov chain $\mathbf{v}_n(t)$ with initial condition $\mathbf{v}_n(0)$ and lightning rate parameter $n^{-1} \ll \lambda(n) \ll 1$. If $\mathbf{v}_n(0) \to \mathbf{v}(0) \in \mathcal{V}_1$ component-wise where $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$ then the sequence of probability measures $\mathbb{P}_n$ converges weakly to the Dirac measure concentrated on the unique solution of the critical forest fire equations* (5)+(6) *with initial condition* $\mathbf{v}(0)$*. In particular*

$$\forall \varepsilon > 0,\, t \geq 0 \ : \quad \lim_{n\to\infty} \mathbb{P}_n \left(|v_{n,k}(t) - v_k(t)| \geq \varepsilon \right) = 0$$

Chapter 2 is devoted to the proof of Theorem 2.1 and Theorem 2.2.

2.4 About the proofs

Now we give an overview of the content of Chapter 2:

In order to prove Theorem 2.2 we want to show that that the sequence of probability measures $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is tight using Prohorov's theorem. From tightness it follows that every subsequence of $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ has a sub-sub-sequence which converges in probability to some random FFE. Then we prove that every subsequential limit of $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is concentrated on the subset of FFEs satisfying (5)+(6) (which is a set with one single element given Theorem 2.1), from which Theorem 2.2 follows.

- In order to show the tightness of the sequence $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ we need to show that for every $\varepsilon > 0$ there exists a compact subset K of FFEs such that

$$\forall\, n \in \mathbb{N} \ : \quad \mathbb{P}_n(K) \geq 1 - \varepsilon. \tag{10}$$

  In Section 2.2 we define suitable compact subsets that satisfy (10) and show that any subsequential limit of $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is concentrated on the subset of FFEs satisfying (5).

  - In order to do so we introduce auxiliary objects called *forest fire flows* in Subsection 2.2.1. The main idea is that we not only register the number of components of size k at time t for each $k \in \mathbb{N}$, but also the number of times when a component of size k merged with a component of size l before t for each $k, l \in \mathbb{N}$ and the number of components of size k that were destroyed by fire before t for each $k \in \mathbb{N}$.
  - Then we precisely define the dynamics of the mean field forest fire model in Subsection 2.2.2: by the mean field property we need not take into account the graph structure of the connected components: the Markov chain that we study is a coagulation-fragmentation model (which is a modification of the Marcus-Lushnikov process, see [32]). Proposition 2.1 proves that $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is tight and any subsequential limit is concentrated on the set of solutions of (5).

- In Subsection 2.2.3 we take the Laplace transform/generating function of $(v_k(t))_{k=1}^{\infty}$:

$$V(t,x) = \sum_{k=1}^{\infty} v_k(t)e^{-kx} - 1$$

If $(v_k(t))_{k=1}^{\infty}$ satisfies (5)+(6) then $V(t,x)$ solves a more tractable controlled PDE which we call the *Burgers control problem*:

Find a control function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$\partial_t V(t,x) = -\frac{1}{2}\partial_x V^2(t,x) + \varphi(t)e^{-x}, \quad V(0,x) = \sum_{k=1}^{\infty} v_k(0)e^{-kx} - 1, \quad V(t,0) \equiv 0. \tag{11}$$

- In Section 2.3 we investigate the behavior of $V(t,x)$ as $x \to 0_+$, which is related to the tail behavior of $(v_k(t))_{k=1}^{\infty}$ by Tauberian theory.
  - In Subsection 2.3.1 we define $X(t,u)$ by $X(t, -V(t,x)) = x$ and formulate various aspects of the fact that if $\partial_u X(t,u)|_{u=0} = 0$ and $\partial^2_{uu} X(t,u) \asymp 1$ for some fixed t as $x \to 0_+$ then we have

$$X(t,u) \asymp u^2 \quad \Longleftrightarrow \quad |V(t,x)| \asymp \sqrt{x} \quad \Longleftrightarrow \quad \sum_{l=k}^{\infty} v_l \asymp k^{-1/2} \tag{12}$$

  - In Subsection 2.3.2 we apply the method of characteristics to the Burgers control problem to show that $\partial^2_{uu} X(0,u) \asymp 1$ implies $\partial^2_{uu} X(t,u) \asymp 1$. We use this to show that the mass contained in the giant component cannot grow too fast: $\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \leq C^*$ for some $C^* < +\infty$.
  - In Subsection 2.3.3 we prove that any subsequential limit of the sequence $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is concentrated on the subset of FFEs satisfying (6). This proof is quite technical: $\frac{1}{n} \ll \lambda(n)$ only guarantees the destruction of components of size comparable to n and one has to work hard proving

$$\lim_{N\to\infty} \lim_{n\to\infty} \sum_{k \geq N} v_{n,k}(t) = 0.$$

    A key element of the proof is that we take the generating function of the random vector $\left(v_{k,n}(t)\right)_{k=1}^{n}$ defined by (1) and derive results about the tail behavior of $\left(v_{k,n}(t)\right)_{k=1}^{n}$ using Laplace transform estimates and the method of (random) characteristics.

- In Section 2.4 we prove Theorem 2.1 by applying methods from the theory of first order nonlinear PDE to the Burgers control problem (11).

– In Subsection 2.4.1 we prove that any solution of (11) satisfies $\partial_x V(t,x)|_{x=0} = -\infty$ for all $t \geq T_{\text{gel}}$. Thus with the notations of Subsection 2.3.1 we have $\partial_u X(t,u)|_{u=0} = 0$ from which (12) follows: for all $t \geq T_{\text{gel}}$ the system is critical.

– In Subsection 2.4.2 we show

$$\lim_{x \to 0} \frac{1}{2} \partial_x V^2(t,x) = \varphi(t) \tag{13}$$

(this fact formally follows from (11)) and derive fine properties of the solution of (11) from $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$, e.g. that $\varphi(t)$ is locally Lipschitz-continuous on $[T_{\text{gel}}, +\infty)$.

– In Subsection 2.4.3 we use the method of characteristics to prove that the solution of (11) is unique from which uniqueness of (5)+(6) and Theorem 2.1 follows.

3 Mean field frozen percolation

3.1 Context

The frozen percolation process on a binary tree was defined by D. J. Aldous in [3]: it is a modification of the dynamical percolation process which makes the following informal description mathematically rigorous: we only occupy an edge if both end-vertices are in a finite cluster. The self-organized critical property of this model manifests in the fact that for $t \geq \frac{1}{2}$, which is the critical time of the corresponding percolation process, a typical finite cluster has the distribution of a critical percolation cluster.

I. Benjamini and O. Schramm showed that it is impossible to define a similar modification of the percolation process on $\mathbb{Z}^2$. An explanation of this non-existence result can be found in Section 3. of [8].

3.2 The model

The definition of the mean field frozen percolation model is the same as that of the mean field forest fire model (between each pair of unconnected vertices an edge appears with rate $1/n$, moreover each vertex is exposed to a Poisson process of lightnings with rate $\lambda(n)$) except that in the frozen percolation model we remove the vertices as well as the edges of burnt connected components.

The two models are in the same universality class: if $\frac{1}{n} \ll \lambda(n) \ll 1$ then we have $v_k^n(t) \to v_k(t)$ in probability as $n \to \infty$ where $(v_k(t))_{k=1}^{\infty}$ solves Stockmayer's coagulation equation:

$$\forall\, k \geq 1 \quad \dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k} v_l(t) v_{k-l}(t) - k \cdot v_k(t) \sum_{l=1}^{\infty} v_l(t). \tag{14}$$

The solution of (14) with initial condition $v_k(0) = \mathbb{1}[k = 1]$ is well-known (see [39]) and has a similar self-similarity property as the frozen percolation process on the binary tree.

We might start the frozen percolation with a non-empty initial graph which corresponds to solving (14) with a general initial condition. The solutions are still explicit and the model exhibits S.O.C.: for all $t \geq T_{gel}$ we have (9) where

$$\varphi(t) := \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=1}^{\infty} v_k(t). \tag{15}$$

The mean field forest fire model and the mean field frozen percolation model are conjectured to be in the same universality class but the differential equations corresponding to the mean field frozen percolation model have a more explicit solution, which allows us to prove more about this model.

In the frozen percolation model on the binary tree, components are burnt/frozen/removed when their size becomes infinite. The question may arise:

What is the typical size of a frozen component in the mean field frozen percolation model?

3.3 A conjecture

In order to precisely formulate this question let $0 \leq t_1 < t_2$ and denote

$$\Phi^n([t_1, t_2], k) := \frac{k}{n} \cdot |\{ \text{ components of size } k \text{ burnt in } [t_1, t_2] \ \}| \, .$$

Define the mass of components burnt in $[t_1, t_2]$ by

$$\Phi^n([t_1, t_2]) := \sum_{k \geq 1} \Phi^n([t_1, t_2], k).$$

Thus $p_k^n[t_1, t_2] := \frac{\Phi^n([t_1,t_2],k)}{\Phi^n([t_1,t_2])}$, $k = 1, 2, \dots$ is a random probability distribution for all N and $t_1 < t_2$.

Conjecture 3.1. *We consider a sequence of frozen percolation processes with convergent initial conditions* $\mathbf{v}_n(0) \to \mathbf{v}(0)$*. If* $\lambda(n) = n^{-\alpha}$ *where* $0 < \alpha < 1$ *and if we define*

$$\beta(\alpha) := \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{3} \\ \frac{\alpha+1}{2} & \text{if } \alpha \geq \frac{1}{3} \end{cases} \tag{16}$$

then for every $\mathbf{v}(0)$, $T_{gel} < t_1 < t_2$ *and* α *there exists a non-defective probability distribution function* $F : (0, \infty) \to (0, 1)$, $\lim_{x \to 0_+} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$ *such that for all* $x \in \mathbb{R}_+$ *we have*

$$\lim_{n \to \infty} \sum_{k \geq 1} \mathbb{1}[\ k \leq x \cdot n^{\beta(\alpha)} \ ] \cdot p_k^n[t_1, t_2] \ = \ F(x) \tag{17}$$

In plain words we might say that after gelation the typical component size of a frozen vertex is of order $n^{\beta(\alpha)}$. This conjecture is supported by heuristic arguments, computer simulations and Theorems 3.1 and 3.2 below. For $0 < \alpha < \frac{1}{3}$ the model is conjectured to behave similarly to the case described in Theorem 3.1, whereas for $\frac{1}{3} < \alpha < 1$ it is conjectured to behave similarly to the case described in Theorem 3.2. Note that $\beta(\frac{1}{3}) = \frac{2}{3}$ and $n^{\frac{2}{3}}$ is the order of the size of the largest component in the critical Erdős-Rényi random graph.

3.4 Theorems

In Chapter 3 of the thesis we investigate the behavior of the frozen percolation model on the "boundary of the critical regime" to prove the following results:

- If the lightning rate is $\lambda(n) \equiv \lambda_*$ then the frozen percolation model is subcritical and $v_k^n(t) \to v_k(t)$ where $(v_k(t))_{k=1}^{\infty}$ is the solution of the λ_*-*subcritical frozen percolation equations* (an infinite system of ODE's similar to (14)). As $\lambda_* \to 0$ the solutions of the subcritical equation converge to the solutions of (14) and the typical size of a frozen component is of order λ_*^{-2} when $\lambda_* \ll 1$, see Theorem 3.1 below.
- If the lightning rate is $\lambda(n) = \frac{\lambda^*}{n}$ then giant components are born and destroyed from time to time and the $n \to \infty$ limit of $v_k^n(t)$ is a *randomly controlled solution of the Smoluchowski coagulation equations* alternating between subcritical and supercritical phase. As $\lambda^* \to +\infty$ the solutions of the randomly controlled alternating equation converge to the solutions of (14) and the typical size of a frozen component is of order $n \cdot (\lambda^*)^{-1/2}$ when $1 \ll \lambda^*$, see Theorem 3.2 below.

Theorem 3.1. *We consider a sequence of frozen percolation processes with* $\lambda(n) \equiv \lambda_*$ *and convergent initial conditions* $\mathbf{v}_n(0) \to \mathbf{v}(0)$*, moreover we assume that there exists an* $M \in \mathbb{N}$ *such that* $\forall\, n, \forall\, k \geq M:\ v_{n,k}(0) = 0$*. Denote by* $p_k^{n,\lambda_*}[t_1, t_2] = \frac{\Phi^n([t_1,t_2],k)}{\Phi^n([t_1,t_2])}$.
Then we have

$$\lim_{\lambda_* \to 0} \lim_{\mathrm{d}t \to 0} \lim_{n \to \infty} \sum_{k \geq 1} \mathbb{1}[\, k \leq x \cdot 2\varphi(t) \cdot (\lambda_*)^{-2}\,] \cdot p_k^{n,\lambda_*}[t, t+dt] \;=\; \int_0^x \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{y}} e^{-y} \mathrm{d}y \qquad (18)$$

where $\varphi(t)$ *is defined by* (15) *using the solution of* (14) *with initial condition* $\mathbf{v}(0)$.

Note that the formulation of this theorem is slightly different in the thesis (see Theorem 3.8 in Subsection 3.1.5).

The r.h.s. of (18) is the distribution function of the $\Gamma(\frac{1}{2}, 1)$ distribution.

The relevance of Theorem 3.1 to Conjecture 3.1 is the following: (18) is a version of (17) in the sense that if $\lambda_* = n^{-\alpha}$ for some small α then (18) suggests that the typical component size of a frozen vertex is of order $(\lambda_*)^{-2} = n^{2\alpha}$, thus $\beta(\alpha) = 2\alpha$, which is in agreement with (16).

We prove Theorem 3.1 in Section 3.5 of the thesis using Laplace transform methods.

Theorem 3.2. *We consider a sequence of frozen percolation processes with* $\lambda(n) = \frac{\lambda^*}{n}$ *and convergent initial conditions* $\mathbf{v}_n(0) \to \mathbf{v}(0)$*, moreover we assume that there exists an* $M \in \mathbb{N}$ *such that* $\forall\, n, \forall\, k \geq M : v_{n,k}(0) = 0$*. Denote by* $p_k^{n,\lambda^*}[t_1, t_2] = \frac{\Phi^n([t_1,t_2],k)}{\Phi^n([t_1,t_2])}$.

Then we have

$$\lim_{dt\to 0} \lim_{\lambda^*\to\infty} \lim_{n\to\infty} \sum_{k\geq 1} \mathbb{1}[k \leq 2\sqrt{\frac{\varphi(t)}{\lambda^*}} \cdot x \cdot n] \cdot p_k^{n,\lambda^*}[t, t+dt] \;=\; \int_0^x \frac{4}{\sqrt{\pi}} y^2 e^{-y^2} dy \qquad (19)$$

where $\varphi(t)$ *is defined by* (15) *using the solution of* (14) *with initial condition* $\mathbf{v}(0)$.

Note that the formulation of this theorem is slightly different in the thesis (see Theorem 3.9 in Subsection 3.1.5).

The r.h.s. of (19) is the distribution function function of the size-biased Rayleigh distribution (see Definition 3.6.3 in Subsection 3.6.1).

The relevance of Theorem 3.2 to Conjecture 3.1 is the following: (19) is a version of (17) in the sense that if $\lambda(n) = \frac{n^\varepsilon}{n}$ for some small ε (that is $\alpha = 1 - \varepsilon$) then (19) suggests that the typical component size of a frozen vertex is of order $\sqrt{\frac{1}{n^\varepsilon}}n = n^{1-\varepsilon/2}$, thus $\beta(\alpha) = \frac{\alpha+1}{2}$, which is in agreement with (16).

We prove Theorem 3.1 in Section 3.6 of the thesis using a coupling argument, the key observation is that the size of the giant component grows linearly right after its birth.

4 The edge reconnecting model

4.1 Context

In recent years a limiting theory has been developed for dense graph sequences (in dense graphs the number of edges is comparable with $|V(G)|^2$). Roughly speaking, a sequence $(G_n)_{n=1}^\infty$ of simple graphs converges if for any fixed *testgraph* F, the density of copies of F found in G_n (called the *homomorphism density*) converges as $n \to \infty$. It was shown in [30] that the limit object can be represented by a symmetric measurable function $W : [0, 1]^2 \to [0, 1]$. Such functions are called *graphons*.

A connection between the theory of dense graph limits and that of infinite exchangeable arrays of random variables was first observed in [18]: if we label the vertices of our graphs with a uniformly chosen permutation then a graph sequence converges if and only if the sequence of the randomly labeled adjacency matrices converge in distribution to an infinite adjacency matrix.

A natural generalization of the theory of dense graph limits to multigraphs (graphs with loop and multiple edges) is given in [28] (joint work with István Kolossváry). We named the limit objects *multigraphons*.

4.2 The model

We introduce the *edge reconnecting model*, a random multigraph (undirected graph with multiple and loop edges) evolving in time. Denote the multigraph at time T by $\mathcal{G}_n(T)$, where $T = 0, 1, 2, \dots$ and $n = |V(\mathcal{G}_n(T))|$ is the number of vertices. We denote by $m = |E(\mathcal{G}_n(T))|$ the number of edges (the number of vertices and edges does not change over time).

Given the multigraph $\mathcal{G}_n(T)$ we get $\mathcal{G}_n(T+1)$ by uniformly choosing an edge in $E(\mathcal{G}_n(T))$, choosing one of the endpoints of that edge with a coin flip and reconnecting the edge to a new endpoint which is chosen using the rule of linear preferential attachment: a vertex v is chosen with probability $\frac{d(v)+\kappa}{2m+n\kappa}$, where $d(v)$ is the degree of vertex v in $\mathcal{G}_n(T)$ and $\kappa \in (0, +\infty)$ is a parameter. We give the formal definition of the edge reconnecting model in Section 4.2 of the thesis.

Our aim is to describe the time evolution of edge reconnecting model $\mathcal{G}_n(T)$ when $1 \ll n$ using the terminology of *dense graph limits*. The notion of convergence of simple graph sequences was defined in [30] (with notations slightly different from what we are using now): we say that a sequence of simple graphs $(G_n)_{n=1}^\infty$ is convergent if for every simple graph F the limit $\lim_{n\to\infty} t_=(F, G_n)$ exists, where

$$t_=(F,G) = \frac{1}{n^{|V(F)|}} \sum_{\varphi: V(F)\to V(G)} \mathbb{1}[\forall\, v, w \in V(F):\ E(v,w) = E(\varphi(v), \varphi(w))] \qquad (20)$$

and $E(v,w)$ denotes the number of edges between v and w. In [30] several equivalent characterizations of *graphons* (limit objects of convergent graph sequences) are given. In [28] a natural generalization of the theory of dense graph limits to multigraphs is given: a sequence of multigraphs $(G_n)_{n=1}^\infty$ is convergent if for every multigraph F the limit $g(F) = \lim_{n\to\infty} t_=(F, G_n)$ exists (moreover $g(\cdot)$ is a "non-defective probability distribution", see Definition 4.3.2 for details) where $t_=(F,G)$ is defined by (20). The limit object of a convergent multigraph sequence is a $W : [0,1] \times [0,1] \times \mathbb{N}_0 \to [0,1]$ function satisfying

$$W(x,y,k) \equiv W(y,x,k), \quad \sum_{k=0}^{\infty} W(x,y,k) \equiv 1, \quad W(x,x,2k+1) \equiv 0.$$

Such functions are called *multigraphons*. We say that $G_n \to W$ if for every $k \in \mathbb{N}$ and every multigraph F with k vertices we have $\lim_{n\to\infty} t_=(F, G_n) = t_=(F, W)$ where

$$t_=(F,W) := \int_{[0,1]^k} \prod_{v \le w \le k} W(x_v, x_w, E(v,w))\, \mathrm{d}x_1\, \mathrm{d}x_2\, \dots\, \mathrm{d}x_k.$$

We give a short survey of the theory of multigraph limits in Section 4.3.

If $\mathcal{G}_n$ is a random multigraph on n vertices for each $n \in \mathbb{N}$ and if for all F multigraphs we have $t_=(F, \mathcal{G}_n) \xrightarrow{\mathrm{d}} t_=(F, W)$ for some multigraphon W as $n \to \infty$, that is

$$\forall\, F\ \forall\, \varepsilon > 0 : \lim_{n\to\infty} \mathbf{P}\big(\ |t_=(F, \mathcal{G}_n) - t_=(F, W)| > \varepsilon\ \big) = 0 \qquad (21)$$

then we say that the sequence $\mathcal{G}_n$ converges to W in probability, $\mathcal{G}_n \xrightarrow{\mathrm{d}} W$.

For a multigraphon W and $x \in [0,1]$ we define the *average degree* of W at x and the edge density of W by

$$D(W,x) := \int_0^1 \sum_{k=0}^{\infty} k \cdot W(x,y,k)\,\mathrm{d}y \tag{22}$$

$$\rho(W) := \int_0^1 \int_0^1 \sum_{k=0}^{\infty} k \cdot W(x,y,k)\,\mathrm{d}y\,\mathrm{d}x \tag{23}$$

If $\rho(W) < +\infty$ then $D(W,x) < +\infty$ for almost all x.

Let G be a multigraph on n vertices. The adjacency matrix of a labeling of the multigraph G with $[n] = \{1,2,\dots,n\}$ is denoted by $(B(i,j))_{i,j=1}^n$, where $B(i,j) \in \mathbb{N}_0 = \{0,1,2,\dots\}$ is the number of edges connecting the vertices labeled by i and j. $B(i,j) = B(j,i)$ since the graph is undirected and $B(i,i)$ is two times the number of loop edges at vertex i (thus $B(i,i)$ is an even number).

We denote the set of adjacency matrices of multigraphs on n nodes by $\mathcal{A}_n$, thus

$$\mathcal{A}_n = \left\{ B \in \mathbb{N}_0^{n\times n} \,:\, B^T = B,\ \forall\, i \in [n]\ \ 2 \,|\, B(i,i) \right\}.$$

4.3 Theorems

Recall the formulas defining the Poisson, Binomial and Gamma distributions:

$$\mathbf{p}(k,\lambda) := e^{-\lambda}\frac{\lambda^k}{k!} \tag{24}$$

$$\mathbf{b}(k,n,p) := \binom{n}{k} p^k (1-p)^{n-k} \tag{25}$$

$$\mathbf{g}(x,\alpha,\beta) := x^{\alpha-1}\frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)}\mathbb{1}[x>0] \tag{26}$$

We describe the evolution of the edge reconnecting model by describing the evolution of the limiting multigraphons. We consider a sequence of initial multigraphs $(G_n)_{n=1}^\infty$ which converge to a multigraphon W. We assume $|V(G_n)| = n$. We denote the adjacency matrix of G_n by $B_n \in \mathcal{A}_n$. We assume that the technical condition

$$\exists\, \lambda > 0,\ C < +\infty\ \ \forall\, n: \quad \frac{1}{\binom{n}{2}} \sum_{i \le j \le n} e^{\lambda B_n(i,j)} \le C, \quad \frac{1}{n}\sum_{i=1}^{n} e^{\lambda B_n(i,i)} \le C \tag{27}$$

holds.

First we state Theorem 4.1 about the evolution of the edge reconnecting model on the $T = \mathcal{O}(n^2)$ timescale.

Theorem 4.1. *Let us fix $\kappa \in (0,+\infty)$. We consider the edge reconnecting model $\mathcal{G}_n(T)$, $T = 0,1,\dots$, with the adjacency matrix of the initial state $\mathcal{G}_n(0)$ being B_n for $n = 1,2,\dots$. We assume $B_n \to W$ for some multigraphon W and that (27) holds.*

Then for all $t \in [0, +\infty)$ we have

$$\mathcal{G}_n\left(\lfloor t \cdot \frac{\rho(W) \cdot n^2}{2} \rfloor\right) \xrightarrow{d} W_t \qquad \text{as} \qquad n \to \infty \tag{28}$$

where

$$W_t(x,y,k) = \sum_{h=0}^{\infty} W(x,y,h) \sum_{l=0}^{k} \mathbf{b}(l,h,e^{-t})) \cdot \mathbf{p}\left(k-l, \frac{D(W,x)\cdot D(W,y)}{\rho(W)}(1-e^{-t})\right) \quad \text{if } x \neq y \tag{29}$$

$$W_t(x,y,k) = \mathbb{1}[2|k] \cdot \sum_{h=0}^{\infty} W(x,y,h) \sum_{l=0}^{\frac{k}{2}} \mathbf{b}(l, \frac{h}{2}, e^{-t})) \cdot \mathbf{p}\left(\frac{k}{2}-l, \frac{D(W,x)\cdot D(W,y)}{2\rho(W)}(1-e^{-t})\right) \quad \text{if } x = y \tag{30}$$

We give an intuitive explanation of Theorem 4.1 in Section 4.6 by relating the evolution of the number of parallel/loop edges between two vertices to the evolution of the queue length of an M/M/∞-queue. We rigorously prove Theorem 4.1 in Section 4.9.

Now we look at the evolution of the edge reconnecting model on the $T = \mathcal{O}(n^3)$ timescale.

Theorem 4.2. *Let us fix $\kappa \in (0, +\infty)$. We consider the edge reconnecting model $\mathcal{G}_n(T)$, $T = 0, 1, \dots$, with the adjacency matrix of the initial state $\mathcal{G}_n(0)$ being B_n for $n = 1, 2, \dots$.*

We assume $B_n \to W$ for some multigraphon W and that (27) holds.

Then for all $t \in (0, +\infty)$ (but not for t=0) we have

$$\mathcal{G}_n\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor\right) \xrightarrow{d} \hat{W}_t \qquad \text{as} \qquad n \to \infty \tag{31}$$

where

$$\hat{W}_t(x,y,k) = \begin{cases} \mathbf{p}(k, \frac{F_t^{-1}(x)F_t^{-1}(y)}{\rho(W)}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F_t^{-1}(x)F_t^{-1}(y)}{2\rho(W)}\right) & \text{if } x = y \end{cases} \tag{32}$$

and F_t^{-1} is the inverse function of $F_t(x) = \int_0^x f(t,y)\,\mathrm{d}y$ where

$$f(t,x) = \int_0^{\infty} \sum_{i=0}^{\infty} \mathbf{p}(i, z\cdot\tau(\alpha,t))\mathbf{g}(x, \kappa+i, \tau(\alpha,t)+\alpha)\,\mathrm{d}F_0(z), \tag{33}$$

$\alpha = \frac{\kappa}{\rho(W)}$, $\tau(\alpha,t) = \frac{\alpha}{\exp(\alpha t)-1}$ *and* $F_0(x) = \int_0^1 \mathbb{1}[D(W,y) \le x]\,\mathrm{d}y$, $x \in [0, +\infty)$.

We prove Theorem 4.2 in Section 4.10, but first we give an intuitive explanation in Section 4.6 by relating the evolution of the degree of a vertex to a diffusion process (the C.I.R. process).

Summarizing Theorem 4.1 and Theorem 4.2, we have

$$\mathcal{G}_n(t \cdot \frac{\rho(W)}{2} \cdot n^2) \xrightarrow{\mathrm{d}} W_t \qquad \text{and} \qquad \mathcal{G}_n(t \cdot \rho(W) \cdot n^3) \xrightarrow{\mathrm{d}} \hat{W}_t \tag{34}$$

where the multigraphons W_t and $\hat{W}_t$ are explicit functions of t, W and the linear preferential attachment parameter κ. Moreover we have

$$\lim_{t\to 0_+} W_t = W, \qquad \lim_{t\to\infty} W_t = \lim_{t\to 0_+} \hat{W}_t, \qquad \lim_{t\to\infty} \hat{W}_t = W_\infty \tag{35}$$

where W_∞ is the limiting multigraphon of the stationary states of the edge reconnecting models: $\mathcal{G}_n(\infty) \xrightarrow{\mathrm{d}} W_\infty$. Thus by (35) the convergence theorems (34) give the *full characterization* of the time evolution of the multigraphons arising as the graph limits of the edge reconnecting model.

Although the above theorems are stated using the "multigraphon" formalism, in their proofs we use the correspondence between the theory of graph limits and that of exchangeable arrays, a connection first observed in [18]. In Section 4.5 we show how the convergence of the values $t_=(F, G_n)$ is related to the convergence in distribution of the minors of the randomly relabeled adjacency matrices of the multigraphs $(G_n)_{n=1}^\infty$.

4.4 About the proofs

In Section 4.6 we give the intuitive explanation of these results using exchangeable arrays. These sketch proofs also serve as an outline of the rigorous proofs. The basic idea is to relate the time evolution of the edge reconnecting model to certain continuous-time stochastic processes using an appropriate rescaling of time:

- If we fix a vertex $v \in V(\mathcal{G}_n(0))$ and denote by $d(T, v)$ the degree of v in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{R}_+$-valued continuous-time stochastic process $\frac{1}{n} d(n^3 \cdot t, v)$ "almost looks like" that of a Cox-Ingersoll-Ross process (a diffusion process that is commonly used in financial mathemathics to model the evolution of interest rates). This fact is rigorously proved using the theory of stochastic differential equations and is used in the proof of $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{d}} \tilde{W}_t$.
- If we fix two vertices $v, w \in V(\mathcal{G}_n(0))$ and denote by $E(T, v, w)$ the number of parallel/loop edges connecting v and w in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{N}$-valued continuous-time stochastic process $E(n^2 \cdot t, v, w)$ "almost looks like" that of the queue length of an M/M/∞-queue. This fact is rigorously proved using a coupling argument and is used in the proof of $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{d}} \breve{W}_t$.

The most interesting property of the edge reconnecting model is the separation of *two different timescales*: the degrees of the vertices only change significantly on the n^3 timescale, whereas the number of parallel (or loop) edges between two vertices evolves on the much faster n^2 timescale. The arrival rate of the M/M/∞-queue describing the evolution of $E(n^2 \cdot t, v, w)$ depends on the current degrees of v and w, but since the degrees evolve on the much slower n^3 timescale, they may be treated as constant background parameters on the n^2 timescale. If we fix k vertices in the model and denote by $\mathcal{G}_n^k(T)$ the random subgraph of $\mathcal{G}_n(T)$ spanned by these vertices then the $\mathcal{M}_k$-valued stochastic process $\mathcal{G}_n^k(n^3 \cdot t + n^2 \cdot s)$ looks stationary in the time variable $s \in \mathbb{R}$ if $t \in (0, +\infty)$ is fixed and $1 \ll n$, but different values of t yield distinct pseudo-stationary distributions since $n^3 \cdot (t_2 - t_1)$ steps are enough for the background variables (degrees) to significantly change.

This phenomenon is called *aging* in statistical physics, see [4] and [9].

In Section 4.7 we explicitly describe the stationary distribution $\mathcal{G}_n(\infty)$ of the edge reconnecting model by relating it to the Pólya urn model and prove $\mathcal{G}_n(\infty) \xrightarrow{\text{d}} W_\infty$. As an intermediate step we describe the multigraph limits of random multigraphs which are uniformly chosen from the set of multigraphs with a given degree sequence (this construction is known as the *configuration model* in the theory of random graphs).

In Section 4.8 we state and prove technical lemmas needed for the proof of Theorem 4.1 in Section 4.9 and the proof of Theorem 4.2 in Section 4.10.

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