Examination of Regular Mosaics

PhD Thesis

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1 Preliminaries

In the 3-dimensional spaces the Schläfli's symbols of the regular mosaics (honeycombs) are $\{p, q, r\}$, where $\{p, q\}$ describes the domain (cell) and $\{q, r\}$ denotes the vertex figure of a mosaic (Coxeter [1]). Thus, an edge of a regular polyhedron $\{p, q\}$ is surrounded by r other polyhedra, so the order of the rotational symmetry along an edge is r ([15]). Coxeter ([1]) examined the higher dimensional honeycombs as well. He proved that there is only the well-known cube mosaic $\{4, 3, 4\}$ in the 3-dimensional Euclidean space and there are 15 regular mosaics in the 3-dimensional hyperbolic space. Four of them have bounded domains ($\{3, 5, 3\}$, $\{4, 3, 5\}$, $\{5, 3, 4\}$, $\{5, 3, 5\}$), while the domains of the further ones are unbounded ($\{3, 4, 4\}$, $\{3, 3, 6\}$, $\{4, 3, 6\}$, $\{5, 3, 6\}$, $\{4, 4, 3\}$, $\{6, 3, 3\}$, $\{6, 3, 4\}$, $\{6, 3, 5\}$, $\{6, 3, 6\}$, $\{4, 4, 4\}$, $\{3, 6, 3\}$). Coxeter ([1]) also showed that there are three honeycombs ($\{3, 3, 4, 3\}$, $\{3, 4, 3, 3\}$, $\{4, 3, 3, 4\}$) in the 4-dimensional Euclidean space and there are only the honeycombs $\{4, 3, ..., 3, 4\}$ in the higher dimensional Euclidean spaces. There are honeycombs yet only in the 4- and 5-dimensional hyperbolic spaces. Among the 7 honeycombs in the 4-dimensional hyperbolic space five have bounded domains ($\{3, 3, 3, 5\}$, $\{4, 3, 3, 5\}$, $\{5, 3, 3, 5\}$, $\{5, 3, 3, 4\}$, $\{5, 3, 3, 3\}$) and two have unbounded domains ($\{3, 4, 3, 4\}$, $\{4, 3, 4, 3\}$). It has already been proved, that in the 5-dimensional hyperbolic space the 5 honeycombs have unbounded domains.

Fejes Tóth, L. [2, 261. p.] examined the area of the circles with common centre in the following way. Let $C(r)$ be the area of the circle with radius r. If $a > 0$, then $\lim\limits_{r\to\infty} \frac{C(r+a)-C(r)}{C(r)}$ is equal to 0 in the Euclidean plane, but it is $e^a - 1$ in the hyperbolic plane. This fact inspired several mathematicians to deal with analogous structures of the hyperbolic space.

In the following we generalize this limit for the regular mosaics.

Let us fix a point P, as a (finite) vertex of the mosaic and create belts around it. The first belt consists of the domains of the mosaic containing P. (The belt 0 is the point P itself. In cases of the mosaics having no finite vertices, the belt 0 is defined as a domain itself.) If the belt i is known, let the belt $(i+1)$ consist of the domains having a common (finite) point (not necessarily a common vertex)with the belt i, but they do not have common points with the belt $(i-1)$. Let Π_i denote the polyhedron solid determined by the outer boundary of the belt i ($\Pi_0 = P$). Π_i contains all the belts j, where $j = 0, \dots, i$.

Let V_i denote the volume of the belt i, and F_i^k ($i \geq 0$, $d \geq k > 0$, d is the dimension of the mosaic) the sum of the volumes of the k-dimensional faces on the surface of Π_i. (If $k = d$, then $F_i^r = V_i$, and F_i^0 is the number of the finite vertices on the surface of Π_i.) Furthermore, let $S_i = \sum_{j=0}^{i} V_j$ be the volume of Π_i.

Kárteszi [4] examined the mosaics with regular triangles $\{3, m\}$ in the hyperbolic plane. He took a triangle as the belt 0 and constructed the other belts around it. He calculated that $\lim\limits_{i\to\infty} \frac{V_i}{S_i} = \frac{\sqrt{(m-4)^2-4}-(m-6)}{2}$, $(m > 6)$. Horváth [3] showed for all the regular mosaics $\{p, q\}$ in the hyperbolic plane that $\lim\limits_{i\to\infty} \frac{V_i}{S_i} = \frac{\sqrt{c^2-4}-(c-2)}{2}$, where $c > 2$ and $c = (p-2)(q-2)-2$. Vermes [11], [12], [14] gave this limit for the mosaics with asymptotic polygons in the hyperbolic plane. Zeitler [16] determined for the cube mosaic $\{4, 3, 5\}$ in the 3-dimensional hyperbolic space, that $\lim\limits_{i\to\infty} \frac{V_i}{S_i} = 4\sqrt{14} - 14 \approx 0.9666$ and $\lim\limits_{i\to\infty} \frac{V_{i+1}}{V_i} = 15 + 4\sqrt{14} \approx 29.96$.

2 Construction of the dissertation

In the dissertation we give the limits $\lim\limits_{i\to\infty} \frac{V_i}{S_i}$ and $\lim\limits_{i\to\infty} \frac{V_{i+1}}{V_i}$ $(i \geq 1)$ for almost all the 3-dimensional and 4-dimensional Euclidean and hyperbolic regular mosaics (Table 1.), and we also show, that $\lim\limits_{i\to\infty} \frac{V_{i+1}}{V_i} = \lim\limits_{i\to\infty} \frac{S_{i+1}}{S_i} = \lim\limits_{i\to\infty} \frac{F_{i+1}^k}{F_i^k}$ $(i \geq 1)$. This limit can be called as *crystal-growing ratio*. If i is large enough, there are about $\lim\limits_{i\to\infty} \frac{V_{i+1}}{V_i}$ -times more domains in the belt $(i+1)$ than in the belt i.

For determining the above limits, we classify the vertices in the belts and sum them up. The numbers of the domains of the belts can be given by the help of these different types of vertices. (Since the domains are congruent for a mosaic, we can take their volumes as if they were units.) We give the number of the vertices in the belt $(i+1)$ by the help of the numbers and types of the vertices in the belt i. The basis of the calculation is, that we classify and calculate the vertices, edges, ..., k-dimensional faces of the vertex-figures of the mosaics. These vertex-figures always are the well-known regular polytopes.

Chapter 2 contains the definitions, introduces the basic theorems and lemmas in connection with the recurrence sequences by which we can simplify the calculation of the limits later. During the proofs we use some algebraic theorems ([9], [10]). The most important theorems in this chapter are:

Let $n \geq 2,\ \ i \geq 1,\ \ j = 1, 2, \ldots, n$ be integers, let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be a regular matrix and let $\mathbf{a}_1 \in \mathbb{R}^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n$. We create the vector-sequence $\{\mathbf{a}_i\}_{i=1}^{\infty}$ recursively:

$$\mathbf{a}_{i+1} = \mathbf{M}\mathbf{a}_i, \tag{1}$$

and the scalar-sequence $\{r_i\}_{i=1}^{\infty}$ with

$$r_i = \boldsymbol{\alpha}^T \mathbf{a}_i. \tag{2}$$

Theorem 2.1.1. *The sequence $\{r_i\}_{i=1}^{\infty}$ is a recurrence sequence of order (at most) n, thus*

$$r_i = \beta_1 r_{i-1} + \beta_2 r_{i-2} + \cdots + \beta_n r_{i-n}, \tag{3}$$

where $\beta_j \in I\!R$, $\beta_n \neq 0$ and $i \geq n+1$.

The *characteristic equation* of (3) (also the characteristic equation of $\mathbf{M}$) is given by $(\beta_n \neq 0)$:

$$z^n = \beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n z^0. \tag{4}$$

Moreover, let

$$z^n - \beta_1 z^{n-1} - \beta_2 z^{n-2} - \cdots - \beta_n z^0 = (z - z_1)^{m_1} \cdots (z - z_h)^{m_h}, \tag{5}$$

be the factorization of the characteristic polynomial, where $z_1, \ldots, z_h$ are the different roots $(m_1 + \cdots + m_h = n,\ 1 \leq h \leq n)$, and because of $\beta_n \neq 0$ they are not equal to zero $(z_l \neq 0,\ l = 1, \ldots, h)$ as well.

Using the fundamental theorem ([10, p.33.]) in the theory of recurrence sequences, any element of the linear recurrence sequence $\{r_i\}_{i=1}^{\infty}$ can be written explicitly:

$$r_i = g_1(i)z_1^i + g_2(i)z_2^i + \cdots + g_h(i)z_h^i, \tag{6}$$

where $g_k(i)$ $(k = 1, \ldots, h)$ are polynomials of degree less than m_k and they depend on the first elements $r_1, r_2, \ldots, r_n$ of the sequence $\{r_i\}_{i=1}^{\infty}$, on m_k and z_k $(k = 1, \ldots, h)$.

In the following let us also suppose that all the roots of the equation (4) are real, $z_k \in \mathbb{R}$, $k = 1 \ldots h \leq n$ and $r_i \neq 0$ $(i \geq 1)$. In cases of the Euclidean mosaics the roots are equal to 1, thus $z_1 = 1$, $h = 1$ and $m_1 = n$. Using (6), $r_i = g_1(i)1^i = g_1(i) \neq 0$, so $g_1(i)$ is not the constant zero polynomial. In cases of the considered hyperbolic mosaics the roots of the equation (4) are real, there are at least two different roots and there exists a root z_1 with $m_1 = 1$ for which $|z_1| > |z_k|$ and $|z_1| = z_1 > 1$ $(k = 2, \ldots, h)$.

Theorem 2.2.2. *In case of* $h = 1$, $z_1 = 1$ *and* $1 < h \leq n$, $|z_1| > |z_k| \neq 0$, $|z_1| = z_1 > 1$, $g_1 \neq 0$ $(k = 2, \ldots, h)$, *we get* $\lim\limits_{1 \leq i \to \infty} \frac{r_{i+1}}{r_i} = z_1$ *and* $\lim\limits_{1 \leq i \to \infty} \frac{r_i}{s_i} = \frac{z_1 - 1}{z_1}$ for the sequences r_i and $s_i = \sum_{j=0}^{i} r_j$ $(i \geq 1)$.

Theorem 2.2.6. *The crystal-growing ratio of a regular mosaic is the biggest real eigenvalue, whose absolute value is also maximal, of the matrix* $\mathbf{M}$ *given by the recursion of the mosaic.*

In the further chapters we give the matrices of the recursion of the mosaics.

First of all we examine the belt 1, then the belt $(i + 1)$ by induction, supposing that we have already known the belt i. By the help of the recurrence sequences first we determine the number of the vertices on the outher surface of belts, afterwards the numbers of the domains and other dimensional faces. For counting up them, we use the fact, that – being a regular mosaic – it has congruent domains and congruent vertex figures as well. Thus it is enough for us to know the environment of any vertices of the mosaics and the positions of the surrounding vertices, and their numbers of course. For counting up these, we use only geometrical methods, taking into consideration the topological properties of the vertex figures.

In chapter 3 we begins with the examination the Euclidean cube-mosaic $\{4,3,4\}$ for didactical reason. Then we examine the hyperbolic cube mosaic $\{4,3,5\}$ (although ZEITLER [16] calculated the limits yet in an other way), the 4-dimensional hypercube mosaics ($\{4,3,3,4\}$, $\{4,3,3,5\}$) and their duals ($\{5,3,4\}$, $\{5,3,3,4\}$) (NÉMETH [5], [6]). We prove the following theorem for dual mosaics.

Theorem 3.6.1. *The limits* $\lim\limits_{1\le i\to\infty}\frac{V_i}{S_i}$, $\lim\limits_{1\le i\to\infty}\frac{V_{i+1}}{V_i}$, $\lim\limits_{1\le i\to\infty}\frac{S_{i+1}}{S_i}$ *and* $\lim\limits_{1\le i\to\infty}\frac{F^k_{i+1}}{F^k_i}$ *are equal in cases of the dual mosaics.*

In chapter 4 we deal with the dodecahedron mosaics $\{5,3,4\}$ and $\{5,3,5\}$ by applying the above method.

In chapter 5 we examine the mosaics with infinite regular polyhedra $\{4,4,3\}$, $\{6,3,3\}$, $\{6,3,4\}$ and $\{6,3,5\}$ having unbounded domains (NÉMETH [7]). These mosaics can be cut onto asymptotic pyramid mosaics (VERMES [13], [14]). Their corresponding 4-dimensional analogue is $\{4,3,4,3\}$. Their duals are $\{3,4,4\}$, $\{3,3,6\}$, $\{4,3,6\}$, $\{5,3,6\}$ and $\{3,4,3,4\}$, respectively (NÉMETH [7]). We use the previous method for the computations.

In the 3-dimensional hyperbolic space there are mosaics with regular prisms, too (VERMES I. [13], [14]). In chapter 6 we apply our counting up method for these mosaics. As the number of the mosaics with regular prisms is infinite and we take them altogether in a parametric form, for the final results computer is necessary, too.

Finally, we managed to find an other method for determining the above limits for mosaics with bounded domains in the 3- and 4-dimensional spaces. The description of the method has been placed into the appendix. For determining the numbers of the domains of the different belts we divide the domains into characteristic simplices. The vertices of the characteristic simplices are the centres of the k-dimensional faces of these mosaics and we examine these vertices in algebraic way. From the belt i to the belt $(i+1)$ we can go forward step by step recursively as well, but the matrix of the recursion can be given with the parameters of Schläfli symbols of the regular mosaics, in general. We use the well-known combinatorial sieve ([8, 41. p.]). Of course, the results gained by the two basically different methods are the same. E.g. the results for the mosaic $\{4,3,5\}$ coincide with those published by ZEITLER [16]. The main theorem for the 3-dimensional honeycombs $\{p,q,r\}$ is as follows.

Definition. We denote by v_i^k the number of the centres of the k-dimensional faces in the union of the belts 1, 2, …, i. Let it be written in a matrix $\mathbf{v}_i = \begin{pmatrix} v_i^0 & v_i^1 & v_i^2 & v_i^3 \end{pmatrix}^T$.

Theorem F.1.4. $\mathbf{v}_{i+1} = \mathbf{M}\mathbf{v}_i$, $i \geq 0$, *where* $\mathbf{M} = \mathbf{G}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, *and*

$$\mathbf{G} = \begin{pmatrix} V\left(\frac{U}{4q^2} + \frac{1}{r} - \frac{p}{4}\right) - 1 & V\left(\frac{U}{8q} + \frac{1}{2r} - \frac{p}{4}\right) & V\left(\frac{U}{4pq} - \frac{1}{4}\right) & V\frac{1}{2q} \\ r\left(\frac{U}{2q} - p\right) + 2 & r\left(\frac{U}{4} - p\right) + 1 & r\left(\frac{U}{2p} - 1\right) & r \\ \frac{U}{q} - p & \frac{U}{2} - p & \frac{U}{p} - 1 & 2 \\ \frac{U}{2q} & \frac{U}{4} & \frac{U}{2p} & 1 \end{pmatrix},$$

where $U = \frac{4}{\frac{1}{p}+\frac{1}{q}-\frac{1}{2}}$ and $V = \frac{4}{\frac{1}{r}+\frac{1}{q}-\frac{1}{2}}$.

This method seems to be fruitful for any d-dimensional honeycombs with bounded and unbounded domains. It probably gives the numbers of k-dimensional faces ($k \leq d$) in the belts of the mosaics.

3 Summary of the results

The tables below show the results for the mosaics examined in the dissertation. Until now, except from the Euclidean mosaics, the limits of the regular mosaic $\{4,3,5\}$ (Zeitler [16]) were known. $\{p,q,r\}_g$ denotes the mosaic with asymptotic regular pyramids gained by dividing the infinite regular polyhedron $\{p,q,r\}$ (chapter 5). The Table 1 summarises the limits for the honeycombs. Tables 2 and 3 present them (only for the cases $p \leq 10$) for the regular prism mosaics (chapter 6).

mosaic	$\lim\limits_{i\to\infty} \frac{V_{i+1}}{V_i}$	$\lim\limits_{i\to\infty} \frac{V_i}{S_i}$
$\{4,3,4\}$	1	0
$\{4,3,5\}$, $\{5,3,4\}$	29.96663	0.96663
$\{5,3,5\}$	166.99401	0.99401
$\{3,5,3\}$	46.97871	0.97871
$\{4,4,3\}_g$, $\{4,4,3\}$, $\{3,4,4\}$	10	0.9
$\{6,3,3\}_g$, $\{6,3,3\}$, $\{3,3,6\}$	6	0.83333
$\{6,3,4\}_g$, $\{6,3,4\}$, $\{4,3,6\}$	21	0.95238
$\{6,3,5\}_g$, $\{6,3,5\}$, $\{5,3,6\}$	76	0.98684
$\{4,3,3,4\}$	1	0
$\{3,3,3,5\}$, $\{5,3,3,3\}$	84.03807	0.98810
$\{4,3,3,5\}$, $\{5,3,3,4\}$	2381.82771	0.99958
$\{5,3,3,5\}$	319483.2496	0.999997
$\{4,3,4,3\}_g$, $\{4,3,4,3\}$, $\{3,4,3,4\}$	141.728617	0.992971

Table 1. *Summarizing table for honeycombs.*

$\lim_{i\to\infty} \frac{V_{i+1}}{V_i}$	$\{p\|3,3\}$	$\{p\|3,4\}$	$\{p\|3,5\}$	$\{p\|4,3\}$	$\{p\|5,3\}$
$p=3$	–	–	–	–	–
$p=4$	–	–	–	–	229.904
$p=5$	–	–	–	91.1299	306.746
$p=6$	–	–	–	116.403	384.746
$p=7$	35.2892	117.827	413.707	141.309	463.059
$p=8$	42.1757	138.482	484.300	166.994	541.449
$p=9$	48.8284	158.870	554.594	190.533	619.837
$p=10$	55.3460	179.081	624.665	215.970	698.198

Table 2. $\lim_{i\to\infty} \frac{V_{i+1}}{V_i}$ *for mosaics with regular prisms.*

$\lim_{i\to\infty} \frac{V_i}{S_i}$	$\{p\|3,3\}$	$\{p\|3,4\}$	$\{p\|3,5\}$	$\{p\|4,3\}$	$\{p\|5,3\}$
$p=3$	–	–	–	–	–
$p=4$	–	–	–	–	0.995650
$p=5$	–	–	–	0.989027	0.996740
$p=6$	–	–	–	0.991409	0.997401
$p=7$	0.971663	0.991514	0.997583	0.992923	0.997840
$p=8$	0.976290	0.992779	0.997935	0.993976	0.998153
$p=9$	0.979520	0.993706	0.998197	0.994752	0.998387
$p=10$	0.981929	0.994416	0.998399	0.995348	0.998567

Table 3. $\lim_{i\to\infty} \frac{V_i}{S_i}$ *for mosaics with regular prisms.*

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