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Markov property in non-commutative probability

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1 Introduction

Markov chains are the simplest mathematical models for random phenomena evolving in time. Their simple structure makes it possible to say a great deal about their behaviour. At the same time, the class of Markov chains is rich enough to serve in many applications, for example in population growth, mathematical genetics, networks of queues, Monte Carlo simulation and in many others. This makes the Markov chains the first and most important examples of random processes. Indeed, the whole of the mathematical study of random processes can be regarded as a generalization in one way or another of the theory of Markov chains. We shall be concerned exclusively with the case where the process can assume only a finite or countable set of states.

A discrete random variable is a function X with values in a finite set $\mathcal{X}$ and its probability mass function is $p(x) = \Pr\{X = x\}$, $x \in \mathcal{X}$. Each $x \in \mathcal{X}$ is called a state and $\mathcal{X}$ is called the state-space. A stochastic process is an indexed sequence of random variables. In general, there can be an arbitrary dependence among these random variables. The process is characterized by the joint probability mass functions $p(x_1, x_2, \ldots, x_n) = \Pr\{(X_1, X_2, \ldots, X_n) = (x_1, x_2, \ldots, x_n)\}$, where $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ for $n = 1, 2, \ldots$. A simple, but important example of a stochastic process with dependence is one in which each random variable depends on the one preceding it and is conditionally independent of all the other preceding random variables. Such a process is said to be Markovian. For the most part we will attend to the case of three random variables, so we say, that the random variables X, Y and Z form a Markov triplet (denoted by $X \to Y \to Z$) if

$$p(x, y, z) = p(x)p(y|x)p(z|y),$$

where

$$p(y|x) = \frac{\Pr\{X = x, Y = y\}}{\Pr\{X = x\}}$$

is the conditional probability. If Z has the interpretation as "future", Y is the "present" and X is the "past", then having the Markov property means that, given the present state, future states are independent of the past states. In other words, the description of the present state fully captures all the information that could influence the future evolution of the process.

It is natural to investigate the Markov triplets from information theoretical point of view. In 1948 the electric engineer *C. Shannon* published a remarkable pair of papers laying the foundations for the modern theory of information and communication. Perhaps the key step taken by Shannon was to mathematically define the concept of information. As a measure of uncertainty of a random variable he proposed the following formula

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$$

called Shannon entropy. If the log is to the base 2, the entropy is expressed in bits, while if the base is e, the unit of entropy is sometimes called nat. The Shannon entropy has many properties that are in accord with the intuitive notion of what a measure of information should be, for example it helps us to express the dependence among the random variables. One of its basic properties is subadditivity, i.e.

$$H(X,Y) \leq H(X) + H(Y),$$

where $H(X,Y) = -\sum_{x\in\mathcal{X},y\in\mathcal{Y}} p(x,y)\log p(x,y)$ is the joint entropy of random variables X and Y, and measures our total uncertainty about the pair (X,Y). The equality holds in the subadditivity if and only if X and Y are independent random variables. The other remarkable property of the Shannon entropy is the strong subadditivity:

$$H(X,Y,Z) + H(Y) \leq H(X,Y) + H(Y,Z),$$

with equality if and only if $X \to Y \to Z$, i.e. X, Y and Z form a Markov triplet. It means that Markov triplets are completely characterized by the strong additivity of their Shannon entropy.

At the turn of the twentieth century a series of crises had arisen in the physics. The problem was that the classical theory were predicting absurdities such as the existence of an "ultraviolet catastrophe" involving infinite energies, or electrons spiraling inexorably into the atomic nucleous. The crisis came to a head in the early 1920's and resulted in the creation of the modern theory of quantum mechanics. *J. von Neumann* worked in Göttingen when *W. Heisenberg* gave the first lectures on the subject. Quantum mechanics motivated the creation of new areas in mathematics, the theory of linear operators on Hilbert spaces was certainly such an area. John von Neumann made an effort towards the mathematical foundations and he initiated the study of what are now called von Neumann algebras. With

F.J. Murray, they made a first classification of such algebras [12]. While the mathematics of classical probability theory was subsumed into classical measure theory by *A.N. Kolmogorov* [9], the quantum or non-commutative probability theory was induced by the quantum theory and was incorporated into the beginnings of non-commutative measure theory by *J. von Neumann* [13].

In this concept, quantization is a process in which classical observables, i.e. real functions on a phase space, are replaced by self-adjoint operators on a Hilbert space $\mathcal{H}$. Similarly, in quantum or non-commutative probability the role of random variables is played by self-adjoint elements affiliated to some C*-algebra $\mathcal{A}$ with unit element $\mathbf{1}$. Probability measures are replaced by states, i.e positive linear functionals ϕ on $\mathcal{A}$ such that $\phi(\mathbf{1}) = 1$. If $\mathcal{A}$ is a non-commutative algebra then we say that $(\mathcal{A}, \phi)$ is an abstract or algebraic non-commutative probability space. This concept means a generalization: as long as one considers a commutative C*-algebra $\mathcal{A}$, quantum probability reduces to classical probability. One usually represents $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$ of bounded operators acting on a complex separable Hilbert space $\mathcal{H}$. If the state ϕ is normal, i.e. positive weakly continous normalized linear functional on the von Neumann algebra $\mathcal{B}(\mathcal{H})$, then it is given by $\phi(A) = \mathrm{Tr}\,(\rho A)$, $A \in \mathcal{B}(\mathcal{H})$, for some unique statistical operator ρ acting on $\mathcal{H}$, i.e. $0 < \rho = \rho^* \in \mathcal{B}(\mathcal{H})$ such that $\mathrm{Tr}\,(\rho) = 1$. If $\mathcal{H}$ is finite dimensional ρ is often called density matrix. From the quantum theoretical point of view the selfadjoint elements of $\mathcal{B}(\mathcal{H})$ are identified with physical observables, while state ϕ represents the state of a physical system.

It is a natural question, that how can we generalize the concept of Markovianity in the non-commutative setting which means the framework of C*-algebras, or matrix algebras in the finite dimensional case.

In von Neumann's unifying scheme for classical and quantum probability an important ingredient was missing: conditioning. In order to study non-trivial statistical dependences, in particular to construct Markov processes, this gap had to be filled. The first step was made in 1962 by the most natural quantum generalization of the notion of conditional expectation, by *H. Umegaki* [18], which is relevant for several problems in operator theory and in quantum probability. By a (Umegaki) conditional expectation $E : \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ we mean a norm-one projection of the C*-algebra $\mathcal{A}$ onto the C*-subalgebra $\mathcal{B}$. The map E is automatically a completely positive identity-preserving $\mathcal{B}$-bimodule map by a theorem of *J. Tomiyama* [17]. E is called compatible with a state ϕ if $\phi \circ E = \phi$. Unfortunately Umegaki's notion is not perfect to express the Markovianity, since the states compatible with norm-one projections tend to be trivial in the extremely non-commutative case. Indeed, a state ϕ on $M_n \otimes M_n$ is compatible with an Umegaki conditional expectation onto $M_n \otimes I$ if and only if it is a product state, which means that our random variables are independent. (Here M_n denotes n by n complex matrices.) To avoid this trivial case *L. Accardi* and *A. Frigerio* proposed the following definition in 1978 [3]. Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of unital C*-algebras. A quasi-conditional or generalized conditional expectation w.r.t the given triplet is a completely positive identity-preserving linear map $E : \mathcal{A} \to \mathcal{B}$

such that

$$E(ca) = cE(a), \qquad a \in \mathcal{A}, c \in \mathcal{C}.$$

The notion of non-commutative or quantum Markov chains was also introduced by Accardi in [1, 2]. Quantum Markov chains are defined on non-commutative C*-algebras, in particular on UHF algebras, and they are determined by an initial state and a sequence of unital completely positive maps, called transition maps. Since in the classical case Markov chains can be defined on abelian C*-algebras and are determined by an initial distribution and a sequence of transition matrices, quantum Markov chains can be regarded as the generalization of the classical ones.

In spite of the abstractness of this definition, several improvements have been made in their applications to physical models. In particular a sub-class of Markov chains, also called finitely correlated states, was shown to coincide with the so-called valence bond states introduced in the late 1980's in the context of antiferromagnetic Heisenberg models. The works of *M. Fannes, B. Nachtergaele* and *R.F. Werner* were appreciable to find the ground states of these modells [7]. As another special class of quantum Markov chains, the notion of quantum Markov states was defined in [3]. A quantum Markov state ϕ is determined by an initial state and a sequence of ϕ-preserving quasi-conditional expectations. If we consider a Markov state with three parts we say that it is a short Markov state or Markov triplet. The question raises, whether similarly to the classical case, there is any characterization of Markov states by the entropy quantities?

If ρ is the density matrix of a normal state ϕ, the von Neumann entropy of the state is defined by

$$S(\phi) \equiv S(\rho) = -\mathrm{Tr}\,\rho\log\rho.$$

Similarly to the classical case the von Neumann entropy plays an important role in the investigations of quantum systems's correlations. The von Neumann entropy is subadditive, i.e.

$$S(\phi_{12}) \leq S(\phi_1) + S(\phi_2),$$

where ϕ_{12} is a normal state of the composite system of $\mathcal{A}_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2$, and the equality holds if and only if ϕ_{12} is product of its marginals, i.e. $\phi_{12} = \phi_1 \otimes \phi_2$, which is the non-commutative analogue of the independent random variables. We also have the remarkable strong subadditivity property which was proved by *E. Lieb* and *M.B. Ruskai* in 1973 [10]. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be subalgebras of $\mathcal{B}(\mathcal{H})$ representing three quantum systems and set $\mathcal{A}_{123} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$, $\mathcal{A}_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{A}_{23} = \mathcal{A}_2 \otimes \mathcal{A}_3$ as their several compositions. For a state ϕ_{123} of $\mathcal{A}_{123}$ we denote its restrictions to $\mathcal{A}_{12}$, $\mathcal{A}_{23}$ and $\mathcal{A}_2$ with ϕ_{12}, ϕ_{23} and ϕ_2, respectively. The strong subadditivity says, that

$$S(\phi_{123}) + S(\phi_2) \leq S(\phi_{12}) + S(\phi_{23}).$$

On the analogy of the classical Markov property it has been shown that the strong subadditivity of the von Neumann entropy is tightly related to the Markov property invented by *L. Accardi.* A state of a three-fold tensor product system is Markovian if and only if it takes the equality of the strong subadditivity of von Neumann entropy, which is referred to as strong additivity of the von Neumann entropy. In other words, a state of a tensor product system is Markovian if and only if the von Neumann entropy increase is constant. The exact structure of a density ρ_{123} with this property was established in 1994 by *P. Hayden, R. Jozsa, D. Petz* and *A. Winter* [8].

Although a pivotal example of quantum composite systems is tensor product of Hilbert spaces, we can see that the definition of Markov property has been given under a very general setting that is not limited to the most familiar case of tensor-product systems. Which means, it does not require in principle any specific algebraic location among systems imbedded in the total system. A very important example from this point of view is the algebra of the canonical anti-commutation relation or briefly CAR algebra, that serves as the description of fermion lattice systems.

The quantum-mechanical states of n identical point particles in the configuration space $\mathbb{R}^\nu$ are given by vectors of the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^{\nu n})$. If $\psi \in \mathcal{H}$ is normalized, then

$$\mathrm{d}p(x_1, \dots, x_n) = |\psi(x_1, \dots, x_n)|^2 \mathrm{d}x_1 \dots \mathrm{d}x_n$$

is the probability density for ψ to describe n particles at the infinitesimal neighborhood of the points $x_1, \dots, x_n$. The normalization of ψ corresponds to the normalization of the total probability to unity. But in microscopic physics identical particles are indistinguishable and this is reflected by the symmetry of the probability density under interchange of the particle coordinates. This interchange defines an unitary representation of the permutation group and the symmetry is assured if the ψ transform under a suitable subrepresentation. There are two cases of paramount importance. The first arises when ψ is symmetric under change of coordinates. Particles whose states transform in this manner are called bosons and are said to satisfy Bose-Einstein statistics. The second case corresponds to anti-symmetry of ψ under interchange of each pair of coordinates. The associated particles are called fermions and said to satisfy Fermi-Dirac statistics. The elementary constituents of matter seem to be fermions, while interactions are mediated by bosons. In the case of fermions the anti-symmetry of the wave function had a deep consequence, namely the Pauli principle: It is impossible to create two fermions in the same state. The main qualitative difference between fermions and bosons is the absence of a Pauli principle for the latter particles. There is no bound on the number of particles which can occupy a given state. Mathematically this is reflected by the unboundedness of the so-called Bose annihilation and creation operators. This unboundedness leads to a large number of technical difficulties which are absent for fermions. These problems can be partially avoided by consideration of bounded functions of the annihilation and creation operators. This idea yields the Weyl operators and their algebra, the algebra of canonical commutation relation (briefly CCR algebra). I investigate both systems from the wievpoint of Markovianity.

Returning to the classical context an important case was solved recently: The characterisation of multivariate normal Markov triplets was given [4]. In classical probability, a Gaussian measure leads to a characteristic function which is the exponential of a quadratic form. Its logarithm is therefore a quadratic polynomial, and all correlations beyond the second order vanish. Following this procedure in some non-commuting systems as the CAR algebra or the CCR algebra it is also possible to define the useful concept of the correlation function (cumulants or truncated function in other words), and we can arrive to the analogues of Gaussian distributions, to the so-called quasi-free states. In these states the n-point functions can be computed from the 2-point functions and in one kind of central limit theorem the limiting state is quasi-free. The quasi-free states are quite tractable, for example the von Neumann entropy has an explicit expression. It is a natural to ask: What can we say about the quasi-free Markov triplets? My goal is the discussion of these questions.

2 New scientific results

1. However the strong subadditive property of von Neumann entropy is well known for a long time ago, all known proofs are quite difficult. I gave a simple proof based on the Golden-Thompson-Lieb inequality. The advantage of the proof, that it works not just for tensor product systems and that the necessary condition of the equality can be read out easily. The condition is sufficient as well.

Theorem 2.1 *Let $\mathcal{A}_{12}, \mathcal{A}_{23}$ and $\mathcal{A}_2$ be subalgebras of $\mathcal{A}_{123}$, where $\mathcal{A}_2 \subset \mathcal{A}_{12}, \mathcal{A}_{23}$, as well. For a state ϕ_{123} of $\mathcal{A}_{123}$ we denote by ϕ_{12}, ϕ_{23} and ϕ_2 its restrictions to $\mathcal{A}_{12}$, $\mathcal{A}_{23}$ and $\mathcal{A}_2$, respectively. If there exist a trace preserving conditional expectations*

$E_{12} : \mathcal{A}_{123} \to \mathcal{A}_{12}$ and $E_{23} : \mathcal{A}_{123} \to \mathcal{A}_{23}$, such that the commuting square condition holds, i.e. $E_{12} \circ E_{23} = E_{23} \circ E_{12}$, then

$$S(\phi_{123}) + S(\phi_2) \le S(\phi_{12}) + S(\phi_{23}),$$

and the equality holds if an only if we get for the density matrices

$$\log D_{123} + \log D_2 = \log D_{12} + \log D_{23}.$$

The same condition for the equality was also gained by *D. Petz* in an other form [15, 16].

2. It has been shown that the Markovianity is tightly related to the strong subadditivity of von Neumann entropy. Namely, a state of a three-composed tensor-product system forms a Markov triplet if and only if it takes the equality for the strong subadditivity inequality of entropy. Moreover the complete characterization of Markov triplets is given in this case [8]. The results can not be used for CAR algebras directly. The root of the problem is the difference between the three-fold tensor product system and the CAR algebra from the point of view of the commutation of the subsystems. Indeed, however when the set I is countable, the CAR algebra $\mathcal{A}(I)$ is isomorphic to the C^*-infinite tensor product $\overline{\otimes_I M_2(\mathbb{C})}^{C^*}$, but the isomorphism does not preserve the natural localization. The elements of the disjoint subsystems do not commute in contrast to the tensor product case. In spite of these difficulties the strong subadditivity of von Neumann entropy also holds for CAR algebras as was proved by Araki and Moriya [5] and as my proof of Theorem 2.1 shows. Let I and J be two arbitrary subsets of $\mathbb{Z}$ and denote $\mathcal{A}(I \cup J)$, $\mathcal{A}(I)$, $\mathcal{A}(J)$

and $\mathcal{A}(I \cap J)$ the CAR algebras corresponding to the sets $I \cup J$, I, J and $I \cap J$, respectively with the states $\phi_{I\cup J}$, ϕ_I, ϕ_J and $\phi_{I\cap J}$, as usual. Then

$$S(\phi_I) + S(\phi_J) \geq S(\phi_{I\cap J}) + S(\phi_{I\cup J}) \tag{1}$$

holds. I prove that the equality case is equivalent with the Markov property also for CAR algebras if we restrict ourself for even states.

Theorem 2.2 *Let $\phi_{I\cup J}$ be an even state on the CAR algebra $\mathcal{A}(I \cup J)$. Then $\phi_{I\cup J}$ is a Markov triplet corresponding to the localization $\{\mathcal{A}(I \setminus J), \mathcal{A}(I), \mathcal{A}(I \cup J)\}$, i.e. there exists a quasi-conditional expectation γ w.r.t the triplet $\mathcal{A}(I \setminus J) \subset \mathcal{A}(I) \subset \mathcal{A}(I \cup J)$ satisfying*

$$\phi_I \circ \gamma = \phi_{I\cup J}, \tag{2}$$

$$E(\mathcal{A}(J)) \subset \mathcal{A}(I \cap J), \tag{3}$$

if and only if it saturates the strong subadditivity inequality of entropy with equality, ie.

$$S(\phi_I) + S(\phi_J) = S(\phi_{I\cap J}) + S(\phi_{I\cup J}). \tag{4}$$

I also give a modification of the above theorem by leaving the condition of evennes:

Theorem 2.3 *Let $\phi_{I\cup J}$ be an arbitrary state on the CAR algebra $\mathcal{A}(I \cup J)$. Then $\phi_{I\cup J}$ is a Markov state corresponding to the localization $\{\mathcal{A}(I \setminus J)^+, \mathcal{A}(I), \mathcal{A}(I \cup J)\}$ if and only if (4) holds.*

Here $\mathcal{A}(I \setminus J)^+$ denotes the even subalgebra of $\mathcal{A}(I \setminus J)$.

3. I give a characterization of quasi-free product states on CAR algebras by their symbol. I show that the quasi-free states on CAR algebra which saturate the strong additivity of von Neumann entropy with equality (ie. Markove states) are product states with respect their localization.

4. Assume that ψ is a state of $CCR(\mathcal{H})$. If

$$C_\psi(f,g) := \psi(B^+(f)B^-(g))$$

can be defined, then it will be called 2-point function of ψ. A positive operator T, defined by

$$\langle g|Tf\rangle = C_\psi(f,g), \tag{5}$$

is called the 2-point operator of ψ. I computed the relative entropy of a state ψ and a Gaussian state ω_A.

Theorem 2.4 *Consider a state ψ on the algebra $CCR(\mathcal{H})$ with a 2-point operator T. Then its relative entropy with respect to the quasi-free state ω_A is given by*

$$S(\psi\|\omega_A) = -S(\psi) - \mathit{Tr}\, T \log A(I+A)^{-1} + \mathit{Tr}\, \log(I+A). \tag{6}$$

As a consequence we get that the quasi-free state ω_A has the largest entropy among states with 2-point operator A.

Theorem 2.1 *Let ψ be a state of* $\mathrm{CCR}(\mathcal{H})$ *such that its 2-point function is $\psi(B^+(f)B^-(g)) = \langle g, Af\rangle$ $(f,g \in \mathcal{H})$ for a positive* *operator* $A \in B(\mathcal{H})$*. Then* $S(\psi) \le S(\omega_A)$ *and equality implies* $\psi = \omega_A$*.*

5. I also investigated the Markov property on CCR algebras. Assume that the Hilbert space $\mathcal{H}$ has the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Then

$$\mathrm{CCR}(\mathcal{H}) = \mathrm{CCR}(\mathcal{H}_1) \otimes \mathrm{CCR}(\mathcal{H}_2) \otimes \mathrm{CCR}(\mathcal{H}_3)$$

and the equality in the strong subadditivity of the von Neumann entropy can be the definition of the Markov property [14]. If φ_{123} is quasi-free (Gaussian), then it is given by a positive operator (corresponding to the 2-point function) and the main goal was to describe the Markov property in terms of this operator. Consider a Gaussian state $\omega_A \equiv \omega_{123}$, where A is a positive operator acting on $\mathcal{H}$. This operator has the block-matrix form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix}.$$

Our aim is to characterize the Markov property in terms of the block-matrix A. Denote by P_i the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_i$, $1 \le i \le 3$. Of course, $P_1 + P_2 + P_3 = I$ and we use also the notation $P_{12} := P_1 + P_2$ and $P_{23} := P_2 + P_3$.

Theorem 2.2 *Assume that* $A \in B(\mathcal{H})$ *is a positive invertible operator and the Gaussian state* $\omega_A \equiv \omega_{123}$ *on* $\mathrm{CCR}(\mathcal{H})$ *has finite von Neumann entropy. Then the following conditions are equivalent.*

(a) $S(\omega_{123}) + S(\omega_2) = S(\omega_{12}) + S(\omega_{23})$

(b) $Tr\,\kappa(A) + Tr\,\kappa(P_2AP_2) = Tr\,\kappa(P_{12}AP_{12}) + Tr\,\kappa(P_{23}AP_{23})$

(c) *There is a projection* $P \in B(\mathcal{H})$ *such that* $P_1 \le P \le P_1 + P_2$ *and* $PA = AP$.

Condition (c) tells that the matrix A has the following block diagonal form:

$$A = \begin{bmatrix} A_{11} & \begin{bmatrix} a & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} a^* \\ 0 \end{bmatrix} & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} & \begin{bmatrix} 0 \\ b \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & b^* \end{bmatrix} & A_{33} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & a \\ a^* & c \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} d & b \\ b^* & A_{33} \end{bmatrix} \end{bmatrix}, \quad (7)$$

where the parameters a, b, c, d (and 0) are operators.

6. I discuss the minimization of the relative entropy with respect a quasi-free state on CCR algebras under some conditions. I show that the minimizer is Markovian similarly to the classical probabilistic case [6].

Theorem 2.3 *Let* $\omega \equiv \omega_A$ *be a Markovian a Gaussian state on the CCR-algebra* $\mathrm{CCR}(\mathcal{H})$ *and let* ψ_1 *be a state of* $\mathrm{CCR}(\mathcal{H}_1)$ *with a 2-point function. If* ψ *is the state minimizing the relative entropy* $S(\psi||\omega_A)$ *under the constraint that* $\psi|\mathrm{CCR}(\mathcal{H}_1) = \psi_1$ *is fixed, then* ψ *is a Markov state.*

Note that the minimizer Markovian ψ has the same conditional expectation than the given state ω. In the probabilistic case the similar statement is well known. I also prove the following.

Theorem 2.4 *Let $\omega \equiv \omega_A$ be a Markovian quasi-free state on the CCR-algebra* $\mathrm{CCR}(\mathcal{H})$. *There exists a state ψ which is minimizing the relative entropy $S(\psi||\omega_A)$ under the constraint that $\psi|\mathcal{A}_1$ has a fixed 2-point operator. Moreover, ψ is a Markov state.*

7. I investigated some analogy between the classical Gaussian and the CCR Gaussian Markov states. I prove that commuting field operators form a classical Gaussian Markov triplet.

3 Publications related to the dissertation

1. V.P. Belavkin, J. Pitrik, Notes on the equality in SSA of entropy on CAR algebra arXiv:math-ph/0602035

2. J. Pitrik, Markovian quasifree states on canonical anticommutation relation algebras, J. Math. Phys. **48**(2007), 112110.

3. A. Jenčová, D. Petz and J. Pitrik, Markov triplets on CCR-algebras, Acta Sci. Math. (Szeged), **76**(2009), 625–648.

4. D. Petz and J. Pitrik, Markov property of Gaussian states of CCR-algebras, *to be published in* J. Math. Phys. **50**

5. D. Petz and J. Pitrik, Gaussian Markov triplets, *to be published in* Proceedings of the Quantum Bio-Informatics III., $QP - PQ$ Quantum Probability and White Noise Analysis

6. J. Pitrik, Markov triplets on CAR algebras, *to be published in* Quantum Probability and Related Topics, Vol. 25.

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