# Asymptotic behavior of random graphs evolving in time

PhD thesis

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Ezen értekezés bírálatai és a védésről készült jegyzőkönyv a későbbiekben a Budapesti Műszaki és Gazdaságtudományi Egyetem Természettudományi Karának Dékáni Hivatalában elérhető.

Alulírott Ráth Balázs hozzájárulok a doktori értekezésem interneten történő korlátozás nélküli nyilvánosságra hozatalához.

Alulírott Ráth Balázs kijelentem, hogy ezt a doktori értekezést magam készítettem, és abban csak a megadott forrásokat használtam fel. Minden olyan részt, amelyet szó szerint, vagy azonos tartalomban, de átfogalmazva más forrásból átvettem, egyértelműen, a forrás megadásával megjelöltem.

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Ráth Balázs

# Contents

# 1 Introduction

In this PhD thesis I investigate the time evolution of certain graph-valued Markov chains: the vertex set and edge set of the graph changes over time: with probabilities that depend on the present structure of the graph we add/delete edges/vertices from the graph. If we consider a sequence of Markov chains with a sequence of initial graphs in which the number of vertices $n$ goes to infinity, but assume that some family of statistics of the initial graphs converge as $n \to \infty$, then with the appropriate scaling of time we are able to translate the microscopic transition rules of the Markov chain into differential equations governing the time evolution of the limiting values of the family of statistics that we consider. By the analysis of the solutions of these differential equations we are able to describe the large-scale evolution and properties of random graphs.

We consider two different families of models in this thesis:

The **mean field forest fire model** and the **mean field frozen percolation model** are two closely related random graph models: in both cases we modify the dynamical Erdős-Rényi random graph model so that large connected components are destroyed, which creates a competition between coagulation and fragmentation. The most important property of these models is that they exhibit *self-organized criticality*, which can be proved by the analysis of the corresponding modifications of the *Smoluchowski coagulation equations*.

The **edge reconnecting model** is a dense multigraph evolving in discrete timesteps: at each step we reconnect one endpoint of a uniformly chosen edge according to the rules of linear preferential attachment. We investigate the model using the notion of *dense graph limits* and give a full description of the time evolution of the limit objects. The number of parallel edges and the degrees evolve on different timescales and because of this the model exhibits *aging*.

This thesis is divided into three chapters in which we provide the analysis of these three random graph models:

The topic of Chapter 2 is the mean field forest fire model and is based on [36].

The topic of Chapter 3 is the mean field frozen percolation and is based on [33].

The topic of Chapter 4 is the edge reconnecting model and is based on [34], joint work with László Szakács.

The rest of this Introduction is also divided into three sections in which we give a short résumé of the context, description, results and relevance of these three models. More detailed introductions to the three topics are given in the first sections of the respective chapters.

## 1.1 Mean field forest fire model

### Context:

The dynamical Bernoulli bond percolation model is a random graph evolving in continuous time according to the following Markovian dynamics:

Fix an infinite homogenous graph (e.g. the lattice $\mathbb{Z}^d$, $d \geq 2$). The edges of the graph can be either "open" or "closed". We start the process from the state where all edges are closed. Edges are independently switched from closed to open with rate 1. As the time parameter $t$ increases, the model undergoes *phase transition*: there is a particular value $t_c \in (0, +\infty)$ (the critical time) such that if we define

$$v_k(t) := \mathbf{P}\big(\text{ the size of the connected open component of the origin is } k \text{ at time } t\ \big)$$

then

- for $t < t_c$ the model is *subcritical*: $\sum_{k=1}^{\infty} v_k(t) = 1$ and $v_k(t)$ decays exponentially in $k$.
- for $t > t_c$ the model is *supercritical*: $\sum_{k=1}^{\infty} v_k(t) = 1 - \theta(t)$ where $\theta(t) > 0$ is the probability that the origin is contained in an infinite component. $v_k(t)$ decays exponentially in $k$.
- for $t = t_c$ the model is *critical*: $\theta(t_c) = 0$ and $v_k(t)$ decays polynomially in $k$.

The mean field version of dynamical bond percolation is the dynamical Erdős-Rényi random graph model: the edges of the complete graph on $n$ vertices are turned from closed to open with rate $\frac{1}{n}$. If we define

$$v_k^n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[\text{ the size of the connected open component of vertex } i \text{ is } k \text{ at time } t\ ] \tag{1}$$

then $v_k^n(t) \to v_k(t)$ in probability as $n \to \infty$ (where $(v_k(t))_{k=1}^{\infty}$ is the solution of the *Smoluchowski coagulation equations* with multiplicative kernel)) and with $t_c = 1$ the phase transition can be described exactly the same way as before (the only difference is that $\theta(t)$ is the density of the *giant component*). The decay rate of component size densities at the critical time is $v_k(t_c) \asymp k^{-3/2}$ in the Erdős-Rényi model.

The critical forest fire model on the lattice $\mathbb{Z}^d$ might be informally defined in the following way: edges are independently switched from closed to open with rate 1, but if an infinite open component appears, we switch its edges to closed instantaneously. It is conjectured by physicists (see [19]) that the forest fire model exhibits self-organized criticality (S.O.C.): for all $t \geq t_c$ the graph is critical: $\theta(t_c) = 0$ and $v_k(t)$ decays polynomially in $k$.

There are few mathematical results describing forest fire models on $\mathbb{Z}^d$: in fact a rigorously defined graph-valued stochastic process satisfying the informal definition of the critical forest fire model has not been constructed yet. In [20] and [21] M. Dürre proves the existence and uniqueness of the subcritical forest fire model satisfying the following informal definition: edges are independently switched from closed to open with rate 1 and all the edges of open connected components of size $k$ are switched to closed with rate $\lambda k$ (a lightning strikes the component with a rate proportional to its size). Heuristically $\lambda \to 0$ yields the critical forest fire model.

## The model:

We modify the dynamical Erdős-Rényi model in a similar fashion to obtain the mean field forest fire model. We start with the empty graph on $n$ vertices and between each pair of unconnected vertices an edge appears with rate $1/n$, moreover each vertex is exposed to a Poisson process of lightnings with rate $\lambda(n)$. If a lightning strikes a vertex, then fire spreads along the edges instantaneously and burns them: connected components of size $k$ burn down with rate $\lambda(n) \cdot k$ and the component is replaced with $k$ isolated vertices. The total number of vertices remains $n$.

We investigate the model in the *critical regime* $\frac{1}{n} \ll \lambda(n) \ll 1$ as $n \to \infty$. Since $\lambda(n) \ll 1$, the fire doesn't do much harm to small components, but by $\frac{1}{n} \ll \lambda(n)$ giant components of size comparable to $n$ burn immediately.

Our main result is the following: if we define $v_k^n(t)$ by (1) then $v_k^n(t) \to v_k(t)$ in probability as $n \to \infty$, where $(v_k(t))_{k=1}^{\infty}$ is the unique solution of the following infinite system of differential equations:

$$\forall\, k \geq 2: \quad \dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k} v_l(t) v_{k-l}(t) - k v_k(t), \qquad v_k(0) = \mathbb{1}[k=1], \qquad \sum_{k=1}^{\infty} v_k(t) \equiv 1 \tag{2}$$

It is worth emphasizing that the limiting functions $(v_k(t))_{k=1}^{\infty}$ do not depend on the exact decay rate of $\lambda(n)$ as long as $\frac{1}{n} \ll \lambda(n) \ll 1$. For $t \leq 1$ the values of $v_k(t)$ are the same as the ones obtained from the Erdős-Rényi model without forest fires, but

$$\forall \ t \geq 1: \quad \theta(t) \equiv 0, \qquad v_k(t) \asymp k^{-3/2},$$

thus the mean field forest fire model indeed exhibits self-organized criticality.

We might start the mean field forest fire model with a non-empty initial graph which corresponds to solving (2) with a general initial condition. The same methods of proof work and the model exhibits S.O.C.

## About the proof:

The system of differential equations (2) is a modification of the Smoluchowski coagulation equations, but the boundary condition $\sum_{k=1}^{\infty} v_k(t) \equiv 1$ makes this system of ODE-s genuinely infinite and the proof of well-posedness highly non-trivial:

If we define the Laplace transform (generating function) of $(v_k(t))_{k=1}^{\infty}$ by $V(t,x) = \sum_{k=1}^{\infty} v_k(t) e^{-kx}$ then (2) is transformed into the following *Burgers control problem*:

Find a control function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$\partial_t V(t,x) = -\frac{1}{2}\partial_x V^2(t,x) + \partial_x V(t,x) + \varphi(t) e^{-x}, \qquad V(0,x) = e^{-x}, \qquad V(t,0) \equiv 1. \tag{3}$$

In order to show that the control problem (3) has a unique solution we use methods from the theory of first order nonlinear PDE (the method of characteristics). The "critical" property $v_k(t) \asymp k^{-3/2}$ is transformed into $|V(t,x)-1| \asymp \sqrt{x}$ by Tauberian theory and proving that all solutions of (3) must be "critical" for all $t \geq 1$ is only an intermediate step in the proof of its well-posedness.

In order to prove $v_k^n(t) \to v_k(t)$ we need to show that the functions $(v_k^n(t))_{k=1}^n$ "asymptotically solve" (2) as $n \to \infty$. It is again the boundary condition $\sum_{k=1}^\infty v_k(t) \equiv 1$ that is the most technical to prove: $\frac{1}{n} \ll \lambda(n)$ only guarantees the destruction of components of size comparable to $n$ and one has to work hard proving $\lim_{N\to\infty} \lim_{n\to\infty} \sum_{k\geq N} v_k^n(t) = 0$.

## 1.2 Mean field frozen percolation

### Context:

The frozen percolation process on a binary tree was defined in [3]: it is a modification of dynamical percolation which makes the following informal description mathematically rigorous: if an infinite open cluster appears, we remove it (including vertices). Started from the empty configuration this model exhibits not only S.O.C., but self-similarity: for all $t \geq t_c$ a typical finite cluster has the distribution of a critical percolation cluster.

### The model:

The definition of the mean field frozen percolation model is the same as that of the mean field forest fire model except that in the frozen percolation model we remove the vertices as well as the edges of burnt connected components. The two models are in the same universality class: we have $v_k^n(t) \to v_k(t)$ in probability as $n \to \infty$ where $(v_k(t))_{k=1}^\infty$ solves Stockmayer's coagulation equation:

$$\forall\, k \geq 1 \quad \dot{v}_k(t) = \frac{k}{2} \sum_{l=1}^{k} v_l(t) v_{k-l}(t) - k v_k(t) \sum_{l=1}^{\infty} v_l(t). \tag{4}$$

The solution of (4) with initial condition $v_k(0) = \mathbb{1}[k=1]$ is well-known (see [39]) and has a similar self-similarity property as the frozen percolation process on the binary tree.

We might start the frozen percolation with a non-empty initial graph which corresponds to solving (4) with a general initial condition. The solutions are still explicit and the model exhibits S.O.C.: for all $t \geq t_c$ we have $v_k(t) \asymp k^{-3/2}$.

In the frozen percolation model on the binary tree, components are burnt/frozen/removed when their size becomes infinite. The question may arise:

*What is the typical size of a frozen component in the mean field frozen percolation model?*

We investigate the behavior of the frozen percolation model on the "boundary of the critical regime" to prove the following results:

- If the lightning rate is $\lambda(n) \equiv \lambda_*$ then the frozen percolation model is subcritical and $v_k^n(t) \to v_k(t)$ where $(v_k(t))_{k=1}^\infty$ is the solution of the $\lambda_*$-*subcritical frozen percolation equations* (an infinite system of ODE's similar to (4)). As $\lambda_* \to 0$ the solutions of the subcritical equation converge to the solutions of (4) and the typical size of a frozen component is of order $\lambda_*^{-2}$ when $\lambda_* \ll 1$.

- If the lightning rate is $\lambda(n) = \frac{\lambda^*}{n}$ then giant components are born and destroyed from time to time and the $n \to \infty$ limit of $v_k^n(t)$ is a *randomly controlled solution of the Smoluchowski coagulation equations* alternating between subcritical and supercritical phase. As $\lambda^* \to +\infty$ the solutions of the randomly controlled alternating equation converge to the solutions of (4) and the typical size of a frozen component is of order $n \cdot (\lambda^*)^{-1/2}$ when $1 \ll \lambda^*$.

## 1.3 Edge reconnecting model

### Context:

In recent years a limiting theory has been developed for dense graph sequences (in dense graphs the number of edges is comparable with $|V(G)|^2$). Roughly speaking, a sequence $(G_n)_{n=1}^\infty$ of simple graphs converges if for any fixed *testgraph* $F$, the density of copies of $F$ found in $G_n$ (called the *homomorphism density*) converges as $n \to \infty$. It was shown in [30] that the limit object can be represented by a symmetric measurable function $W : [0,1]^2 \to [0,1]$. Such functions are called *graphons*.

A connection between the theory of dense graph limits and that of infinite exchangeable arrays of random variables was first observed in [18]: if we label the vertices of our graphs with a uniformly chosen permutation then a graph sequence converges if and only if the sequence of the randomly labeled adjacency matrices converge in distribution to an infinite adjacency matrix.

A natural generalization of the theory of dense graph limits to multigraphs (graphs with loop and multiple edges) is given in [28] (joint work with István Kolossváry). We named the limit objects *multigraphons*.

### The model:

The edge reconnecting model is random multigraph evolving in discrete timesteps. At $T = 0$ we start from a multigraph $\mathcal{G}_n(0)$ with $n$ vertices and $m \approx \rho \frac{n^2}{2}$ edges. Given $\mathcal{G}_n(T)$ we obtain $\mathcal{G}_n(T+1)$ by choosing an edge of $\mathcal{G}_n(T)$ uniformly and reconnect one of its endpoints: we choose the new endvertex using linear preferential attachment: the probability that we choose $v$ is $\frac{d(v)+\kappa}{2m+\kappa n}$ where $d(v)$ is the degree of $v$ in $\mathcal{G}_n(T)$.

We characterize the time evolution of the edge reconnecting model using the notion of dense graph limits, thus solving a research problem posed by László Lovász. We consider a sequence of edge reconnecting models for which $\mathcal{G}_n(0) \to W$ as $n \to \infty$ (where $W$ is a multigraphon). The time evolution of the limiting multigraphons can be summarized in two theorems: if we fix $t \in \mathbb{R}_+$, then

$$\mathcal{G}_n(t \cdot n^2) \xrightarrow{\text{d}} \breve{W}_t \qquad \text{and} \qquad \mathcal{G}_n(t \cdot n^3) \xrightarrow{\text{d}} \tilde{W}_t$$

where $\breve{W}_t$ and $\tilde{W}_t$ are determined by $W$, $t$, $\rho$ and $\kappa$. These two theorems give the full characterization of the evolution of the graph limits, since

$$\lim_{t\to 0_+} \breve{W}_t = W, \qquad \lim_{t\to\infty} \breve{W}_t = \lim_{t\to 0_+} \tilde{W}_t, \qquad \lim_{t\to\infty} \tilde{W}_t = W_\infty$$

where $\mathcal{G}_n(\infty) \xrightarrow{d} W_\infty$ is the graph limit of the stationary states.

## About the proofs:

The basic idea is to relate the time evolution of the edge reconnecting model to certain continuous-time stochastic processes using an appropriate rescaling of time:

- If we fix a vertex $v \in V(\mathcal{G}_n(0))$ and denote by $d(T, v)$ the degree of $v$ in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{R}_+$-valued continuous-time stochastic process $\frac{1}{n} d(n^3 \cdot t, v)$ "almost looks like" that of a Cox-Ingersoll-Ross process (a diffusion process that is commonly used in financial mathemathics to model the evolution of interest rates). This fact is rigorously proved using the theory of stochastic differential equations and is used in the proof of $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\text{d}} \tilde{W}_t$.
- If we fix two vertices $v, w \in V(\mathcal{G}_n(0))$ and denote by $E(T, v, w)$ the number of parallel/loop edges connecting $v$ and $w$ in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{N}$-valued continuous-time stochastic process $E(n^2 \cdot t, v, w)$ "almost looks like" that of the queue length of an M/M/$\infty$-queue. This fact is rigorously proved using a coupling argument and is used in the proof of $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\text{d}} \breve{W}_t$.

## Aging:

The most interesting property of the edge reconnecting model is the separation of *two different timescales*: the degrees of the vertices only change significantly on the $n^3$ timescale, whereas the number of parallel (or loop) edges between two vertices evolves on the much faster $n^2$ timescale. The arrival rate of the M/M/$\infty$-queue describing the evolution of $E(n^2 \cdot t, v, w)$ depends on the current degrees of $v$ and $w$, but since the degrees evolve on the much slower $n^3$ timescale, they may be treated as constant background parameters on the $n^2$ timescale. If we fix $k$ vertices in the model and denote by $\mathcal{G}_n^k(T)$ the random subgraph of $\mathcal{G}_n(T)$ spanned by these vertices then the $\mathcal{M}_k$-valued stochastic process $\mathcal{G}_n^k(n^3 \cdot t + n^2 \cdot s)$ looks stationary in the time variable $s \in \mathbb{R}$ if $t \in (0, +\infty)$ is fixed and $1 \ll n$, but different values of $t$ yield distinct pseudo-stationary distributions since $n^3 \cdot (t_2 - t_1)$ steps are enough for the background variables (degrees) to significantly change.

This phenomenon is called *aging* in statistical physics, see [4] and [9].

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# 2 Mean field forest fire model

## 2.1 Introduction

### 2.1.1 Context

In conventional models of equilibrium statistical physics, such as Bernoulli percolation, random cluster models, the Ising model or the Heisenberg model there is always a parameter which controls the character of the equilibrium Gibbs measure: in percolation and random cluster-type models this is the density of open sites/edges, in the Ising or Heisenberg models the inverse temperature. Typically the following happens: tuning the control parameter at a particular value (the critical density or the critical inverse temperature) the system exhibits critical behavior in the thermodynamical limit, manifesting e.g. in power law rather than exponential decay of the upper tail of the distribution of the size of connected clusters. Off this particular critical value of the control parameter these distributions decay exponentially. We emphasize here that the critical behavior is observed only at this particular critical value of the control parameter.

As opposed to this, in some dynamically defined models of interacting microscopic units one expects the following robust manifestation of criticality: In some systems dynamics defined naturally in terms of local interactions some effects can propagate instantaneously through macroscopic distances in the system. This behavior may have dramatic effects on the global behavior, driving the system to a permanent critical state. The point is that without tuning finely some parameter of the interaction the dynamics drives the system to criticality. This kind of behavior is called *self-organized criticality (SOC)* in the physics literature. The two best known examples are the sandpile models where so called avalanches spread over macroscopic distances instantaneously, and the forest fire models where beside the Poissonian flow of switching sites/edges from "empty" to "occupied" state (i.e. trees being grown), at some instants connected clusters of occupied sites/edges (forests of trees) are turned from "occupied" to "empty" state instantaneously (i.e. forests hit by lightnings are burnt down on a much faster time scale than the growth of trees). These models and these phenomena prove to be difficult to analyze mathematically rigorously due to the following two facts:

- There are always two competing components of the dynamics (in the forest fire models: growing trees and burning down forests) causing lack of any kind of monotonicity of the models.
- Long range effects due to instantaneous propagation of short range interactions are very difficult to be controlled.

Regarding forest fire models there are very few mathematically rigorous results describing SOC. The best known and most studied model of forest fires is the so-called Drossel-Schwabl model. For the original formulation see [19], or the more recent survey [37]. We formulate here a related variant.

Let $\Lambda_n := \mathbb{Z}^d \cap [-n, n]^d$. The state space of the model of size $n$ is $\Omega_n := \{0, 1\}^{\Lambda_n}$: sites of $\Lambda_n$ can be occupied by a tree (1) or empty (0). The dynamics consists of two competing mechanisms:

(A) Empty (0) sites turn occupied (1) with rate one, independently of whatever else happens in the system.

(B) Sites get hit by "lightnings" with rate $\lambda(n)$, independently of whatever else happens in the system. When site is hit by lightning its whole connected cluster of occupied sites turns instantaneously from "occupied" (1) to "empty" (0) state. (That is: when a tree is hit by lightning the whole forest to which it belongs burns down instantaneously.)

The dynamics goes on indefinitely.

As long as $n$ is kept fixed the mechanism $A + B$ defines a decent finite state Markov process – though a rather complicated one. The main question is: what happens in the thermodynamic limit, when $n \to \infty$, $\Lambda_n \nearrow \mathbb{Z}^d$? Can one specify a dynamics on the state space $\Omega_\infty := \{0, 1\}^{\mathbb{Z}^d}$ which could be identified with the infinite volume limit of the systems defined above?

In order to make some guesses, one has first to specify the lightning rate $\lambda(n)$. Intuitively one expects four regimes of the rate $\lambda(n)$ with essentially different asymptotic behavior of the system in the limit of infinite volume:

I. If $\lambda(n) \ll |\Lambda_n|^{-1}$ then the effect of lightning is simply not felt in the thermodynamic limit: in macroscopic time intervals of any fixed length no lightning will hit the entire system. Thus, in this regime the system will simply be the dynamical formulation of Bernoulli percolation.

II. If $\lambda(n) = |\Lambda_n|^{-1}\lambda$ with some fixed $\lambda \in (0, \infty)$ then one expects in the thermodynamic limit the following dynamics (described in plain, non-technical terms). The system evolves as dynamical site percolation, with independent Poisson evolutions on sites, and with rate $\lambda\theta(t)$, where $\theta(t)$ is the density of the (unique) infinite cluster, the sites of this (unique) infinite cluster are turned from occupied to empty. After this forest fire the system keeps on evolving like dynamical percolation until a new infinite component is born, and the dynamics goes on indefinitely.

III. If $|\Lambda_n|^{-1} \ll \lambda(n) \ll 1$ then in the infinite volume limit - if it makes any sense - something really interesting must happen: The lightning rate is too small to hit finite clusters within any finite horizon. But it is too large to let the infinite percolating cluster to be born. One can expect (somewhat naively) that in this regime in the thermodynamic limit a dynamics will be defined on $\Omega_\infty$ in which *in plain words* the following happens:

- empty (0) sites turn occupied (1) with rate one, independently of whatever else happens in the system;

- when the *incipient infinite percolating cluster* is about to be born, it is switched from "occupied" (1) to "empty" (0) state;
- the dynamics goes on indefinitely.

In this way this presumed infinitely extended dynamics would stick to a permanent critical state when the infinite incipient critical cluster is always about to be born, but not let to grow beyond criticality.

IV. If $\lambda(n) = \lambda \in (0, \infty)$ then lightning will hit regularly even small clusters and thus, one may expect that - if the infinitely extended dynamics is well defined - the system will stay subcritical indefinitely.

There is no problem with the mathematically rigorous definition of the infinitely extended dynamics in regimes I. and II. But these plain descriptions don't necessarily make mathematical sense and it is not at all clear that such infinitely extended critical forest fire models can at all be defined in a mathematically satisfactory way.

In our understanding, the most interesting open questions are the existence and characterization of the infinitely extended dynamics in regime III. and/or the $\lambda \to \infty$ limit in regime II. and/or the $\lambda \to 0$ limit in regime IV., after the thermodynamic limit.

There are however some deep results regarding these (or some other related) models of forest fires, though clarification of the above questions seems to be far out of reach at present.

Here follows a (necessarily incomplete) list of some important results related to these questions:

– M. Dürre proves existence of infinitely extended forest fire dynamics in a related model in the subcritical regime IV. , [20]. In a companion paper he also proves that under some regularity conditions assumed the dynamics is uniquely defined, [21].

– J. van den Berg and R. Brouwer, respectively R. Brouwer consider the so called *self-destructive percolation* model, which is very closely related to what we called regime II. above. They prove various deep technical results and formulate some intriguing conjectures related to the $\lambda \to \infty$ limit in regime II. (of the already infinitely extended dynamics), see [5], [6], [14]

– J. van den Berg and A. Járai analyze the $\lambda \to 0$ asymptotics of the (infinitely extended) model in regime IV. in dimension 1, [7].

– J. van den Berg and B. Tóth consider an *inhomogeneous* one dimensional model which indeed exhibits SOC, see [8]. (In one dimensional space-homogeneous models of course there is no critical behavior)

### 2.1.2 The model

We investigate a modification of the dynamical formulation of the Erdős-Rényi random graph model, adding "forest fires" caused by "lightning" to the conventional Erdős-Rényi coagulation mechanism. Actually our model will be a particular coagulation-fragmentation dynamics exhibiting robust self-organized criticality.

Let $\mathcal{S}_n := \{1, 2, \ldots, n\}$ and $\mathcal{B}_n := \{(i,j) = (j,i) : i, j \in \mathcal{S}_n, i \neq j\}$ be the set of vertices, respectively, unoriented edges of the complete graph $\mathcal{K}_n$. We define a dynamical random graph model as follows. The state space of our Markov process is $\{0,1\}^{\mathcal{B}_n}$.

Edges $(i,j)$ of $\mathcal{K}_n$ will be called occupied or empty according whether $\omega(i,j) = 1$ or $\omega(i,j) = 0$. As usual, we call clusters the maximal subsets connected by occupied edges.

Assume that initially, at time $t = 0$, all edges are empty. The dynamics consists of the following

(A) Empty edges turn occupied with rate $1/n$, independently of whatever else happens in the system.

(B) Sites of $\mathcal{K}_n$ get hit by lightnings with rate $\lambda(n)$, independently of whatever else happens in the system. When a site is hit by lightning, all edges which belong to its connected occupied cluster turn instantaneously empty.

In this way a random graph dynamics is defined. The coagulation mechanism (A) alone defines the well understood Erdős-Rényi random graph model. For basic facts and refined details of the Erdős-Rényi random graph problem see [22], [12], [26]. As we shall see soon, adding the fragmentation mechanism (B) may cause essential changes in the behavior of the system.

We are interested of course in the asymptotic behavior of the system when $n \to \infty$. In order to formulate our problem first have to introduce the proper spaces on which our processes are defined.

We denote

$$\mathcal{V} := \big\{ \mathbf{v} = \big(v_k\big)_{k\in\mathbb{N}} \,:\, v_k \geq 0, \quad \sum_{k\in\mathbb{N}} v_k \leq 1 \big\}, \quad \theta(\mathbf{v}) := 1 - \sum_{k\in\mathbb{N}} v_k, \tag{5}$$

$$\mathcal{V}_1 := \big\{ \mathbf{v} \in \mathcal{V} \,:\, \theta(\mathbf{v}) = 0 \big\}. \tag{6}$$

We endow $\mathcal{V}$ with the (weak) topology of component-wise convergence. We may interpret $\theta$ as the density of the giant component.

A map $[0,\infty) \ni t \mapsto \mathbf{v}(t) \in \mathcal{V}$ which is component-wise of bounded variation on compact intervals of time and continuous from the left in $[0,\infty)$, will be called *a forest fire evolution (FFE).* If $\mathbf{v}(t) \in \mathcal{V}_1$ for all $t \in [0,\infty)$ we call the FFE *conservative.* Denote the space of FFE-s and conservative FFE-s by $\mathcal{E}$, respectively, $\mathcal{E}_1$. The space $\mathcal{E}$ is endowed with the topology of component-wise weak convergence of the signed measures corresponding to the functions $v_k(\cdot)$ on compact intervals of time. This topology is metrizable and the space $\mathcal{E}$ endowed with this topology is complete and separable.

Now, we define the *cluster size distribution* in our random graph process as follows

$$v_{n,k}(t) := n^{-1}\#\{j \in \mathcal{S}_n : j \text{ belongs to a cluster of size } k \text{ at time } t\} =: n^{-1}V_{n,k}(t), \quad (7)$$

$$\mathbf{v}_n(t) := \big(v_{n,k}(t)\big)_{k\in\mathbb{N}}. \quad (8)$$

This means that given the state of the system at time $t$ the sequence $\mathbf{v}_n(t)$ is the cluster size distribution of a uniformly selected site from $\mathcal{S}_n$. We refer to $v_{n,k}(t)$ as the mass contained in components of size $k$ at time $t$.

Clearly, the random trajectory $t \mapsto \mathbf{v}_n(t)$ is a (conservative) FFE. We consider the left-continuous version of $t \mapsto \mathbf{v}_n(t)$ instead of the traditional c.à.d.l.à.g., for technical reasons discussed in Subsection 2.2.1.

We investigate the asymptotics of this process, as $n \to \infty$.

It is well known (see e.g. [15], [16], [2]) that in the Erdős-Rényi case – that is: if $\lambda(n) = 0$

$$\mathbf{v}_n(\cdot) \xrightarrow{\mathbf{P}} \mathbf{v}(\cdot) = \big(v_k(\cdot)\big)_{k\in\mathbb{N}} \quad \text{as} \quad n \to \infty, \quad (9)$$

where the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of ODE-s

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t), \qquad k \geq 1, \quad (10)$$

with initial conditions

$$v_k(0) = \delta_{k,1}. \quad (11)$$

The infinite system of ODE-s (10) are the *Smoluchowski coagulation equations*, the initial conditions (11) are usually called *monodisperse.*

A short and *non-rigorous* explanation of (10) is the following: assume a new edge appears at time $t$ in the graph. We might assume that the endvertices of the edge are chosen independently and uniformly. The probability that the left endvertex falls in a component of size $k_1$ and the right endvertex falls in a component of size $k_2$ is $v_{k_1}(t) \cdot v_{k_2}(t)$. Thus the probability that two components whose size adds up to $k$ merges to create a component of size $k$ when we draw the new edge is $\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t)$ and the expected number of components of size $k$ that disappear because they are connected to other components by the new edge is $2v_k(t)$. If we want to calculate the expected change of the number of components of size $k$, we have to add up these two terms with the appropriate signs. What we get is almost (10): the factor $k$ in appears because the mass of vertices contained in components of size $k$ changes by $\frac{k}{n}$, the factor $\frac{1}{2}$ appears because the probability that a new edge appears between $t$ and $t + \mathrm{d}t$ is $\frac{n}{2}\mathrm{d}t$. The precise version of this argument is Proposition 2.1 of Subsection 2.2.2.

The system (10) is actually not very scary: it can be solved one-by-one for $k = 1, 2, \ldots$ in turn. For the initial conditions (11) the solution is known explicitly:

$$v_k(t) = \frac{k^{k-1}}{k!}e^{-kt}t^{k-1}. \quad (12)$$

$(v_k(t))_{k=1}^{\infty} \in \mathcal{V}$ is a (possibly defected) probability distribution called the Borel distribution: in a Galton-Watson branching process with offspring distribution POI($t$) the resulting random tree has $k$ vertices with probability $v_k(t)$. Thus the branching process is subcritical, critical and supercritical for $t < 1$, $t = 1$ and $t > 1$, respectively.

For general initial conditions $v_k(0)$ satisfying

$$\sum_{k=1}^{\infty} v_k(0) = 1, \qquad \sum_{k=1}^{\infty} k^2 v_k(0) < \infty,$$

the qualitative behavior of the solution of (10) is similar: Define the *gelation time*

$$T_{\text{gel}} := \Big(\sum_{k=1}^{\infty} k v_k(0)\Big)^{-1} \tag{13}$$

- For $0 \leq t < T_{\text{gel}}$ the system is subcritical: $\theta(\mathbf{v}(t)) = 0$ and, $k \mapsto v_k(t)$ decay exponentially with $k$.
- For $T_{\text{gel}} < t < \infty$ the system is supercritical: $\theta(\mathbf{v}(t)) > 0$ and $k \mapsto v_k(t)$ decay exponentially with $k$. Further on: $t \mapsto \theta(\mathbf{v}(t))$ is smooth and strictly increasing with $\lim_{t\to\infty} \theta(\mathbf{v}(t)) = \sum_{k=1}^{\infty} v_k(0)$.
- Finally, at $t = T_{\text{gel}}$ the system is critical: $\theta(\mathbf{v}(T_{\text{gel}})) = 0$ and

$$\sum_{l=k}^{\infty} v_l(T_{\text{gel}}) \asymp k^{-1/2} \quad \text{as} \quad k \to \infty. \tag{14}$$

Our aim is to understand in similar terms the asymptotic behavior of the system when, beside the Erdős-Rényi coagulation mechanism, the fragmentation due to forest fires also takes place.

Similarly to the Drossel-Schwabl case presented in subsection 2.1.1 we have four regimes of the lightning rate $\lambda(n)$, in which the asymptotic behavior is different:

| | |
|---|---|
| I.: | $\lambda(n) \ll n^{-1}$, |
| II.: | $\lambda(n) = n^{-1}\lambda, \qquad \lambda \in (0,\infty)$, |
| III.: | $n^{-1} \ll \lambda(n) \ll 1$, |
| IV.: | $\lambda(n) = \lambda \in (0,\infty)$. |

The $n \to \infty$ asymptotics of the processes $t \mapsto \mathbf{v}_n(t)$ in the four regimes is summarized as follows:

I. The effect of lightnings is simply not felt in the $n \to \infty$ limit. In this regime the system will be the dynamical formulation of the Erdős-Rényi random graph model, the asymptotic description presented in the previous paragraph is valid.

II. In the $n \to \infty$ limit the sequence of processes $t \mapsto \mathbf{v}_n(t)$ converges weakly (in distribution) in the topology of the space $\mathcal{E}$ to a process $t \mapsto \mathbf{v}(t)$ described as follows: The process $t \mapsto \mathbf{v}(t)$ evolves deterministically, driven by the Smoluchovski equations (10) (exactly as in the limit of the dynamical Erdős-Rényi model) with the following Markovian random jumps added to the dynamics:

$$\mathbf{P}\big(\mathbf{v}(t+\mathrm{d}t) = J\mathbf{v} \,\big|\, \mathbf{v}(t) = \mathbf{v}\big) = \lambda\theta(\mathbf{v})\mathrm{d}t + o(\mathrm{d}t) \tag{15}$$

$$\text{where} \quad J : \mathcal{V} \to \mathcal{V}, \quad (J\mathbf{v})_k = v_k + \delta_{k,1}\theta(\mathbf{v}). \tag{16}$$

In plain words: with rate $\lambda\theta(\mathbf{v}(t))$ the amount of mass $\theta(\mathbf{v}(t))$ contained in the gel (i.e. the unique giant component) is instantaneously pushed into the singletons.

III. This is the most interesting regime and *technically the content of the present chapter.* In the $n \to \infty$ limit (9) holds, where now the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of *constrained ODE-s*

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t), \qquad k \geq 2, \tag{17}$$

$$\sum_{k\in\mathbb{N}} v_k(t) = 1, \tag{18}$$

with the initial conditions (11). Mind the difference between the system (10) at one hand and the constrained system (17)+(18) at the other: the first equation from (10) is replaced by the global constraint (18). A first consequence is that it is no more true that the ODE-s in (17) can be solved for $k = 1, 2, \dots,$ one-by-one, in turn. The system of ODE-s is *genuinely infinite.* Up to $T_{\text{gel}}$ the solutions of (10), respectively, of (17)+(18) coincide, of course. But dramatic differences arise beyond this critical time. We prove that the system (17)+(18) admits a *unique solution* and for $t \geq T_{\text{gel}}$

$$\sum_{l=k}^{\infty} v_l(t) \sim \sqrt{\frac{2\varphi(t)}{\pi}}k^{-1/2}, \qquad \text{as} \qquad k \to \infty, \tag{19}$$

where $[T_{\text{gel}}, \infty) \ni t \mapsto \varphi(t)$ is strictly positive, bounded and Lipschitz continuous. This shows that in this regime the random graph dynamics exhibits indeed *self-organized critical behavior*: beyond the critical time $T_{\text{gel}}$ it stays critical for ever. The unique stationary solution of the system (17)+(18) is easily found

$$v_k(\infty) = 2\binom{2k-2}{k-1}\frac{1}{k}4^{-k} \approx \frac{1}{\sqrt{4\pi}}k^{-3/2}. \tag{20}$$

IV. In the $n \to \infty$ limit (9) holds again, where now the deterministic functions $t \mapsto v_k(t)$ are solutions of the infinite system of ODE-s

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t) - \lambda k v_k(t) + \lambda\delta_{k,1}\sum_{l=1}^{\infty} l v_l(t), \qquad k \geq 1, \tag{21}$$

with the initial conditions in $\mathcal{V}_1$. The system (21) is again a genuine infinite system (it can't be solved one-by-one for $k = 1, 2, \ldots$ in turn). The Cauchy problem (21) with initial condition in $\mathcal{V}_1$ has a unique solution, which stays *subcritical*, i.e. for any $t \in (0, \infty)$ $k \mapsto v_k(t)$ decays exponentially. The unique stationary solution is closely related to that of (20):

$$v_{\lambda,k}(\infty) = (\lambda + 1)\left(1 - \frac{\lambda^2}{(1+\lambda)^2}\right)^k v_k(\infty)$$

### 2.1.3 The main results, overview

We present the results formulated and proved only for the regime III: $n^{-1} \ll \lambda(n) \ll 1$, which shows *self-organized critical* asymptotic behaviour. The methods developed along the proofs are sufficient to prove the asymptotic behaviour in the other regimes, described in items I, II and IV but we omit these (in our opinion less interesting) details.

**Theorem 2.1.** *If the initial condition* $\mathbf{v}(0) \in \mathcal{V}_1$ *is such that* $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$, *and* $T_{gel}$ *is defined by* (13) *then the critical forest fire equations* (17)+(18) *have a unique solution with the following properties:*

1. *For* $t \leq T_{gel}$ *the solution coincides with that of* (10).
2. *For* $t \geq T_{gel}$ *there exists a positive, locally Lipschitz-continuous function* $\varphi$ *such that*

$$\dot{v}_1(t) = -v_1(t) + \varphi(t) \tag{22}$$

*and* (19) *holds.*

**Theorem 2.2.** *Let* $\mathbb{P}_n$ *denote the law of the random FFE of the forest fire Markov chain* $\mathbf{v}_n(t)$ *with initial condition* $\mathbf{v}_n(0)$ *and lightning rate parameter* $n^{-1} \ll \lambda(n) \ll 1$. *If* $\mathbf{v}_n(0) \to \mathbf{v}(0) \in \mathcal{V}_1$ *component-wise where* $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$ *then the sequence of probability measures* $\mathbb{P}_n$ *converges weakly to the Dirac measure concentrated on the unique solution of the critical forest fire equations* (17)+(18) *with initial condition* $\mathbf{v}(0)$. *In particular*

$$\forall \varepsilon > 0,\ t \geq 0\ :\quad \lim_{n\to\infty} \mathbb{P}_n\left(|v_{n,k}(t) - v_k(t)| \geq \varepsilon\right) = 0.$$

The rest of this Chapter is devoted to the proof of Theorem 2.1 and Theorem 2.2. Now we give an overview of the content of the forthcoming sections:

In order to prove Theorem 2.2 we want to show that that the sequence of probability measures $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is tight using Prohorov's theorem. From tightness it follows that every subsequence of $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ has a sub-sub-sequence which converges in probability to some random FFE. Then we prove that every subsequential limit of $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is concentrated on the subset of FFEs satisfying (17)+(18) (which is a set with one single element given Theorem 2.1), from which Theorem 2.2 follows.

- In order to show the tightness of the sequence $(\mathbb{P}_n)_{n=1}^{\infty}$ we need to show that for every $\varepsilon > 0$ there exists a compact subset $K$ of FFEs such that

$$\forall\, n \in \mathbb{N} : \quad \mathbb{P}_n(K) \geq 1 - \varepsilon. \tag{23}$$

  In Section 2.2 we define suitable compact subsets that satisfy (23) and show that any subsequential limit of $(\mathbb{P}_n)_{n=1}^{\infty}$ is concentrated on the subset of FFEs satisfying (17).

  - In order to do so we introduce auxiliary objects called *forest fire flows* in Subsection 2.2.1. The main idea is that we not only register the number of components of size $k$ at time $t$ for each $k \in \mathbb{N}$, but also the number of times when a component of size $k$ merged with a component of size $l$ before $t$ for each $k, l \in \mathbb{N}$ and the number of components of size $k$ that were destroyed by fire before $t$ for each $k \in \mathbb{N}$.
  - Then we precisely define the dynamics of the mean field forest fire model in Subsection 2.2.2: by the mean field property we need not take into account the graph structure of the connected components: the Markov chain that we study is a coagulation-fragmentation model (which is a modification of the Marcus-Lushnikov process, see [32]). Proposition 2.1 proves that $(\mathbb{P}_n)_{n=1}^{\infty}$ is tight and any subsequential limit is concentrated on the set of solutions of (17).
  - In Subsection 2.2.3 we take the Laplace transform/generating function of $(v_k(t))_{k=1}^{\infty}$:

$$V(t,x) = \sum_{k=1}^{\infty} v_k(t) e^{-kx} - 1$$

    If $(v_k(t))_{k=1}^{\infty}$ satisfies (17)+(18) then $V(t,x)$ solves a more tractable controlled PDE which we call the *Burgers control problem*:
    Find a control function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$\partial_t V(t,x) = -\frac{1}{2}\partial_x V^2(t,x) + \varphi(t)e^{-x}, \quad V(0,x) = \sum_{k=1}^{\infty} v_k(0)e^{-kx} - 1, \quad V(t,0) \equiv 0. \tag{24}$$

- In Section 2.3 we investigate the behavior of $V(t,x)$ as $x \to 0_+$, which is related to the tail behavior of $(v_k(t))_{k=1}^{\infty}$ by Tauberian theory.

  - In Subsection 2.3.1 we define $X(t,u)$ by $X(t, -V(t,x)) = x$: reversing the roles of the independent and the dependent variables is known as the *hodograph transform* and is used in to transform quasilinear PDE into linear PDE, see Section 4.4.3. of [23]. In particular, the Burgers equation is transformed into a much simpler linear equation by the hodograph transform.

In Subsection 2.3.1 formulate various aspects of the fact that if $\partial_u X(t,u)|_{u=0} = 0$ and $\partial^2_{uu} X(t,u) \asymp 1$ for some fixed $t$ as $x \to 0_+$ then we have

$$X(t,u) \asymp u^2 \qquad \Longleftrightarrow \qquad |V(t,x)| \asymp \sqrt{x} \qquad \Longleftrightarrow \qquad \sum_{l=k}^{\infty} v_l \asymp k^{-1/2} \tag{25}$$

- In Subsection 2.3.2 we apply the method of characteristics to the Burgers control problem to show that $\partial^2_{uu} X(0,u) \asymp 1$ implies $\partial^2_{uu} X(t,u) \asymp 1$. We use this to show that the mass contained in the giant component cannot grow too fast: $\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) \le C^*$ for some $C^* < +\infty$.
- In Subsection 2.3.3 we prove that any subsequential limit of the sequence $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is concentrated on the subset of FFEs satisfying (18). This proof is quite technical: $\frac{1}{n} \ll \lambda(n)$ only guarantees the destruction of components of size comparable to $n$ and one has to work hard proving

$$\lim_{N\to\infty} \lim_{n\to\infty} \sum_{k \ge N} v_{n,k}(t) = 0.$$

  A key element of the proof is that we take the generating function of the random vector $\left(v_{k,n}(t)\right)_{k=1}^{n}$ defined by (7) and derive results about the tail behavior of $\left(v_{k,n}(t)\right)_{k=1}^{n}$ using Laplace transform estimates and the method of (random) characteristics.

- In Section 2.4 we prove Theorem 2.1 by applying methods from the theory of first order nonlinear PDE to the Burgers control problem (24).
  - In Subsection 2.4.1 we prove that any solution of (24) satisfies $\partial_x V(t,x)|_{x=0} = -\infty$ for all $t \ge T_{\mathrm{gel}}$. Thus with the notations of Subsection 2.3.1 we have $\partial_u X(t,u)|_{u=0} = 0$ from which (25) follows: for all $t \ge T_{\mathrm{gel}}$ the system is critical.
  - In Subsection 2.4.2 we show

$$\lim_{x\to 0} \frac{1}{2} \partial_x V^2(t,x) = \varphi(t) \tag{26}$$

    (this fact formally follows from (24)) and derive fine properties of the solution of (24) from $\sum_{k=1}^{\infty} k^3 v_k(0) < +\infty$, e.g. that $\varphi(t)$ is locally Lipschitz-continuous on $[T_{\mathrm{gel}}, +\infty)$.
  - In Subsection 2.4.3 we use the method of characteristics to prove that the solution of (24) is unique from which uniqueness of (17)+(18) and Theorem 2.1 follows.

## 2.2 Coagulation and fragmentation

### 2.2.1 Forest fire flows

In this section we investigate the underlying structure of forest fire evolutions arising from the coagulation-fragmentation dynamics of our model on $n$ vertices. We only introduce randomness in the next subsection, thus the properties we derive in this section hold almost surely for the FFEs obtained from the mean field forest fire model.

We define auxiliary objects called forest fire flows: let $q_{n,k,l}(t)$ denote $n^{-1}$ times the number of $(k,l)$-coagulation events (a component of size $k$ merges with a component of size $l$) up to time $t$. Let $r_{n,k}(t)$ denote $n^{-1} \cdot k$ times the number of $k$-burning events (a component of size $k$ burns) up to time $t$. For the precise definitions see (36), (37), (39) and (40).

In Subsection 2.2.1 and Subsection 2.2.2 we precisely formulate and prove lemmas based on the following heuristic ideas:

- The state $\mathbf{v}_n(t)$ of the forest fire process on $n$ vertices (see (8)) can be recovered if we know the initial state $\mathbf{v}_n(0)$, $q_{n,k,l}(t)$ for all $k,l \in \mathbb{N}$ and $r_{n,k}(t)$ for all $k$ by the relation

$$\mathrm{d}v_{n,k}(t) = \frac{k}{2}\sum_{l=1}^{k-1} \mathrm{d}q_{n,l,k-l}(t) - k\sum_{l\geq 1} \mathrm{d}q_{n,k,l}(t) - \mathrm{d}r_{n,k}(t) + \mathbb{1}[k=1]\sum_{l\geq 1} \mathrm{d}r_{n,l}(t).$$

  The heuristic explanation of this formula is similar to that of (10): the mass contained in components of size $k$ increases if smaller components whose size add up to $k$ merge and decreases when a component of size $k$ merges with anything else or burns (the burnt mass goes to $v_1(t)$). The precise formula is (28).

- (28) is similar to the equations (21). This will help us proving Theorem 2.2: if $1 \ll n$ and $n^{-1} \ll \lambda(n) \ll 1$ then the random forest fire evolution $\mathbf{v}_n(t)$ "almost" satisfies the equations (17)+(18) that uniquely determine the deterministic limiting object $\mathbf{v}(t)$. We essentially prove that (17) is satisfied in the $n \to \infty$ limit in Proposition 2.1 of Subsection 2.2.2. We prove that (18) is satisfied in the limit in Subsection 2.3.3.

- We want to prove that the conditions (23) of Prokhorov's theorem for probability measures on the space of forest fire evolutions hold. We show useful compact subsets of the space of forest fire flows in Lemma 2.2.2. The key idea here is that a FFE behaves wildly (i.e. the total variation of the functions $v_k(t)$ is big) if a lot of creation and destruction happens simultaneously, but we cannot have big forest fires unless we have had a lot of coagulations before the fire, and $n \cdot \sum_{k\leq l} q_{n,k,l}(t)$ (the total number of coagulations up to time $t$) is less then the number of edges that turned from empty to occupied up to time $t$.

We define the moments of $\mathbf{v} \in \mathcal{V}$ as

$$m_0 = \sum_{k=1}^{\infty} v_k, \quad m_1 = \sum_{k=1}^{\infty} k \cdot v_k, \quad m_2 = \sum_{k=1}^{\infty} k^2 \cdot v_k, \quad m_3 = \sum_{k=1}^{\infty} k^3 \cdot v_k$$

By (5) and (6) $m_0 = 1$ if and only if $\mathbf{v} \in \mathcal{V}_1$.

Fix $T \in (0, \infty)$. A map $[0,T] \ni t \mapsto \mathbf{v}(t) \in \mathcal{V}$ is a *a forest fire evolution (FFE)* on $[0,T]$ if $v_k(\cdot)$, $k \in \mathbb{N}$ is of bounded variation and continuous from the left in $(0,T]$. Denote the space of FFE-s on $[0,T]$ by $\mathcal{E}[0,T]$ and the space of FFE-s with initial condition $\mathbf{v}(0) = \mathbf{v} \in \mathcal{V}$ on $[0,T]$ by $\mathcal{E}_{\mathbf{v}}[0,T]$. Note that a priori $\theta(\cdot) = 1 - \sum_{k\in\mathbb{N}} v_k(\cdot)$ need not be of bounded variation.

If $\mathbf{v}_n(\cdot) \in \mathcal{E}[0,T]$ is a sequence of FFE-s then we say that $\mathbf{v}_n(\cdot) \to \mathbf{v}(\cdot)$ if $v_{n,k}(\cdot) \Rightarrow v_k(\cdot)$ for all $k \in \mathbb{N}$ where "$\Rightarrow$" denotes weak convergence of the finite signed measures on $[0,T]$ corresponding to the functions $v_{n,k}(\cdot)$ and $v_k(\cdot)$. Note that we did not require the convergence of $\theta_n(\cdot)$ to $\theta(\cdot)$.

This topology is metrizable and the spaces $\mathcal{E}[0,T]$ and $\mathcal{E}_{\mathbf{v}}[0,T]$ endowed with this topology are separable and complete (by Fatou's lemma, $\lim_{n\to\infty} \mathbf{v}_n(t)$ stays in $\mathcal{V}$).

Denote $\mathbb{N} := \{1, 2, \dots\}$ and $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

A *forest fire flow (FFF)* is a collection of maps $[0,T] \ni t \mapsto \big(\mathbf{q}(t), \mathbf{r}(t)\big)$ where for $0 \le s \le t \le T$

$$0 = q_{k,l}(0) \le q_{k,l}(s) \le q_{k,l}(t), \qquad \mathbf{q}(t) = \big(q_{k,l}(t)\big)_{k,l\in\bar{\mathbb{N}}}, \qquad q_{k,l}(t) = q_{l,k}(t),$$

$$0 = r_k(0) \le r_k(s) \le r_k(t), \qquad \mathbf{r}(t) = (r_k(t))_{k\in\bar{\mathbb{N}}}, \qquad r_1(t) \equiv 0$$

We define

$$q_k(t) := \sum_{l\in\bar{\mathbb{N}}} q_{k,l}(t), \qquad q(t) := \sum_{k\in\bar{\mathbb{N}}} q_k(t), \qquad r(t) := \sum_{k\in\bar{\mathbb{N}}} r_k(t) \tag{27}$$

and assume the *finiteness conditions* $q(T) < +\infty$, $r(T) < +\infty$. All functions involved are continuous from the left in $(0,T]$. This is why we have chosen to consider the left-continuous versions of these functions rather than the traditional c.à.d.l.à.g.: the supremum of increasing left-continuous functions is itself left-continuous, thus the left-continuity of $q_k$, $q$ and $r$ automatically follows from the left-continuity of $q_{k,l}$ and $r_k$.

We say that the FFF $[0,T] \ni t \mapsto \big(\mathbf{q}(t), \mathbf{r}(t)\big)$ is *consistent with the initial condition* $\mathbf{v}(0) = \mathbf{v} \in \mathcal{V}$ if $t \mapsto \mathbf{v}(t)$ defined by

$$v_k(t) = v_k(0) + \frac{k}{2} \sum_{l=1}^{k-1} q_{l,k-l}(t) - k q_k(t) - r_k(t) + \mathbb{1}[k=1] r(t), \qquad k \in \mathbb{N}. \tag{28}$$

is in $\mathcal{E}_{\mathbf{v}}[0,T]$. That is: for all $t \in [0,T]$ and $k \in \mathbb{N}$ $v_k(t) \ge 0$ and $\sum_{k\in\mathbb{N}} v_k(t) \le 1$ holds. In this case we say that the FFF $\big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$ *generates* the FFE $\mathbf{v}(\cdot)$.

We denote by $\mathcal{F}_{\mathbf{v}}[0,T]$ the space of FFF-s consistent with the initial condition $\mathbf{v}(0) = \mathbf{v} \in \mathcal{V}$. For any $\mathbf{v} \in \mathcal{V}$, $\mathcal{F}_{\mathbf{v}}[0,T] \ne \emptyset$, since the zero flow is consistent with any initial condition.

At this point we mention that later we are going to obtain a FFF $\big(\mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big)$ from a realization of our forest fire model on $n$ vertices by (36), (37), (39) and (40). There is a FFF corresponding to the limit object as well: for the solution of the critical forest fire equations (17)+(18) (the uniqueness of the solution is stated in Theorem 2.1) we define $\big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$ by

$$\dot{q}_{k,l}(t) = v_k(t)v_l(t), \quad q_{\infty,k}(t) \equiv q_{\infty,\infty}(t) \equiv 0, \quad r_k(t) \equiv 0, \quad \dot{r}_\infty(t) = \varphi(t) \tag{29}$$

with the $\varphi(t)$ of (22). In Definition 2.2.1 we define a topology on the space of FFFs. In later sections we are going to prove that

$$\big(\mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \xrightarrow{\mathbf{P}} \big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$$

from which Theorem 2.2 will follow.

Summing (28) for $k \in \mathbb{N}$ we obtain a formula for the evolution of $\theta(\cdot)$ defined in (5): for $s \leq t$

$$\theta(t) = \theta(s) + \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{l=K-k+1}^{\infty} k \cdot (q_{k,l}(t) - q_{k,l}(s)) + \\ \sum_{k=1}^{\infty} k \cdot (q_{k,\infty}(t) - q_{k,\infty}(s)) - (r_\infty(t) - r_\infty(s)) \tag{30}$$

Later we will see that the term $\lim_{K\to\infty} \sum_{k=1}^{K} \sum_{l=K-k+1}^{\infty} k \cdot (q_{k,l}(t) - q_{k,l}(s))$ does not vanish for the FFF defined by (29) for the unique solution $\mathbf{v}(t)$ of (17)+(18) if $T_{\text{gel}} \leq s < t$: by (29) we have $q_{\infty,k}(t) \equiv 0$ and $r_\infty(t) - r_\infty(s) = \int_s^t \varphi(u)\,\mathrm{d}u$, moreover (18) is equivalent to $\theta(t) \equiv 0$, thus (30) becomes

$$0 = 0 + \lim_{K\to\infty} \sum_{k=1}^{K} \sum_{l=K-k+1}^{\infty} k \cdot (q_{k,l}(t) - q_{k,l}(s)) - \int_s^t \varphi(u)\,\mathrm{d}u$$

This phenomenon is related to (26).

If $\big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$ is a FFF then the functions $q_{k,l}$, $q_k$, $q$, $r_k$ and $r$ (where $k, l \in \bar{\mathbb{N}}$) are continuous from the left and increasing with initial condition 0: such functions are the distribution functions of nonnegative measures on $[0, T]$. By $q(T) < +\infty$ and $r(T) < +\infty$ these measures are finite. We denote by "$\Rightarrow$" the weak convergence of measures on $[0, T]$, which can alternatively be defined by point-wise convergence of the distribution functions at the continuity points of the limiting function.

**Definition 2.2.1.** *Let*

$$\big(\mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) = \Big( \left(q_{n,k,l}(\cdot)\right)_{k,l\in\bar{\mathbb{N}}}, \left(r_{n,k}(\cdot)\right)_{k\in\bar{\mathbb{N}}} \Big), \quad n = 1, 2, \ldots$$

*be a sequence of FFFs. Define $q_{n,k}(\cdot)$, $q_n(\cdot)$ and $r_n(\cdot)$ for all $n$ by (27).*

*We say that* $\big(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\big)\to\big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)$ *as* $n\to\infty$ *if*

$$\begin{aligned}
\forall\, k,l\in\mathbb{N}\ :&\quad q_{n,k,l}(\cdot)\Rightarrow q_{k,l}(\cdot)\\
\forall\, k\in\mathbb{N}\ :&\quad q_{n,k}(\cdot)\Rightarrow q_k(\cdot)\\
&\quad q_n(\cdot)\Rightarrow q(\cdot)\\
\forall\, k\in\mathbb{N}\ :&\quad r_{n,k}(\cdot)\Rightarrow r_k(\cdot)\\
&\quad r_n(\cdot)\Rightarrow r(\cdot)
\end{aligned}$$

Note that we do not require $r_{n,\infty}(\cdot)\Rightarrow r_\infty(\cdot)$ and $q_{n,k,\infty}(\cdot)\Rightarrow q_{k,\infty}$ for $k\in\bar{\mathbb{N}}$. Nevertheless these "missing" ingredients of the limit flow $\big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)$ of convergent flows are uniquely determined by the convergent ones if we rearrange the relations (27):

$$q_{k,\infty}(t):=q_k(t)-\sum_{l\in\mathbb{N}}q_{k,l}(t), \tag{31}$$

$$r_\infty(t):=r(t)-\sum_{k\in\mathbb{N}}r_k(t), \tag{32}$$

$$q_{\infty,\infty}(t):=q(t)-2\sum_{k\in\mathbb{N}}q_k(t)+\sum_{k,l\in\mathbb{N}}q_{k,l}(t). \tag{33}$$

In fact, $r_{n,\infty}(\cdot)\not\Rightarrow r_\infty(\cdot)$ and $q_{n,k,\infty}(\cdot)\not\Rightarrow q_{k,\infty}$ have a physical meaning in the forest fire model if $\big(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\big)$ is defined by (39) and (40):

- In the $\lambda(n)=\mathcal{O}(n^{-1})$ regime $0\equiv q_{n,k,\infty}(\cdot)\not\Rightarrow q_{k,\infty}(\cdot)\not\equiv 0$ indicates the presence of a giant component. The precise formulation of this fact for the Erdős-Rényi model is (45).
- If $\lambda(n)\ll 1$ then only "large" components burn. Indeed in Proposition 2.1 we are going to prove that for all $k\in\mathbb{N}$ $r_{n,k}(\cdot)$ converges to 0 in probability an $n\to\infty$. Thus by (32) we have $r(\cdot)=r_\infty(\cdot)$ in the limit. But Theorem 2.1, Theorem 2.2 and (29) imply that $0=r_{n,\infty}(t)\not\Rightarrow r_\infty(t)=\int_0^t\varphi(s)\,\mathrm{d}s>0$ for $t>T_{\text{gel}}$.

$\mathcal{F}_{\mathbf{v}}[0,T]$ endowed with the topology of Definition 2.2.1 is a complete separable metric space:

**Lemma 2.2.1.** *If* $\big(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\big)\in\mathcal{F}_{\mathbf{v}}[0,T]$ *for all* $n\in\mathbb{N}$ *and* $\big(\mathbf{q}_n(\cdot),\mathbf{r}_n(\cdot)\big)\to\big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)$, *then* $\big(\mathbf{q}(\cdot),\mathbf{r}(\cdot)\big)\in\mathcal{F}_{\mathbf{v}}[0,T]$.

*Proof.* By the definition of weak convergence, $q_{k,l},q_k,q,r_k,r$ are increasing left-continuous functions with initial value 0. We need to check that the functions $r_\infty$, $q_{k,\infty}$, and $q_{\infty,\infty}$ (defined by (32), (31) and (33), respectively) are increasing. We may assume that $0\le s\le t\le T$ are continuity points of $q_{k,l}$, $q_k$, $q$, $r_k$ and $r$ for all $k,l\in\bar{\mathbb{N}}$.

By Fatou's lemma we get

$$
\begin{aligned}
r_\infty(t) - r_\infty(s) &= \lim_{n\to\infty} (r_n(t) - r_n(s)) - \sum_{k\in\mathbb{N}} \lim_{n\to\infty} (r_{n,k}(t) - r_{n,k}(s)) \\
&\geq \limsup_{n\to\infty} \left( r_n(t) - r_n(s) - \sum_{k\in\mathbb{N}} (r_{n,k}(t) - r_{n,k}(s)) \right) \\
&= \limsup_{n\to\infty} (r_{n,\infty}(t) - r_{n,\infty}(s)) \geq 0.
\end{aligned}
$$

One can prove similarly that $q_{k,\infty}$ is increasing for $k \in \mathbb{N}$. In order to prove that

$$
q_{\infty,\infty}(t) - q_{\infty,\infty}(s) \geq \limsup_{n\to\infty} (q_{n,\infty,\infty}(t) - q_{n,\infty,\infty}(s))
$$

let $\alpha_{n,k,l} := q_{n,k,l}(t) - q_{n,k,l}(s)$ for $k, l \in \bar{\mathbb{N}}$. By (33) we only need to check

$$
\lim_{n\to\infty} \sum_{k,l\in\bar{\mathbb{N}}} \alpha_{n,k,l} - \limsup_{n\to\infty} \alpha_{n,\infty,\infty} \geq 2 \sum_{k\in\mathbb{N}} \lim_{n\to\infty} \sum_{l\in\bar{\mathbb{N}}} \alpha_{n,k,l} - \sum_{k,l\in\mathbb{N}} \lim_{n\to\infty} \alpha_{n,k,l}. \tag{34}
$$

Let

$$
K_m := \{(k,l) : (k \geq m \text{ and } l = m) \text{ or } (l \geq m \text{ and } k = m)\} \cup \{(m,\infty)\} \cup \{(\infty, m)\}.
$$

The left hand side of (34) is $\liminf_{n\to\infty} \sum_{m\in\mathbb{N}} \beta_{n,m}$, the right hand side is $\sum_{m\in\mathbb{N}} \lim_{n\to\infty} \beta_{n,m}$, where $\beta_{n,m} := \sum_{(k,l)\in K_m} \alpha_{n,k,l}$, and the inequality follows from Fatou's lemma.

Now that we have proved that the limit of convergent flows is itself a flow, we only need to check that the limit flow is consistent with the initial condition $\mathbf{v}$, but this follows from the facts that $\mathcal{E}_{\mathbf{v}}[0,T]$ is a closed metric space and the mapping from $\mathcal{F}_{\mathbf{v}}[0,T]$ to $\mathcal{E}_{\mathbf{v}}[0,T]$ defined by (28) is continuous with respect to the corresponding topologies.

□

Finally we define the space of all FFF-s as follows:

$$
\mathcal{D}[0,T] := \big\{ \big(\mathbf{v}, \mathbf{q}(\cdot), \mathbf{r}(\cdot)\big) \,:\, \mathbf{v} \in \mathcal{V},\ \big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big) \in \mathcal{F}_{\mathbf{v}}[0,T] \big\}.
$$

This space is again a complete and separable metric space if we define $\big(\mathbf{v}_n, \mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \to \big(\mathbf{v}, \mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$ by requiring $\mathbf{v}_n \to \mathbf{v}$ (coordinate-wise) and $\big(\mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \to \big(\mathbf{q}(\cdot), \mathbf{r}(\cdot)\big)$.

**Lemma 2.2.2.** *For any $C < \infty$ the subset*

$$
\mathcal{K}_C[0,T] := \big\{ \big(\mathbf{v}, \mathbf{q}(\cdot), \mathbf{r}(\cdot)\big) \in \mathcal{D}[0,T] \,:\, q(T) \leq C \big\}
$$

*is compact in $\mathcal{D}[0,T]$.*

*Proof.*

$$\lim_{K\to\infty}\left[\frac{1}{2}\sum_{k=1}^{K}\sum_{l=1}^{k-1} q_{l,k-l}(T)-\sum_{k=1}^{K} q_k(T)\right]=-\frac{1}{2}q(T)+\frac{1}{2}q_{\infty,\infty}(T)$$

by $q(T)\le C$, dominated convergence and $q_{k,l}=q_{l,k}$. Thus summing the equations (28) with coefficients $\frac{1}{k}$ we get

$$\sum_{k=1}^{\infty}\frac{1}{k}v_k(T)-\sum_{k=1}^{\infty}\frac{1}{k}v_k(0)+\frac{1}{2}q(T)=\sum_{k=2}^{\infty}\frac{k-1}{k}r_k(T)+r_\infty(T)+\frac{1}{2}q_{\infty,\infty}(T).$$

The inequalities

$$r(T)\le 2+C,\quad r_\infty(T)\le 1+\frac{1}{2}C,\quad r_k(T)\le (1+\frac{C}{2})\frac{k}{k-1} \tag{35}$$

follow from $\mathbf{v}(T)\in\mathcal{V}$ and $q(T)\le C$.

By Helly's selection theorem and a diagonal argument we can choose a convergent subsequence from any sequence of elements of $\mathcal{K}_C[0,T]$ with the limiting FFF itself being an element of $\mathcal{K}_C[0,T]$.

□

### 2.2.2 The Markov process

It is easy to see that in order to prove Theorem 2.2 we do not need to know anything about the graph structure of the connected components: by the mean field property of the dynamics the stochastic process $\mathbf{v}_n(t)$ defined by (7) and (8) is itself a Markov chain.

In Proposition 2.1 we show that $\left(\mathbb{P}_n\right)_{n=1}^{\infty}$ is tight and that any subsequential limit is concentrated on the set of solutions of the integral equation corresponding to (17).

The state space of the Markov chain $t\mapsto \mathbf{V}_n(t)$ is:

$$\Omega_n := \left\{\mathbf{V}=\left(V_k\right)_{k\in\mathbb{N}} \,:\, V_k\in\{0,k,2k,\dots\},\ \sum_{k\ge 1}V_k=n\right\}$$

$V_k$ is the number of vertices contained in components of size $k$.

The allowed jumps of the Markov chain are described by the following jump transformations for $i\le j$:

$$\sigma_{i,j}:\left\{\mathbf{V}\in\Omega_n \,:\, V_i\big(V_j-j\mathbb{1}[i=j]\big)>0\right\}\to\Omega_n,$$

$$\left(\sigma_{i,j}\mathbf{V}\right)_k := V_k-i\mathbb{1}[k=i]-j\mathbb{1}[k=j]+(i+j)\mathbb{1}[k=i+j],$$

$$\tau_i:\left\{\mathbf{V}\in\Omega_n : V_i>0\right\}\to\Omega_n,\qquad \left(\tau_i\mathbf{V}\right)_k := V_k+i\mathbb{1}[k=1]-i\mathbb{1}[k=i]$$

The transformation $\sigma_{i,j}$ occurs when an edge connects a component of size $i$ to a component of size $j$. The transformation $\tau_i$ occurs when a lightning hits a component of size $i$.

The corresponding jump rates are $a_{n,i,j}, b_{n,i} : \Omega_n \to \mathbb{R}_+$:

$$a_{n,i,j}(\mathbf{V}) := ((1+\mathbb{1}[i=j])n)^{-1} V_i\big(V_j - j\mathbb{1}[i=j]\big), \qquad b_{n,i}(\mathbf{V}) := \lambda(n)V_i.$$

The infinitesimal generator of the chain is :

$$L_n f(\mathbf{V}) = \sum_{i\le j} a_{n,i,j}(\mathbf{V})\big(f(\sigma_{i,j}\mathbf{V}) - f(\mathbf{V})\big) + \sum_i b_{n,i}(\mathbf{V})\big(f(\tau_i\mathbf{V}) - f(\mathbf{V})\big).$$

We denote by $Q_{n,k,l}(t)$ and by $R_{n,k}(t)$ the number of $\sigma_{k,l}$-jumps, respectively $k$-times the number of $\tau_k$-jumps occurred in the time interval $[0,t]$:

$$Q_{n,k,l}(t) := \big(1+\mathbb{1}[k=l]\big)\cdot\big|\big\{s\in[0,t] \,:\, \mathbf{V}_n(s+0) = \big(\sigma_{k,l}\mathbf{V}_n\big)(s-0)\big\}\big|, \tag{36}$$

$$R_{n,k}(t) := \mathbb{1}[k\neq 1]\cdot k\cdot\big|\big\{s\in[0,t] \,:\, \mathbf{V}_n(s+0) = \big(\tau_k\mathbf{V}_n\big)(s-0)\big\}\big|. \tag{37}$$

Finally, the scaled objects are

$$v_{n,k}(t) := n^{-1}V_{n,k}(t), \qquad\qquad \mathbf{v}_n(t) := \big(v_{n,k}(t)\big)_{k\in\mathbb{N}}, \tag{38}$$

$$q_{n,k,l}(t) := n^{-1}Q_{n,k,l}(t), \qquad q_{n,k,\infty}(t)\equiv 0, \qquad \mathbf{q}_n(t) := \big(q_{n,k,l}(t)\big)_{k,l\in\bar{\mathbb{N}}}, \tag{39}$$

$$r_{n,k}(t) := n^{-1}R_{n,k}(t), \qquad r_{n,\infty}(t)\equiv 0, \qquad \mathbf{r}_n(t) := \big(r_{n,k}(t)\big)_{k\in\bar{\mathbb{N}}} \tag{40}$$

Now, given $T\in(0,\infty)$ and some initial conditions $\mathbf{v}_n(0) = \mathbf{v}_n \in \mathcal{V}_1$, clearly $t\mapsto \mathbf{v}_n(t) \in \mathcal{V}_1$ is a conservative FFE, generated by the FFF $\big(\mathbf{v}_n, \mathbf{q}_n(\cdot), \mathbf{r}_n(\cdot)\big) \in \mathcal{D}[0,T]$ through (28). We denote by $\mathbb{P}_n$ the probability distribution of this process on $\mathcal{D}[0,T]$. We will always assume that the initial conditions converge, as $n\to\infty$, to a deterministic element of $\mathcal{V}_1$:

$$\lim_{n\to\infty} v_{n,k}(0) = v_k, \qquad \mathbf{v} := (v_k)_{k\in\mathbb{N}} \in \mathcal{V}_1. \tag{41}$$

**Proposition 2.1.** *The sequence of probability measures $\mathbb{P}_n$ is tight on $\mathcal{D}[0,T]$. If $\lambda(n) \ll 1$, then any weak limit point $\mathbb{P}$ of the sequence $\mathbb{P}_n$ is concentrated on that subset of $\mathcal{D}[0,T]$ for which the following hold for $k,l\in\mathbb{N}$:*

$$q_{k,l}(t) = \int_0^t v_k(s)v_l(s)\,\mathrm{d}s, \quad q_k(t) = \int_0^t v_k(s)\,\mathrm{d}s, \quad q(t)\le t, \quad r_k(t)\equiv 0 \tag{42}$$

$$\mathbf{v}(0) = \mathbf{v}. \tag{43}$$

*Proof.* There is nothing to prove about the initial condition (43): it was a priori assumed in (41).

In order to prove the validity of the integral equations (42), note first that it is straightforward that the processes $\widetilde{q}_{n,k,l}(t)$, $\langle\widetilde{q}_{n,k,l}\rangle(t)$, $\widetilde{q}_{n,k}(t)$, $\langle\widetilde{q}_{n,k}\rangle(t)$, $\widetilde{r}_{n,k}(t)$, $\langle\widetilde{r}_{n,k}\rangle(t)$, defined below are martingales:

$$
\begin{aligned}
\widetilde{q}_{n,k,l}(t) &:= q_{n,k,l}(t) - \int_0^t v_{n,k}(s) v_{n,l}(s)\,\mathrm{d}s + \frac{k\mathbb{1}[k=l]}{n}\int_0^t v_{n,k}(s)\,\mathrm{d}s,\\
\langle \widetilde{q}_{n,k,l}\rangle(t) &:= \widetilde{q}_{n,k,l}(t)^2 - \frac{\mathbb{1}[k\neq l] + 2\mathbb{1}[k=l]}{n}\int_0^t v_{n,k}(s) v_{n,l}(s)\,\mathrm{d}s + \frac{2k\mathbb{1}[k=l]}{n^2}\int_0^t v_{n,k}(s)\,\mathrm{d}s,\\
\widetilde{q}_{n,k}(t) &:= q_{n,k}(t) - \int_0^t v_{n,k}(s)\,\mathrm{d}s + \frac{k}{n}\int_0^t v_{n,k}(s)\,\mathrm{d}s,\\
\langle \widetilde{q}_{n,k}\rangle(t) &:= \widetilde{q}_{n,k}(t)^2 - \frac{1}{n}\int_0^t \big(v_{n,k}(s)^2 + v_{n,k}(s)\big)\,\mathrm{d}s + \frac{2k}{n^2}\int_0^t v_{n,k}(s)\,\mathrm{d}s,\\
\widetilde{q}_n(t) &:= q_n(t) - t + \frac{1}{n}\int_0^t m_{n,1}(s)\,\mathrm{d}s,\\
\langle \widetilde{q}_n\rangle(t) &:= \widetilde{q}_n(t)^2 - \frac{1}{n}\left(t + \int_0^t \sum_{k=1}^n v_{n,k}(s)^2 \mathrm{d}s\right) + \frac{2}{n^2}\int_0^t m_{n,1}(s)\,\mathrm{d}s,\\
\widetilde{r}_{n,k}(t) &:= r_{n,k}(t) - \lambda(n) k \int_0^t v_{n,k}(s)\,\mathrm{d}s,\\
\langle \widetilde{r}_{n,k}\rangle(t) &:= \widetilde{r}_{n,k}(t)^2 - \frac{\lambda(n)k^2}{n}\int_0^t v_{n,k}(s)\,\mathrm{d}s.
\end{aligned}
$$

From Doob's maximal inequality it readily follows that for any $k, l \in \mathbb{N}$ and $\varepsilon > 0$

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}\big( \sup_{0\le t\le T} \big|\, q_{n,k,l}(t) - \int_0^t v_{n,k}(s) v_{n,l}(s)\,\mathrm{d}s \,\big| > \varepsilon \big) = 0,\\
&\lim_{n\to\infty} \mathbf{P}\big( \sup_{0\le t\le T} \big|\, q_{n,k}(t) - \int_0^t v_{n,k}(s)\,\mathrm{d}s \,\big| > \varepsilon \big) = 0,\\
&\lim_{n\to\infty} \mathbf{P}\big( \sup_{0\le t\le T} \; q_n(t) - t \; > \varepsilon \big) = 0,\\
&\lim_{n\to\infty} \mathbf{P}\big( \sup_{0\le t\le T} \big|\, r_{n,k}(t) \,\big| > \varepsilon \big) = 0.
\end{aligned}
$$

Hence (42). Tightness follows from

$$
\mathbf{E}\big(\, q_n(T) \,\big) \le T, \tag{44}
$$

Markov's inequality and Lemma 2.2.2. □

If we consider the case $\lambda(n) \equiv 0$ (this is the dynamical Erdős-Rényi model) then

(9)+(10) follows from Proposition 2.1 since (28) becomes

$$v_k(t) = v_k(0) + \frac{k}{2}\sum_{l=1}^{k-1} q_{l,k-l}(t) - kq_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - kv_k(s)\,\mathrm{d}s$$

which is the integral form of (10). Plugging (42) into (31) we get for $t > T_{\text{gel}}$

$$q_{k,\infty}(t) = \int_0^t v_k(s)\theta(s)\,\mathrm{d}s > 0. \tag{45}$$

### 2.2.3 The integrated Burgers control problem

In this subsection we precisely describe the subset of the solutions of the integral version of (17) that we are interested in using Proposition 2.1.

We get (17) by dropping the first equation of the well-posed problem (10). Thus there is one extra degree of freedom in the solutions of (17) which manifests in the form of the control function $r(t)$ in (46) below.

Then we take the Laplace transform/generating function of (46) to obtain the integral equation (52) below. With a tricky timechange we can transform the integral equation (52) into the differential equation (59) below.

If $\mathbf{v}(\cdot) \in \mathcal{E}_{\mathbf{v}_0}[0,T]$ is generated by a FFF satisfying (42) through (28), then

$$r(\cdot) = \sum_{k\in\bar{\mathbb{N}}} r_k(\cdot) = \sum_{k=1}^{\infty} r_k(\cdot) + r_\infty(\cdot) = \sum_{k=1}^{\infty} 0 + r_\infty(\cdot) = r_\infty(\cdot)$$

and $\mathbf{v}(\cdot)$ is a solution of the *controlled Smoluchowski integral equations* with control function $r(\cdot)$:

$$v_k(t) = v_k(0) + \frac{k}{2}\sum_{l=1}^{k-1}\int_0^t v_l(s)v_{k-l}(s)\,\mathrm{d}s - k\int_0^t v_k(s)\,\mathrm{d}s + \mathbb{1}[k=1]r(t), \qquad k\in\mathbb{N} \tag{46}$$

$$v_k(t) \geq 0, \quad \sum_{k=1}^{\infty} v_k(t) \leq 1 \tag{47}$$

$$\mathbf{v}(0) = \mathbf{v}_0 \in \mathcal{V}_1. \tag{48}$$

By $q(T) \leq T$, $r_\infty(\cdot) = r(\cdot)$ and (35) we get

$$0 = r(0) \leq r(s) \leq r(t) \quad \text{for} \quad 0 \leq s \leq t \leq T, \quad r(T) \leq 1 + \frac{T}{2}. \tag{49}$$

Using induction on $k$ one can see that the initial condition $\mathbf{v}_0$ and the control function $r(\cdot)$ determines the solution of (46), (48) uniquely.

For $\mathbf{v} \in \mathcal{V}$ we introduce the generating function

$$V : [0, \infty) \to [-1, 0], \qquad V(x) := \sum_{k=1}^{\infty} v_k e^{-kx} - 1. \tag{50}$$

$x \mapsto V(x)$ is analytic on $(0, \infty)$ and has the following straightforward properties:

$$\lim_{x\to\infty} V(x) = -1, \qquad V'(x) \leq 0, \qquad V''(x) \geq 0. \tag{51}$$

It is easy to see that if $t \mapsto \mathbf{v}(t)$ is a solution of (46), (47), (48) then the corresponding generating functions $t \mapsto V(t, \cdot)$ will solve the *integrated Burgers control problem*

$$V(t, x) - V(0, x) + \int_0^t V(s, x)V'(s, x)\,\mathrm{d}s = e^{-x}r(t), \tag{52}$$

$$-1 \leq V(t, 0) \leq 0 \tag{53}$$

$$V(0, x) = V_0(x). \tag{54}$$

The control function $r(\cdot)$ was defined to be continuous from the left in (27), but it need not be continuous: when $\lambda(n) = n^{-1}\lambda$ then the FFE obtained as the $n \to \infty$ limit satisfies (46), (47), (48), but the control function $r(\cdot)$ evolves randomly according to the rules (15), (16):

$$\mathbf{P}\big( r(t + \mathrm{d}t) = r(t) + \theta(t) \,\big|\, \mathcal{F}(t) \big) = \lambda\theta(t)\mathrm{d}t + o(\mathrm{d}t)$$

Thus $r(\cdot)$ is a random step function in this case.

In order to rewrite (52) as a differential equation we introduce a new time variable $\tau$:

$$t(\tau) := \max\{t : t + r(t) \leq \tau\} \tag{55}$$

It is easily seen that $t(\tau)$ is increasing and Lipschitz-continuous:

$$t(\tau) = \int_0^\tau \alpha(s)\,\mathrm{d}s \qquad 0 \leq \alpha(\cdot) \leq 1. \tag{56}$$

Given a solution $V(t, x)$ of (52), (53), (54) define

$$\mathbf{V}(\tau, x) := V(t(\tau), x) + (\tau - t(\tau) - r(t(\tau)))\, e^{-x} \tag{57}$$

Then by (52) we have

$$\mathbf{V}(\tau, x) = V(0, x) - \int_0^{t(\tau)} V(s, x)V'(s, x)\,\mathrm{d}s + (\tau - t(\tau))e^{-x}. \tag{58}$$

Now we show that for all $\tau \geq 0$, $x > 0$ and $t \geq 0$ we have

$$\partial_\tau \mathbf{V}(\tau, x) = -\mathbf{V}(\tau, x)\mathbf{V}'(\tau, x)\alpha(\tau) + (1-\alpha(\tau))e^{-x} \tag{59}$$

$$-1 \leq \mathbf{V}(\tau, 0) \leq 0 \tag{60}$$

$$\mathbf{V}(0, x) = V_0(x) \tag{61}$$

$$\mathbf{V}(t + r(t), x) = V(t, x) \tag{62}$$

First note that the fact

$$\mathbf{V}(\tau, x) \neq V(t(\tau), x) \qquad \Longrightarrow \qquad \alpha(\tau) = 0 \tag{63}$$

follows directly from (55), (56) and (57): if $r(t_+) \neq r(t)$, then $\alpha(\tau) = 0$ for all $t + r(t) < \tau \leq t + r(t_+)$. The differential equation (59) follows from (56), (58) and (63). The boundary inequality (60) follows from

$$-1 \leq V(t(\tau), x) \leq \mathbf{V}(\tau, x) \leq V(t(\tau)_+, x) \leq 0.$$

The initial conditions (54) and (61) are equivalent, and (62) follows from (57) and (55).

From the definition of Lebesgue-Stieltjes integration it follows that for all $t_1 \leq t_2$ we have

$$\int_{t_1 + r(t_1)}^{t_2 + r(t_2)} f(t(\tau))(1 - \alpha(\tau))\,\mathrm{d}\tau = \int_{t_1}^{t_2} f(t)\,\mathrm{d}r(t) \tag{64}$$

## 2.3 Boundary behavior

### 2.3.1 Elementary facts about generating functions

In this subsection we collect some elementary facts about generating functions, which will be used along the proof of Theorem 2.1 and Theorem 2.2. For $\mathbf{v} \in \mathcal{V}$ we introduce the generating function $V(x)$ defined in (50) which has the straightforward properties listed in (51). It is also easy to see that for any $\mathbf{v} \in \mathcal{V}$ and any $x > 0$

$$|V'(x)| \leq \frac{1}{e} x^{-1}, \qquad V''(x) \leq \left(\frac{2}{e}\right)^2 x^{-2}, \qquad |V'''(x)| \leq \left(\frac{3}{e}\right)^3 x^{-3}. \tag{65}$$

We define the functions $E : (0, \infty) \to (0, \infty)$, $E^* : [0, \infty) \to (0, \infty]$, $E_* : [0, \infty) \to [0, \infty)$ as follows:

$$E(x) := -\frac{V'(x)^3}{V''(x)}, \quad E^*(x) := \sup_{0<y\leq x} E(y), \quad E_*(x) := \inf_{0<y\leq x} E(y) \tag{66}$$

Note that these functions are continuous on their domain of definition.

**Lemma 2.3.1.** *Let* $\mathbf{v} \in \mathcal{V}_1$.

*1. For any $x > 0$*

$$0 < \quad V(x)V'(x) \quad \le E^*(x). \tag{67}$$

*2. If in addition*

$$V'(0) := \lim_{x\to 0} V'(x) = -\infty \tag{68}$$

*then the following bounds hold*

$$2^{1/2}E_*(x)^{1/2}x^{1/2} \le \quad -V(x) \quad \le 2^{1/2}E^*(x)^{1/2}x^{1/2} \tag{69}$$

$$2^{-1/2}E_*(x)E^*(x)^{-1/2}x^{-1/2} \le \quad -V'(x) \quad \le 2^{-1/2}E^*(x)E_*(x)^{-1/2}x^{-1/2} \tag{70}$$

$$2^{-3/2}E_*(x)^3E^*(x)^{-5/2}x^{-3/2} \le \quad V''(x) \quad \le 2^{-3/2}E^*(x)^3E_*(x)^{-5/2}x^{-3/2}$$

$$E_*(x) \le \quad V(x)V'(x) \quad \le E^*(x). \tag{71}$$

*Proof.* Since $\mathbf{v} \in \mathcal{V}_1$ we have $V(0) = 0$. Denote the inverse function of $-V(x)$ by $X(u)$: $X(-V(x)) = x$. Note that

$$E(x) = \frac{1}{X''(-V(x))}, \tag{72}$$

and thus

$$X(0) = 0, \qquad X'(0) = -V'(0)^{-1}, \qquad X''(u) = E(X(u))^{-1}.$$

It follows that for $u \in [0, -V(x)]$:

$$-V'(0)^{-1} + E^*(x)^{-1}u \le \quad X'(u) \quad \le -V'(0)^{-1} + E_*(x)^{-1}u,$$

$$-V'(0)^{-1}u + E^*(x)^{-1}\frac{u^2}{2} \le \quad X(u) \quad \le -V'(0)^{-1}u + E_*(x)^{-1}\frac{u^2}{2}.$$

Hence, all the bounds of the Lemma follow directly. □

### 2.3.2 Bounds on $E$

We assume given a solution of the *integrated Burgers control problem*: (52), (53), (54) with a control function $r(\cdot)$ satisfying (49).

We fix $\bar{t} \in (0, \infty)$, $\bar{x} \in (0, \infty)$. All estimates will be valid uniformly in the domain $(t, x) \in [0, \bar{t}] \times [0, \bar{x}]$. The various constants appearing in the forthcoming estimates will depend only on the initial conditions $V(0, x)$ and on the choice of $(\bar{t}, \bar{x})$. The notation

$$A(t, x) \asymp B(t, x)$$

means that there exists a constant $1 < C < +\infty$ which depends only on the initial conditions (54) and the choice of $(\bar{t}, \bar{x})$, such that for any $(t, x) \in [0, \bar{t}] \times [0, \bar{x}]$

$$C^{-1}B(t, x) \le A(t, x) \le CB(t, x). \tag{73}$$

The notation $A(t,x) = \mathcal{O}(B(t,x))$ means that the upper bound of (73) holds.

In the sequel we denote the derivative of functions $f(t,x)$ with respect to the time and space variables by $\dot{f}(t,x)$ and $f'(t,x)$, respectively.

We define (similarly to (66))

$$E(t,x) := -\frac{\partial_x V(t,x)^3}{\partial_x^2 V(t,x)}, \qquad E^*(t,x) := \sup_{0<y\le x} E(t,y), \qquad E_*(t,x) := \inf_{0<y\le x} E(t,y),$$
$$\mathbf{E}(\tau,x) := -\frac{\partial_x \mathbf{V}(\tau,x)^3}{\partial_x^2 \mathbf{V}(\tau,x)}, \qquad \mathbf{E}^*(\tau,x) := \sup_{0<y\le x} \mathbf{E}(\tau,y), \qquad \mathbf{E}_*(\tau,x) := \inf_{0<y\le x} \mathbf{E}(\tau,y).$$

In this subsection we prove the following two lemmas:

**Lemma 2.3.2.** *If $m_2(0) = \sum_{k=1}^{\infty} k^2 \cdot v_k(0) < +\infty$, then for any solution of the integrated Burgers control problem* (52), (54), (53) *with a control function satisfying* (49) *and for $(t,x) \in [0,\overline{t}] \times (0,\overline{x}]$ we have*

$$E(t,x) \asymp 1. \tag{74}$$

**Lemma 2.3.3.** *If $m_2(0) < +\infty$, then for any solution of the integrated Burgers control problem* (52), (53), (54) *with a control function satisfying* (49) *there is a constant $C^*$ which depends only on the initial conditions and $T$ such that for $T_{gel} \le t_1 \le t_2 \le T$ we have*

$$\theta(t_2) - \theta(t_1) \le C^* \cdot (t_2 - t_1). \tag{75}$$

Before proving these lemmas we define the *characteristics* given a solution of (52), (54), (53): for $t \ge 0, x > 0$ let $[0,t] \ni s \mapsto \xi_{t,x}(s)$ be the *unique solution* of the integral equation

$$\xi_{t,x}(s) = x - V(t,x)(t-s) + \int_s^t (u-s) e^{-\xi_{t,x}(u)} \mathrm{d}r(u). \tag{76}$$

Existence and uniqueness of the solution of (76) follow from a simple fixed point argument. Now we prove that (given $(t,x)$ fixed) $s \mapsto \xi_{t,x}(s)$ is also solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}s}\xi_{t,x}(s) =: \dot{\xi}_{t,x}(s) = V(s, \xi_{t,x}(s)), \qquad \xi_{t,x}(t) = x. \tag{77}$$

In order to prove this we define $\mathbf{V}(\tau,x)$ by (57). Thus from (63) it follows that that the solution of (77) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_{t,x}(t(\tau)) = V(t(\tau), \xi_{t,x}(t(\tau)))\alpha(\tau) = \mathbf{V}(\tau, \xi_{t,x}(t(\tau)))\alpha(\tau) \tag{78}$$

From this and (59) we get that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{V}(\tau, \xi_{t,x}(t(\tau))) = \dot{\mathbf{V}}(\tau, \xi_{t,x}(t(\tau))) + \mathbf{V}'(\tau, \xi_{t,x}(t(\tau))) \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}\xi_{t,x}(t(\tau)) = (1-\alpha(\tau))e^{-\xi_{t,x}(t(\tau))}$$

Integrating this and using $\xi_{t,x}(t) = x$ and (62) we get for all $\tau_1 \le t + r(t)$

$$\mathbf{V}(\tau_1, \xi_{t,x}(t(\tau_1))) = V(t,x) - \int_{\tau_1}^{t+r(t)} (1-\alpha(\tau))e^{-\xi_{t,x}(t(\tau))}\,\mathrm{d}\tau$$

Substituting this into the r.h.s. of (78), integrating and using (56) we get for all $\tau_2 \le t + r(t)$

$$\xi_{t,x}(t(\tau_2)) = x - V(t,x)(t - t(\tau_2)) + \int_{\tau_2}^{t+r(t)} (t(\tau) - t(\tau_2))e^{-\xi_{t,x}(t(\tau))}(1-\alpha(\tau))\,\mathrm{d}\tau$$

Now (76) follows from this by substituting $\tau_2 = s + r(s)$ and using (64).

Differentiating (59) with respect to $x$ we get

$$\dot{\mathbf{V}}'(\tau,x) = -\mathbf{V}'(\tau,x)^2\alpha(\tau) - \mathbf{V}(\tau,x)\mathbf{V}''(\tau,x)\alpha(\tau) - (1-\alpha(\tau))e^{-x} \tag{79}$$

$$\dot{\mathbf{V}}''(\tau,x) = -3\mathbf{V}'(\tau,x)\mathbf{V}''(\tau,x)\alpha(\tau) - \mathbf{V}(\tau,x)\mathbf{V}'''(\tau,x)\alpha(\tau) + (1-\alpha(\tau))e^{-x} \tag{80}$$

Using this and (78) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{E}(\tau, \xi_{t,x}(t(\tau))) = \left(3\frac{\mathbf{V}'(\tau,\xi_{t,x}(t(\tau)))^2}{\mathbf{V}''(\tau,\xi_{t,x}(t(\tau)))} + \frac{\mathbf{V}'(\tau,\xi_{t,x}(t(\tau)))^3}{\mathbf{V}''(\tau,\xi_{t,x}(t(\tau)))^2}\right) e^{-\xi_{t,x}(t(\tau))}\,(1-\alpha(\tau)). \tag{81}$$

*Proof of Lemma 2.3.2.* $E(0,x) = \mathbf{E}(0,x) \asymp 1$ follows from $m_2(0) < +\infty$. For $t \ge 0$ we use the formula (81) to show that $0 \le \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{E}(\tau, \xi_{t,x}(t(\tau))) \le 3$. Since $0 \le e^{-\xi_{t,x}(t(\tau))}(1 - \alpha(\tau)) \le 1$ by (56) we only need to show

$$0 \le \frac{V'(x)^2}{V''(x)^2}(3V''(x) + V'(x)) = 3\frac{V'(x)^2}{V''(x)} + \frac{V'(x)^3}{V''(x)^2} \le 3\frac{V'(x)^2}{V''(x)} \le 3. \tag{82}$$

The lower bound follows from $3V''(x) + V'(x) = \sum_{k=1}^{\infty}(3k^2 - k)v_k e^{-kx} > 0$.

The upper bound follows from Schwarz's inequality:

$$\frac{V'(x)^2}{V''(x)} = \frac{\left(\sum_{k=1}^{\infty} k \cdot v_k e^{-kx}\right)^2}{\sum_{k=1}^{\infty} k^2 \cdot v_k e^{-kx}} \le \sum_{k=1}^{\infty} v_k e^{-kx} \le m_0 \le 1.$$

Integrating (81), using $0 \le \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{E}(\tau, \xi_{t,x}(t(\tau))) \le 3$, (62), (64) and the last inequality in (49) we obtain

$$E(0, \xi_{t,x}(0)) \le E(t,x) \le E(0, \xi_{t,x}(0)) + 3(t/2 + 1).$$

Next we observe that $x \le \xi_{t,x}(0) \le x + t$ by (77) and $-1 < V(t,x) \le 0$.

The last two bounds yield for $(t,x) \in [0,\overline{t}] \times (0,\overline{x}]$

$$0 < E_*(0, \overline{x} + \overline{t}) \le E(t,x) \le E^*(0, \overline{x} + \overline{t}) + 3(\overline{t}/2 + 1) < \infty.$$

$\square$

*Proof of Lemma 2.3.3.* $\theta(t) = -V(t, 0_+)$. Since $V(t,x)$ arises from (50), we assume $-1 < V(t,x) \leq 0$, $V'(t,x) < 0$ for all $x > 0$.

Let us pick an arbitrary $\overline{x} > 0$. Let $C$ be a constant such that $E(t,x) \leq C$ for $(t,x) \in [0,T] \times (0, \overline{x}]$.

First we are going to show that

$$\forall\, 0 \leq t \leq T\,, 0 < x \leq \overline{x}\ : \qquad V'V(t,x) := V'(t,x)V(t,x) \leq C^* := \max\{1, 2C\} \tag{83}$$

Note that we cannot use (67) here since that bound uses $V(t,0) = 0$. But $V(0,0) = 0$ holds, thus (83) holds for $t = 0$. From (59) and (79) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathbf{V}'\mathbf{V}(\tau,x)\right) = \\
&\left(-2\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)^2 - \mathbf{V}(\tau,x)^2\mathbf{V}''(\tau,x)\right)\alpha(\tau) + \left(\mathbf{V}'(\tau,x) - \mathbf{V}(\tau,x)\right)e^{-x}(1-\alpha(\tau)) \leq \\
&\qquad -\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)^2\left(2 - \frac{1}{C}\mathbf{V}'\mathbf{V}(\tau,x)\right)\alpha(\tau) + \left(\mathbf{V}'(\tau,x) - \mathbf{V}(\tau,x)\right)e^{-x}(1-\alpha(\tau))
\end{aligned}$$

From (60) we get

$$\mathbf{V}'\mathbf{V}(\tau,x) \geq 1 \quad \Longrightarrow \quad \mathbf{V}'(\tau,x) \leq \frac{1}{\mathbf{V}(\tau,x)} \leq -1 \leq \mathbf{V}(\tau,x)$$

Thus by (56) we get

$$\begin{aligned}
\mathbf{V}'\mathbf{V}(\tau,x) \geq 1 \quad &\Longrightarrow \quad \left(\mathbf{V}'(\tau,x) - \mathbf{V}(\tau,x)\right)e^{-x}(1-\alpha(\tau)) \leq 0 \\
\mathbf{V}'\mathbf{V}(\tau,x) \geq 2C \quad &\Longrightarrow \quad -\mathbf{V}(\tau,x)\mathbf{V}'(\tau,x)^2\left(2 - \frac{1}{C}\mathbf{V}'\mathbf{V}(\tau,x)\right)\alpha(\tau) \leq 0 \\
\mathbf{V}'\mathbf{V}(\tau,x) \geq C^* \quad &\Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathbf{V}'\mathbf{V}(\tau,x)\right) \leq 0
\end{aligned}$$

From $\mathbf{V}'\mathbf{V}(0,x) \leq C^*$ and the last differential inequality it easily follows by a "forbidden region"-argument that $\mathbf{V}'\mathbf{V}(\tau,x) \leq C^*$ for all $0 < x < \overline{x}$ and $0 \leq \tau \leq T + r(T)$. This and (62) implies (83).

By (52) and (83) we have

$$V(t_1,x) - V(t_2,x) \leq \int_{t_1}^{t_2} V(s,x)V'(s,x)\,\mathrm{d}s \leq C^* \cdot (t_2 - t_1)$$

for every $0 < x < \bar{x}$. Letting $x \to 0_+$ implies the claim of the Lemma. □

### 2.3.3 No giant component in the limit

The aim of this subsection is to prove the following proposition:

**Proposition 2.2.** *If $n^{-1} \ll \lambda(n) \ll 1$ and $m_2(0) < +\infty$ holds for $\mathbf{v}(0)$ on the right-hand side of (41) then any weak limit point $\mathbb{P}$ of the sequence of probability measures $\mathbb{P}_n$ is concentrated on the set of conservative forest fire evolutions:*

$$\mathbb{P}\big(\sum_{k=1}^{\infty} v_k(t) \equiv 1\big) = 1 \tag{84}$$

We are going to prove Proposition 2.2 by contradiction: in Lemma 2.3.4 we show that if $\theta(\cdot) \not\equiv 0$ in the limit, then there is a positive time interval such that $\theta(t)$ has a positive lower bound, and that this implies that even in the convergent sequence of finite-volume models, a lot of mass is contained in arbitrarily big components on this interval. Than in subsequent Lemmas we prove that these big components indeed burn, which produces such a big increase in the value of the burnt mass $r(\cdot)$ that is in contradiction with $\mathbf{E}\big(\, r(T)\,\big) \leq 2 + \mathbf{E}\big(\, q(T)\,\big) \leq 2 + T$.

By Proposition 2.1 the random FFE obtained as a weak limit point is almost deterministic: (46) holds with a possibly random control function $r(\cdot)$. Also, by (42) we $\mathbb{P}$-almost surely have $q(t) \leq t$ from which (49) follows. Thus (74) and (75) hold $\mathbb{P}$-almost surely for the random flow obtained as a weak limit point with a deterministic constant $C^*$.

**Lemma 2.3.4.** *If $\mathbb{P}_n \Rightarrow \mathbb{P}$ where $\mathbb{P}$ does not satisfy (84) on $[0, T]$, then there exist $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3 > 0$ and a deterministic $t^* \in [\varepsilon_1, T]$ such that for every $K < +\infty$, every $m < +\infty$ and every sequence*

$$t^* - \varepsilon_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t^*$$

*there exists an $n_0 < +\infty$ such that for every $n \geq n_0$ and $1 \leq i \leq m$ we have*

$$\mathbb{P}_n\left( \max_{\alpha_i \leq t \leq \beta_i} 1 - \sum_{k=1}^{K-1} v_{n,k}(t) > \varepsilon_2 \right) > \varepsilon_3. \tag{85}$$

**Lemma 2.3.5.** *If $n^{-1} \ll \lambda(n)$ then for every $\varepsilon_2 > 0$ there is a $\varepsilon_4 > 0$ such that for every $\tilde{t} > 0$ there is a $K$ and an $n_1$ such that for all $n \geq n_1$ $1 - \sum_{k=1}^{K-1} v_{n,k}(0) \geq \varepsilon_2$ implies*

$$\mathbb{E}_n\left(r_n(\tilde{t})\right) \geq \varepsilon_4 \tag{86}$$

Lemma 2.3.5 states that if initially a lot of mass is contained in big components, then in a short time a lot of mass burns. We prove this statement in two steps: in Lemma 2.3.6 we prove that if we start with a lot of mass contained in big components, then in a short time either a lot of this mass is burnt or the big components coagulate, so a lot of mass is contained in components of size $n^{1/3}$ (the same proof works if we replace the exponent $\alpha = 1/3$ by any $0 < \alpha < 1/2$). Then in Lemma 2.3.7 we prove that if we start with a lot of components of size $n^{1/3}$ then in a short time a lot of mass burns.

**Lemma 2.3.6.** *There are constants $C_1 < +\infty$, $C_2 > 0$, $C_3 > 0$ such that if*

$$1 - \sum_{k=1}^{K-1} v_{n,k}(0) \geq \varepsilon_2 \tag{87}$$

*for all $n$ then*

$$\lim_{n\to\infty} \mathbf{P}\Big( \sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}\left(\bar{t}\right) + r_n\left(\bar{t}\right) \geq C_2\varepsilon_2 \Big) = 1 \tag{88}$$

*where $\bar{t} = \frac{C_1}{K\varepsilon_2}$.*

**Lemma 2.3.7.** *There are constants $C_4 < +\infty$, $C_5 > 0$ such that if*

$$\sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(0) \geq C_2\varepsilon_2/2$$

*for all $n$ then with*

$$\bar{t}_n := C_4\varepsilon_2^{-2}\big(n^{-1/3}\log(n) + (n\lambda(n))^{-1}\big) \tag{89}$$

*we have*

$$\lim_{n\to\infty} \mathbf{E}\big( r_n(\bar{t}_n) \big) \geq C_5\varepsilon_2. \tag{90}$$

**Remark.** *The upper bound* (89) *is technical: on one hand it is not optimal, on the other hand, for the proof of Lemma 2.3.5 we only need $\bar{t}_n \ll 1$ as $n \to \infty$.*

The proof of Lemma 2.3.5 directly follows as a consequence of the Lemmas 2.3.6 and 2.3.7.

In the rest of this subsection we prove the proposition and lemmas stated above.

*Proof of Proposition 2.2.* We are going to show that if there is a sequence $\mathbb{P}_n$ such that the weak limit point $\mathbb{P}$ violates (84) then for some $n$ we have

$$\mathbb{E}_n\left(r_n(T)\right) > T + 2 \tag{91}$$

which is in contradiction with (44) and (35). In fact, $T+2$ could be replaced with any finite constant in (91), but $T+2$ is big enough to have a contradiction.

We define $\varepsilon_1$, $\varepsilon_2$, $\varepsilon_3 > 0$ and $t^*$ using Lemma 2.3.4. Next, we define $\varepsilon_4$ using this $\varepsilon_2$ and Lemma 2.3.5. Given these, we choose $\tilde{t}$ be so small that

$$\left\lfloor \frac{\varepsilon_1}{2\tilde{t}} \right\rfloor \varepsilon_3\varepsilon_4 > T + 2.$$

We choose $K$ and $n_1$ big enough so that (86) holds. Further on, we fix the intervals $[\alpha_i, \beta_i]$, $1 \leq i \leq m = \lfloor \frac{\varepsilon_1}{2\tilde{t}} \rfloor$ so that $\alpha_{i+1} - \beta_i > \tilde{t}$ holds for all $i$ and also $T - \beta_m > \tilde{t}$ holds. We choose $n_0$ such that (85) holds and let $n := \max\{n_0, n_1\}$.

Finally, we define the stopping times $\tau_1, \tau_2, \dots, \tau_m$ by

$$\tau_i := \beta_i \wedge \min\{t : t \geq \alpha_i \text{ and } 1 - \sum_{k=1}^{K-1} v_{n,k}(t) \geq \varepsilon_2\}.$$

We have $\tau_i + t^* \leq \beta_i + t^* < \alpha_{i+1} \leq \tau_{i+1}$.

Using the strong Markov property, (86) and (85), the inequality (91) follows:

$$\mathbf{E}\big(\, r_n(T)\,\big) \geq \sum_{i=1}^{m} \mathbf{E}\big(\, r_n(\tau_i + \tilde{t}) - r_n(\tau_i) \,\big|\, \tau_i < \beta_i \,\big)\mathbf{P}\big(\, \tau_i < \beta_i \,\big) \geq m\varepsilon_4\varepsilon_3.$$

□

*Proof of Lemma 2.3.4.* First we prove that if $\mathbb{P}$ does not satisfy (84) then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $\varepsilon_1 \leq t^* \leq T$ such that

$$\mathbb{P}\big( \inf_{t^*-\varepsilon_1 \leq t \leq t^*} \theta(t) > \varepsilon_2 \big) > \varepsilon_3. \tag{92}$$

Since (84) is violated, we have $\mathbb{P}\big( \sup_{0 \leq t \leq T} \theta(t) > \varepsilon \big) > \varepsilon$ for some $\varepsilon > 0$.

Let $L := \lfloor \frac{2C^*T}{\varepsilon} \rfloor$ and $t_i := \frac{\varepsilon i}{2C^*}$ for $1 \leq i \leq L$ where $C^*$ is the constant in (75). Since $\theta(0) = 0$ we have

$$\big\{ \sup_{0 \leq t \leq T} \theta(t) > \varepsilon \big\} \subseteq \bigcup_{i=1}^{L} \big\{ \theta(t_i) > \frac{\varepsilon}{2} \big\}$$

almost surely with respect to $\mathbb{P}$. Thus $\mathbb{P}\big(\theta(t^*) > \frac{\varepsilon}{2}\big) > \frac{\varepsilon}{L}$ for some $t^* \in \{t_1, \dots t_L\}$. Using (75) again (92) follows with $\varepsilon_1 := \frac{\varepsilon}{4C^*}$, $\varepsilon_2 := \frac{\varepsilon}{4}$, $\varepsilon_3 = \frac{\varepsilon}{L}$.

Now given $K$ and the intervals $[\alpha_i, \beta_i]$, $1 \leq i \leq m$ we define the continuous functionals $f_i : \mathcal{D}[0, T] \to \mathbb{R}$ by

$$f_i\left(\mathbf{v}(0), \mathbf{q}(\cdot), \mathbf{r}(\cdot)\right) := \frac{1}{\beta_i - \alpha_i} \int_{\alpha_i}^{\beta_i} \big(1 - \sum_{k=1}^{K} v_k(t)\big)\, \mathrm{d}t$$

where $v_k(t)$ is defined by (28). Thus for all $i$

$$H_i := \{(\mathbf{v}(0), \mathbf{q}(\cdot), \mathbf{r}(\cdot)) \in \mathcal{D}[0, T] : f_i\left(\mathbf{v}(0), \mathbf{q}(\cdot), \mathbf{r}(\cdot)\right) > \varepsilon_2\}$$

is an open subset of $\mathcal{D}[0, T]$ with respect to the topology of Definition 2.2.1. Thus by the definition of weak convergence of probability measures we have

$$\lim_{n \to \infty} \mathbb{P}_n(H_i) \geq \mathbb{P}(H_i) \geq \mathbb{P}\left( \inf_{t^*-\varepsilon_1 \leq t \leq t^*} \theta(t) > \varepsilon_2 \right) > \varepsilon_3$$

from which the claim of the lemma easily follows. □

We will make use of the following generating function estimates in the proof of Lemma 2.3.6. If $V(x)$ is defined as in (50) and if $\mathbf{v} \in \mathcal{V}_1$ then for $\varepsilon \leq \frac{1}{2}$

$$1 - \sum_{k=1}^{K-1} v_k \geq \varepsilon \quad \Longrightarrow \quad V(1/K) \leq (e^{-1} - 1)\varepsilon \tag{93}$$

$$V(1/K) \leq -\varepsilon \quad \Longrightarrow \quad 1 - \sum_{k=1}^{\varepsilon K/2} v_k \geq \varepsilon/4. \tag{94}$$

*Sketch proof of Lemma 2.3.6.* If we let $n \to \infty$ immediately, we get that the limiting functions $v_1(t), v_2(t), \dots$ solve (46), (47), (48) with a possibly random control function $r(t) \equiv r_\infty(t)$.

The $n \to \infty$ limit of (88) is

$$\theta(\bar{t}) + r(\bar{t}) \geq C_2 \varepsilon_2 \tag{95}$$

Now we prove that if $\mathbf{v}(\cdot)$ is a solution of (46), (47), (48) then $1 - \sum_{k=1}^{K-1} v_k(0) \geq \varepsilon_2$ implies (95) with $C_1 = 4$ and $C_2 = \frac{1}{4}$. This proof will also serve as an outline of the proof of Lemma 2.3.6.

In order to prove (95) define $V(t,x)$ by (50). Thus $V(t,x)$ solves the integrated Burgers control problem (52), (53), (54).

Define $U(t,x) := V(t,x) - r(t)e^{-x}$. Thus $U'(t,x) = V'(t,x) + r(t)e^{-x}$ and by (52) we have $\dot{U}(t,x) = -V(t,x)V'(t,x)$. Define the characteristic curve $\xi(\cdot)$ by

$$\dot{\xi}(t) = V(t, \xi(t)) \qquad \xi(0) = \frac{1}{K} \tag{96}$$

Let $u(t) := U(t,\xi(t)) - V(0, \frac{1}{K})$. Thus $u(0) = 0$, and

$$\begin{aligned} \dot{u}(t) = \dot{U}(t,\xi(t)) + U'(t,\xi(t))\dot{\xi}(t) = -V(t,\xi(t))V'(t,\xi(t)) + \\ \left(V'(t,\xi(t)) + r(t)e^{-\xi(t)}\right) V(t,\xi(t)) = r(t)e^{-\xi(t)} V(t,\xi(t)) \leq 0. \end{aligned} \tag{97}$$

Thus $u(t) \leq 0$, moreover

$$V(t,\xi(t)) = V(0,\frac{1}{K}) + r(t)e^{-\xi(t)} + u(t) \leq V(0,\frac{1}{K}) + r(t), \tag{98}$$

$$\xi(t) = \frac{1}{K} + \int_0^t u(s)\,\mathrm{d}s + \int_0^t r(s)e^{-\xi(s)}\,\mathrm{d}s + tV(0,\frac{1}{K}) \leq \frac{1}{K} + t \cdot r(t) + tV(0,\frac{1}{K}). \tag{99}$$

By (93) we have $V(0,\frac{1}{K}) \leq -\frac{1}{2}\varepsilon_2$. In order to prove that $\theta(\bar{t}) + r(\bar{t}) \geq \frac{1}{4}\varepsilon_2$ with $\bar{t} = \frac{4}{K\varepsilon_2}$ we consider two cases:

If $r(\bar{t}) \geq \frac{1}{4}\varepsilon_2$ then we are done. If $r(\bar{t}) < \frac{1}{4}\varepsilon_2$ define $\tau := \min\{t : \xi(t) = 0\}$. By (99) we have

$$\xi(\bar{t}) \leq \frac{1}{K} + \bar{t} \cdot r(\bar{t}) + \bar{t} \cdot \left(-\frac{1}{2}\varepsilon_2\right) < \frac{1}{K} + \frac{1}{K} - \frac{2}{K} = 0$$

Thus $\tau \leq \bar{t}$. By (98) we get

$$-\theta(\tau) = V(\tau, 0) = V(\tau, \xi(\tau)) \leq -\frac{1}{2}\varepsilon_2 + \frac{1}{4}\varepsilon_2 = -\frac{1}{4}\varepsilon_2$$

Thus $\frac{1}{4}\varepsilon_2 \leq \theta(\tau) \leq \theta(\tau) + r(\tau) \leq \theta(\bar{t}) + r(\bar{t})$ because by (30) the function $\theta(t) + r(t)$ is increasing. $\square$

To make this proof work for Lemma 2.3.6 we have to deal with the fluctuations caused by randomness, combinatorial error terms and the fact that $\lambda(n)$ only disappears in the limit.

*Proof of Lemma 2.3.6.* Given a FFF obtained from a forest fire Markov process by (38),(39) and (40), define

$$U_n(t, x) := \sum_{k=1}^{n} \left[ v_{n,k}(0) + \frac{k}{2} \sum_{l=1}^{k-1} q_{n,l,k-l}(t) - k q_{n,k}(t) - r_{n,k}(t) \right] e^{-kx} - 1 - \lambda(n)$$

By (28) we have

$$U_n(t, x) + r_n(t) e^{-x} = \sum_{k=1}^{n} v_{n,k}(t) e^{-kx} - 1 - \lambda(n) =: V_n(t, x) - \lambda(n) =: W_n(t, x).$$

$$\begin{aligned}
W'(t, x) &= -\sum_{k \geq 1} k \cdot v_{n,k}(t) e^{-kx} \\
-\frac{1}{2} \partial_x \left( W(t, x) + 1 + \lambda(n) \right)^2 &= \sum_{k \geq 1} \frac{k}{2} \sum_{l=1}^{k-1} v_{n,l}(t) v_{n,k-l}(t) e^{-kx} \\
W''(t, x) &= \sum_{k \geq 1} k^2 \cdot v_{n,k}(t) e^{-kx} \\
W''(t, 2x) &= \sum_{k \geq 1} \left( \frac{k}{2} \right)^2 \cdot \mathbb{1}[2 \mid k] \cdot v_{n,\frac{k}{2}}(t) e^{-kx}
\end{aligned}$$

If $X(t)$ is a process adapted to the filtration $\mathcal{F}(t)$, let

$$L\, X(t) := \lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbf{E}\big( X(t + \mathrm{d}t) - X(t) \,\big|\, \mathcal{F}_t \big)$$

Using the martingales of Proposition 2.1 we get

$$L\,U_n(t,x) = \sum_{k\geq 1}\left[\frac{k}{2}\sum_{l=1}^{k-1} L\,q_{n,l,k-l}(t) - k\cdot L\,q_{n,k}(t) - L\,r_{n,k}(t)\right]e^{-kx} =$$

$$\sum_{k\geq 1}\left[\frac{k}{2}\sum_{l=1}^{k-1}\left(v_{n,l}(t)v_{n,k-l}(t) - \frac{l\cdot \mathbb{1}[2l=k]}{n}v_{n,l}(t)\right) - \right.$$

$$\left. k\cdot\left(v_{n,k}(t) - \frac{k}{n}v_{n,k}(t)\right) - (\lambda(n)\cdot k\cdot v_{n,k}(t))\right]e^{-kx} =$$

$$-\frac{1}{2}\partial_x\left(W(t,x)+1+\lambda(n)\right)^2 - \frac{1}{n}W''(t,2x) +$$

$$W'(t,x) + \frac{1}{n}W''(t,x) + \lambda(n)W'(t,x) =$$

$$-W_n'(t,x)W_n(t,x) + \frac{1}{n}\left(W_n''(t,x) - W_n''(t,2x)\right) \quad (100)$$

Given the random function $W_n(t,x)$ we define the random characteristic curve $\xi_n(t)$ similarly to (96):

$$\dot{\xi}_n(t) = W_n(t,\xi_n(t)), \qquad \xi_n(0) := \frac{1}{K} \quad (101)$$

This ODE is well-defined although $W_n(t,x)$ is not continuous in $t$, but almost surely it is a step function with finitely many steps which is a sufficient condition to have well-posedness for the solution of (101). Define $u_n(t) := U_n(t,\xi_n(t)) - W_n(0,\frac{1}{K})$. Thus $u_n(0)=0$ and

$$u_n(t) = W_n(t,\xi_n(t)) - W_n(0,\frac{1}{K}) - r_n(t)e^{-\xi_n(t)} = V_n(t,\xi_n(t)) - V_n(0,\frac{1}{K}) - r_n(t)e^{-\xi_n(t)} \quad (102)$$

The solution of (101) is

$$\xi_n(t) = \frac{1}{K} + \int_0^t u_n(s)\,\mathrm{d}s + \int_0^t r_n(s)e^{-\xi_n(s)}\,\mathrm{d}s + tW_n(0,\frac{1}{K}) \quad (103)$$

Putting together (100) and (101) similarly to (97) and using (65) we get

$$L\,u_n(t) \leq \frac{1}{n}\left(W_n''(t,\xi_n(t)) - W_n''(t,2\xi_n(t))\right) \leq n^{-1}\cdot\xi_n(t)^{-2} \quad (104)$$

Now $\widetilde{u}_n(t) = u_n(t) - \int_0^t L\,u_n(s)\,\mathrm{d}s$ is a martingale and

$$L\,\widetilde{u}_n(t)^2 = \lim_{h\to 0_+}\frac{1}{h}\mathbf{E}\Big(\big(U_n(t+h,\xi_n(t)) - U_n(t,\xi_n(t))\big)^2\,\Big|\,\mathcal{F}_t\Big) \leq$$

$$\frac{1}{2}\sum_{k,l=1}^{n}\left(\frac{k+l}{n}e^{-(k+l)\xi_n(t)} - \frac{k}{n}e^{-k\xi_n(t)} - \frac{l}{n}e^{-l\xi_n(t)}\right)^2 v_{n,k}(t)v_{n,l}(t)n$$

$$+\sum_{l=1}^{n}\left(\frac{l}{n}e^{-l\xi_n(t)}\right)^2\lambda(n)v_{n,l}(t)n = \mathcal{O}\left(\frac{1}{n}W_n''(t,\xi_n(t))\right) = \mathcal{O}\left(n^{-1}\cdot\xi_n(t)^{-2}\right) \quad (105)$$

Define the stopping time

$$\tau_n := \min\{t : \xi_n(t) = n^{-\alpha}\} \quad \alpha = 1/3.$$

In fact any $0 < \alpha < 1/2$ would be just as good to make the right-hand side of (104) and (105) disappear when $t \le \tau_n$ and $n \to \infty$.

It follows from (105) and Doob's maximal inequality that

$$\sup_t |\widetilde{u}_n(t \wedge \tau_n \wedge T)| \Rightarrow 0 \quad \text{as} \quad n \to \infty$$

By (104) we have $\widetilde{u}_n(t) + \int_0^t n^{-1} \cdot \xi_n(s)^{-2}\, \mathrm{d}s \ge u_n(t)$ thus

$$\sup_t u_n(t \wedge \tau_n \wedge T) \Rightarrow 0 \quad \text{as} \quad n \to \infty \tag{106}$$

By (93) and (87) we have

$$V_n(0, \frac{1}{K}) \le (e^{-1} - 1)\varepsilon_2 =: -\varepsilon_5 \tag{107}$$

Define the events $A_n$, $B_n$ and the time $\bar{t}_n$ by

$$\begin{aligned}
A_n &:= \big\{ \sup_{t \le \tau_n \wedge T} \int_0^t u_n(s)\, \mathrm{d}s \le \frac{1}{K} \big\} \cap \big\{ u_n(\tau_n \wedge T) \le \varepsilon_5/3 \big\}, \\
B_n &:= \big\{ r_n(\tau_n) \le \varepsilon_5/3 \big\}, \\
\bar{t}_n &:= \frac{3}{K\, |W_n(0, \xi_n(0))|} \le \frac{3}{K\varepsilon_5},
\end{aligned}$$

We are going to show that that there are constants $C_2, C_3 < +\infty$ such that

$$A_n \subseteq \big\{ \sum_{k = C_3 \varepsilon_2 n^{1/3}}^{n} v_{n,k}\,(\bar{t}) + r_n\,(\bar{t}) \ge C_2 \varepsilon_2 \big\} \tag{108}$$

which, since (106) implies that $\lim_{n\to\infty} \mathbf{P}\big( A_n \big) = 1$, gives (88).

First we show that

$$A_n \cap B_n \subseteq \{\tau_n \le \bar{t}_n\}. \tag{109}$$

If we assume indirectly that $A_n$, $B_n$ and $\tau_n > \bar{t}_n$ hold then $\int_0^{\bar{t}_n} u_n(s)\, \mathrm{d}s \le \frac{1}{K}$, so by (103) we get

$$\xi_n(\bar{t}_n) \le \frac{1}{K} + \frac{1}{K} + \int_0^{\bar{t}_n} r_n(s) e^{-\xi_n(s)}\, \mathrm{d}s + \bar{t}_n W_n(0, \xi_n(0)) \le -\frac{1}{K} + \bar{t}_n \cdot r_n(\tau_n) \le 0.$$

But $\xi_n(\bar{t}_n) \le 0$ is in contradiction with $\tau_n > \bar{t}_n$, thus (109) holds.

Now, by (102) we have $V_n(\tau_n, n^{-1/3}) = u_n(\tau_n) + V_n(0, \frac{1}{K}) + r_n(\tau_n)e^{-n^{-1/3}}$. Thus by (107), the definition of $A_n$ and $B_n$ and (94) we get

$$A_n \cap B_n \subseteq \big\{u_n(\tau_n) \le \frac{\varepsilon_5}{3}\big\} \cap \big\{V_n(0, \frac{1}{K}) \le -\varepsilon_5\big\} \cap \big\{r_n(\tau_n)e^{-n^{-1/3}} \le \frac{\varepsilon_5}{3}\big\} \subseteq$$
$$\big\{V_n(\tau_n, n^{-1/3}) \le \frac{-\varepsilon_5}{3}\big\} \subseteq \big\{ \sum_{k=n^{1/3}\varepsilon_5/6}^{n} v_{n,k}(\tau_n) \ge \varepsilon_5/12\big\}$$

Thus we have

$$A_n \subseteq (A_n \cap B_n) \cup B_n^c \subseteq \big\{ \sum_{k=n^{1/3}\varepsilon_5/6}^{n} v_{n,k}(\tau_n) \ge \varepsilon_5/12\big\} \cup \big\{r_n(\tau_n) > \varepsilon_5/3\big\} \subseteq$$
$$\big\{ \sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(\tau_n) + r_n(\tau_n) \ge C_2\varepsilon_2\big\}$$

with $C_3 = (1 - e^{-1})/6$ and $C_2 = (1 - e^{-1})/12$. But $\sum_{k=C_3\varepsilon_2 n^{1/3}}^{n} v_{n,k}(t) + r_n(t)$ increases with time, from which (108) follows. □

*Proof of Lemma 2.3.7.* If $v$ is a vertex of the graph $G(n, t)$ let $\mathcal{C}_n(v, t)$ denote the connected component of $v$ at time $t$. Denote by $\tau_b(v)$ the first burning time of $v$:

$$\tau_b(v) := \inf\{t \,:\, |\mathcal{C}_n(v, t_+)| < |\mathcal{C}_n(v, t_-)|\}$$

Of course $|\mathcal{C}_n(v, \tau_b(v)_+)| = 1$. Define $\bar{n} := C_3\varepsilon_2 n^{1/3}$ and

$$\mathcal{H}_n(t) := \{v \,:\, |\mathcal{C}_n(v, 0)| \ge \bar{n} \text{ and } \tau_b(v) > t\}$$

Fix a vertex $v \in \mathcal{H}_n(0)$.

$$\begin{aligned} c_n(t) &:= \frac{1}{n}\,|\mathcal{C}_n(v, (t \wedge \tau_b(v))_-)| \\ w_n(t) &:= \frac{1}{n}\,|\mathcal{H}_n(t)| \\ z_n(t) &:= \frac{1}{n} \sum_{w \in \mathcal{H}_n(0)} \mathbb{1}[\tau_b(w) \le t] = w_n(0) - w_n(t) \end{aligned}$$

Thus $c_n(t)$ is an increasing process (we "freeze" $c_n(t)$ when it burns). We consider the right-continuous versions of the processes $c_n(t), w_n(t), z_n(t)$.

$$w_n(0) \ge C_2\varepsilon_2/2 =: \varepsilon_6.$$

We are going to prove that there are constants $C_4 < +\infty$, $C_5 > 0$ such that

$$\lim_{n\to\infty} \mathbf{E}\big(\, z_n(\bar{t}_n)\,\big) \ge C_5\varepsilon_2 \tag{110}$$

which implies (90).

Define the stopping times

$$\tau_w := \inf\{t : w_n(t) < \varepsilon_6/2\}$$
$$\tau_g := \inf\{t : c_n(t) > \varepsilon_6/4\}$$
$$\tau := \tau_b(v) \wedge \tau_w \wedge \tau_g$$

Since $v \in \mathcal{H}_n(0)$ we have

$$c_n(t) \geq c_n(0) = \frac{|\mathcal{C}_n(v,0)|}{n} \geq \frac{\bar{n}}{n}$$

If $\mathcal{C}_n(v,t)$ is connected to a vertex in $\mathcal{H}_n(t)$ by a new edge at time $t$ then

$$c_n(t_+) - c_n(t_-) \geq \frac{\bar{n}}{n}, \quad \log(c_n(t_+)) - \log(c_n(t_-)) \geq \log\left(1 + \frac{\bar{n}}{nc_n(t_-)}\right) \geq \frac{\log(2)\bar{n}}{nc_n(t_-)}$$

$$L\,\log(c_n(t)) \geq \frac{\log(2)\bar{n}}{nc_n(t)} \lim_{\mathrm{d}t\to 0} \frac{1}{\mathrm{d}t}\mathbf{P}\big(\, c_n(t+\mathrm{d}t) - c_n(t) \geq \frac{\bar{n}}{n} \,\big|\, \mathcal{F}_t \,\big) \geq$$
$$\frac{\log(2)\bar{n}}{nc_n(t)} \cdot \frac{1}{n} |\mathcal{C}_n(v,t)| \left(|\mathcal{H}_n(t)| - |\mathcal{C}_n(v,t)|\right) \mathbb{1}[t \leq \tau_b(v)] \geq \log(2)\bar{n}\cdot(w_n(t) - c_n(t))\, \mathbb{1}[t \leq \tau_b(v)] \geq$$
$$\log(2)\bar{n}\frac{\varepsilon_6}{4}\mathbb{1}[t \leq \tau] = n^{1/3}\frac{\log(2)}{8} \cdot C_2 \cdot C_3 \cdot (\varepsilon_2)^2 \cdot \mathbb{1}[t \leq \tau] =: n^{1/3}\varepsilon_7 \mathbb{1}[t \leq \tau]$$

Thus $\log(c_n(t)) - \varepsilon_7 \cdot n^{1/3}(t \wedge \tau)$ is a submartingale. Using the optional sampling theorem we get

$$-\varepsilon_7 \cdot n^{1/3}\mathbf{E}\big(\,\tau\,\big) \geq \mathbf{E}\big(\,\log(c_n(\tau))\,\big) - \varepsilon_7 \cdot n^{1/3}\mathbf{E}\big(\,\tau\,\big) \geq \log(c_n(0)) \geq -\log(n)$$

By Markov's inequality we obtain that for some constant $C < +\infty$

$$\mathbf{P}\big(\,\tau \leq Cn^{-1/3}\varepsilon_2^{-2}\log(n)\,\big) \geq \frac{1}{2}$$

If $\tau_g \leq \tau_b(v) \wedge \tau_w$, then $\mathcal{C}_n(v,\tau_g) > \frac{\varepsilon_6}{4}n$, so $\mathbf{E}\big(\,\tau_b(v) - \tau_g\,\big) \leq (n\lambda(n))^{-1}\frac{4}{\varepsilon_6}$, which implies

$$\mathbf{P}\big(\,\tau_w \wedge \tau_b \leq Cn^{-1/3}\varepsilon_2^{-2}\log(n) + C'(n\lambda(n))^{-1}\varepsilon_2^{-1}\,\big) \geq \frac{1}{4}.$$

for some constant $C'$. We define $\bar{t}$ of (89) with $C_4 := \max\{C, C'\}$. Using the linearity of expectation we get

$$\mathbf{E}\big(\,z_n(\bar{t})\,\big) = \mathbf{E}\big(\,\frac{1}{n}\sum_{w \in \mathcal{H}_n(0)} \mathbb{1}[\tau_b(w) \leq \bar{t}]\,\big) \geq \varepsilon_6 \mathbf{P}\big(\,\tau_b(v) \leq \bar{t}\,\big).$$

The inequality $\mathbb{1}[\tau_w \leq \bar{t}]\frac{\varepsilon_6}{2} \leq z_n(\bar{t})$ follows from the definition of $\tau_w$.

$$\frac{1}{4} \leq \mathbf{P}\big(\,\tau_w \wedge \tau_b \leq \bar{t}\,\big) \leq \mathbf{P}\big(\,\tau_w \leq \bar{t}\,\big) + \mathbf{P}\big(\,\tau_b \leq \bar{t}\,\big) \leq \mathbf{E}\big(\,z_n(\bar{t})\,\big)\frac{2}{\varepsilon_6} + \mathbf{E}\big(\,z_n(\bar{t})\,\big)\frac{1}{\varepsilon_6}$$

From this (110) follows. □

## 2.4 The critical equation

The aim of this section is to prove Theorem 2.1 by applying methods from the theory of first order nonlinear PDE to the Burgers control problem.

### 2.4.1 Permanent criticality

In this subsection we prove that any solution of the integrated Burgers control problem satisfying the boundary condition $V(t,0) \equiv 0$ also satisfies $V'(t,0) = -\infty$ for all $t \geq T_{\text{gel}}$. Thus by Lemma 2.3.1 for all $t \geq T_{\text{gel}}$ the system is critical, i.e. (25) holds. At the end of the subsection we summarize the results in Lemma 2.4.2.

Existence to the solutions of (46), (48) with initial condition satisfying $m_2(0) < +\infty$ and boundary condition

$$\sum_{k=1}^{\infty} v_k(t) \equiv 1 \tag{111}$$

follows as corollary to Propositions 2.1 and 2.2: indeed for any initial condition $\mathbf{v}_0 \in \mathcal{V}_1$ we can prepare a sequence of initial conditions of the random graph problem such that (41) holds as $n \to \infty$ (we do not need to assume convergence of $m_{n,2}(0)$ to $m_2(0)$). If $n^{-1} \ll \lambda(n) \ll 1$ then any weak limit of the probability measures $\mathbb{P}_n$ is concentrated on a subset of FFFs which generate a FFE satisfying (46), (111).

Moreover it is easily seen that (111) implies that $r(\cdot)$ must be continuous, and for $k \geq 2$, the functions $t \mapsto v_k(t)$ solving (46) are differentiable. Thus $\mathbf{v}(\cdot)$ solves (17), (18).

Note that assuming that $\mathbf{v}(\cdot) \in \mathcal{E}_{\mathbf{v}_0}[0,T]$ is a solution of (17),(18) one can deduce only from these equations that (46) holds with a control function $r(\cdot)$ satisfying (49): one has to define a FFF using (42) and $q_{k,\infty}(\cdot) \equiv 0$: plugging $\theta(t) \equiv 0$ into (30) we can see that the function $r(\cdot)$ is increasing.

Taking the generating function of a solution of (46), (48), (111) with initial condition satisfying $m_2(0) < +\infty$ we get a solution of (52), (54) satisfying the boundary condition $V(t,0) \equiv 0$.

In this case the increasing function $t \mapsto r(t)$ is absolutely continuous with respect to Lebesgue measure: its Radon-Nykodim derivative $\dot{r}(t) = \varphi(t)$ is a.e. bounded in compact domains:

Taking the limit $x \to 0$ in (52) and using (74), (67) (which holds because $V(t,0) \equiv 0$) we find

$$r(t_2) - r(t_1) = \lim_{x \to 0} \frac{1}{2} \int_{t_1}^{t_2} V(s,x) V'(s,x) \, \mathrm{d}s \leq C \cdot (t_2 - t_1). \tag{112}$$

Thus in the sequel we assume given a solution of the *critical Burgers control problem*

$$\dot{V}(t,x) = -V'(t,x) V(t,x) + e^{-x} \varphi(t), \tag{113}$$

$$V(t,0) \equiv 0 \tag{114}$$

$$V(0,x) = V_0(x) \tag{115}$$

where $\varphi(t)$ is nonnegative and bounded on $[0,T]$, and $V(t,x)$ is of the form (50).

**Lemma 2.4.1.** *For any solution of* (113)*,* (115)*,* (114) *with* $V''(0) < +\infty$ *and for any* $t \geq T_{gel}$ *(see* (13)*) we have* $V'(t,0) := \lim_{x\to 0} V'(t,x) = -\infty$.

*Proof.* We actually prove that for any $\bar{t} < \infty$, $\bar{x} < \infty$ there exists a constant $C = C(\bar{t},\bar{x}) > 0$ such that for any $(t,x) \in [T_{\mathrm{gel}},\bar{t}] \times (0,\bar{x}]$, $-V'(t,x) \geq C/\sqrt{x}$.

One can prove the upper bound of (69) for all $V(x)$ satisfying $V(0) = 0$ without the assumption (68) (the same proof works).

From (74) and the upper bound of (69) it follows that there exists a constant $\widetilde{C} < \infty$ such that for $(t,x) \in [T_{\mathrm{gel}},\bar{t}] \times (0,\bar{x}]$

$$E(t,x)^{-1} \leq \widetilde{C}, \qquad\qquad -V(t,x) \leq \widetilde{C} x^{1/2}.$$

Differentiating with respect to $x$ in (113) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-V'(t,x)\right) = V'(t,x)^2 + V(t,x)V''(t,x) + e^{-x}\varphi(t) =$$

$$V'(t,x)^2 \cdot \left(1 - \frac{V(t,x)V'(t,x)}{E(t,x)}\right) + e^{-x}\varphi(t) \geq V'(t,x)^2 \left(1 - \widetilde{C}^2 x^{1/2} \cdot (-V'(t,x))\right) \quad (116)$$

There exists a $0 < \widehat{C}$ such that for $x \in (0,\bar{x}]$ we have

$$-V'(T_{\mathrm{gel}},x) \geq \widehat{C}/\sqrt{x} \quad (117)$$

by (70) and (74), since $V'(T_{\mathrm{gel}},0) = -\infty \iff m_1(T_{\mathrm{gel}}) = +\infty$ follows from the fact that for $t \leq T_{\mathrm{gel}}$ the solutions of (10) and (17)+(18) coincide, and it is well-known from the theory of the Smoluchowski coagulation equations that we have (14) for the solution of (10).

From the differential inequality (116) it follows that

$$-V'(t,x) \leq \frac{1}{\widetilde{C}} x^{-1/2} \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(-V'(t,x)\right) \geq 0 \quad (118)$$

Let $C := \min\{\widehat{C}, \widetilde{C}^{-1}\}$. For $(t,x) \in [T_{\mathrm{gel}},\bar{t}] \times (0,\bar{x}]$ the inequality

$$-V'(t,x) \geq C/\sqrt{x}.$$

follows from (117) and (118) by a "forbidden region"-argument.

□

Summarizing: from Lemmas 2.3.1, 2.3.2, 2.4.1 and (112) it follows

**Lemma 2.4.2.** *For* $(t,x) \in [T_{gel},\bar{t}] \times (0,\bar{x}]$

$$-V(t,x) \asymp x^{1/2}, \quad (119)$$

$$-V'(t,x) \asymp x^{-1/2}, \quad (120)$$

$$V''(t,x) \asymp x^{-3/2}, \quad (121)$$

$$V(t,x)V'(t,x) \asymp 1, \quad (122)$$

$$\varphi(t) \asymp 1. \quad (123)$$

### 2.4.2 Bounds on $E'$

In this subsection we assume given a solution of (113), (114), (115) satisfying $|V'''(0,0)| < +\infty$. All of the results of the previous subsection are valid for $V(t,x)$.

**Lemma 2.4.3.**

$$E'(T_{gel}, x) = \mathcal{O}(x^{-1/2}) \tag{124}$$

From now on, we consider the solution of (113), (114), (115) for $t \geq T_{\text{gel}}$, that is we assume that $T_{\text{gel}} = 0$.

**Lemma 2.4.4.** *The function $(t,x) \mapsto E(t,x)$ is continuous on the domain $(t,x) \in [0,\bar{t}] \times [0,\bar{x}]$, and*

$$\varphi(t) = \lim_{x\to 0} V'(t,x)V(t,x) = E(t,0). \tag{125}$$

**Lemma 2.4.5.**

*(i) The function $x \mapsto E(t,x)$ is Hölder-$1/2$ at $x \to 0$:*

$$E(t,x) = \varphi(t)\big(1 + \mathcal{O}(x^{1/2})\big). \tag{126}$$

*(ii) The function $t \mapsto \varphi(t)$ is Lipschitz continuous: there exists a constant $C < +\infty$ (which depends only on the initial conditions (115) and the choice of $\bar{t}$ such that for any $t_1, t_2 \in [0,\bar{t}]$*

$$|\varphi(t_1) - \varphi(t_2)| \leq C|t_1 - t_2|. \tag{127}$$

In the rest of this subsection we prove the lemmas stated above. At the end of the subsection we summarize our results on the boundary behavior of the solution of (113), (114), (115) satisfying (124) in Proposition 2.3.

*Proof of Lemma 2.4.3.* We consider the function $X(t,u)$ defined for every $t$ as in the proof of Lemma 2.3.1. $X'''(0,u) = \mathcal{O}(1)$ for $u \in [0,\bar{u}]$ by $m_1(0) > 0$ and $m_3(0) < +\infty$. For $t \leq T_{\text{gel}}$ we have $\varphi(t) \equiv 0$ thus $V(t,x)$ satisfies the Burgers equation

$$\dot{V}(t,x) + V(t,x)V'(t,x) = 0$$

from which

$$X(t,u) = X(0,u) - tu$$

follows. This trick (transforming a quasilinear PDE into a linear PDE by exchanging the roles of independent and dependent variables) is known as the hodograph transform, see 4.4.3. of [23].

Differentiating (72) with respect to $x$ we get

$$E'(T_{\text{gel}}, x) = E(T_{\text{gel}}, x)^2 X'''(0, -V(T_{\text{gel}}, x))V'(T_{\text{gel}}, x).$$

Now (124) follows from (74) and (70). $\square$

Since the function $r(t)$ is continuous we get that $t(\tau)$ defined by (55) is the inverse function of $t + r(t)$ which by (57) implies $\mathbf{V}(\tau, x) \equiv V(t(\tau), x)$. Integrating (81) and using (62), (64) we get for $0 \le t_1 \le t_2 < \infty$

$$E(t_2, x) = E(t_1, \xi_{t_2,x}(t_1)) + \int_{t_1}^{t_2} \{3\frac{V'(s, \xi_{t_2,x}(s))^2}{V''(s, \xi_{t_2,x}(s))} + \frac{V'(s, \xi_{t_2,x}(s))^3}{V''(s, \xi_{t_2,x}(s))^2}\} e^{-\xi_{t_2,x}(s)} \varphi(s)\, \mathrm{d}s \tag{128}$$

$$= E(t_1, \xi_{t_2,x}(t_1)) + \int_{t_1}^{t_2} \{ - 3\frac{E(s, \xi_{t_2,x}(s))}{V'(s, \xi_{t_2,x}(s))} + \frac{E(s, \xi_{t_2,x}(s))^2}{V'(s, \xi_{t_2,x}(s))^3}\} e^{-\xi_{t_2,x}(s)} \varphi(s)\, \mathrm{d}s. \tag{129}$$

*Proof of Lemma 2.4.4.* From (123) and (76) it follows that the characteristic curves $\xi_{t,x}(s)$ are jointly continuous in the variables $\{(t, x, s) : 0 \le s \le t,\ \ 0 \le x\}$. And hence, further on, from (128) and (82), by dominated convergence it follows that $(t, x) \mapsto E(t, x)$ is jointly continuous in $\{(t, x) : 0 \le t,\ \ 0 \le x\}$. Further, from (71) it follows that

$$\lim_{x \to 0} V(t, x)V'(t, x) = \lim_{x \to 0} E(t, x) =: E(t, 0)$$

Hence, (125) follows from (112) again by dominated convergence. □

*Proof of Lemma 2.4.5.*

*(i) We prove $|E'(t, x)| = \mathcal{O}(x^{-1/2})$. In this order we shall use the following a priori estimates*

$$\xi_{t,x}(s) \asymp \left(x^{1/2} + (t - s)\right)^2 \tag{130}$$

$$\xi'_{t,x}(s) := \partial_x \xi_{t,x}(s) = \mathcal{O}\left(\left(x^{1/2} + (t - s)\right)x^{-1/2}\right). \tag{131}$$

*Indeed:* (130) *follows from* (76), (119) *and* (123), *and we get* (131) *from* (120) *and from the fact that characteristics do not intersect (thus $0 \le \xi'_{t,x}(s)$) by differentiating* (76) *w.r.t. $x$:*

$$0 \le \xi'_{t,x}(s) \le 1 - V'(t, x)(t - s)$$

*The a priori bound*

$$|E'(t, x)| = \mathcal{O}(x^{-1}). \tag{132}$$

*follows from*

$$E'(t, x) = -3V'(t, x)^2 + E(t, x)\frac{-V'''(t, x)}{V''(t, x)} = \mathcal{O}((x^{-1/2})^2) + \mathcal{O}(x^{-1})$$

*by* (120), (74) *and*

$$-\frac{x}{2}V'''(t, x) \le \int_{\frac{x}{2}}^{x} V'''(y)\, \mathrm{d}y \le V''(\frac{x}{2}) = \mathcal{O}(x^{-3/2})$$

*using both the upper and lower bounds of* (121).

*Differentiating with respect to $x$ in* (129) *yields*

$$E'(t,x) = E'(0,\xi_{t,x}(0))\xi'_{t,x}(0) + \tag{133}$$
$$+ \int_0^t \Big\{ -3\frac{E'(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))} + 3\frac{E(s,\xi_{t,x}(s))V''(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))^2}$$
$$+ 2\frac{E(s,\xi_{t,x}(s))E'(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))^3} - 3\frac{E(s,\xi_{t,x}(s))^2V''(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))^4}$$
$$+ 3\frac{E(s,\xi_{t,x}(s))}{V'(s,\xi_{t,x}(s))} - \frac{E(s,\xi_{t,x}(s))^2}{V'(s,\xi_{t,x}(s))^3}\Big\}\xi'_{t,x}(s)e^{-\xi_{t,x}(s)}\varphi(s)\,\mathrm{d}s.$$

*Next using* (132) *bound we estimate the expression of $E'(t,x)$ given in* (133). *Using* (74), (120), (121), (124), (130), *and* (131) *we conclude that if* (132) *holds then actually*

$$|E'(t,x)| = \mathcal{O}(x^{-1/2}). \tag{134}$$

*The dominating order is given by the first term (outside the integral) and the first two terms under the integral on the right hand side of* (133).

*Finally,* (126) *follows from* (125) *and* (134).

*(ii) In order to prove* (127) *we note that from* (128) *and* (125) *it follows that for* $0 \le t_1 \le t_2 \le \bar{t}$

$$\varphi(t_1) - \varphi(t_2) = E(t_1,0) - E(t_1,\xi_{t_2,0}(t_1))$$
$$- \int_{t_1}^{t_2} \Big\{3\frac{V'(s,\xi_{t_2,0}(s))^2}{V''(s,\xi_{t_2,0}(s))} + \frac{V'(s,\xi_{t_2,0}(s))^3}{V''(s,\xi_{t_2,0}(s))^2}\Big\}e^{-\xi_{t_2,0}(s)}\varphi(s)\,\mathrm{d}s$$

*Hence, by* (126), (130) *and* (82) *we obtain directly* (127). □

Summarizing again, from Lemmas 2.3.1, 2.3.2, 2.4.1, 2.4.4 and 2.4.5 it follows

**Proposition 2.3.** *For a solution of* (113), (115), (114) *with initial condition satisfying*

$T_{gel} = 0$, (74) *and* (124) *and for* $(t,x) \in [0,\bar{t}] \times (0,\bar{x}]$

$$-V(t,x) = \sqrt{2\varphi(t)}x^{1/2}\big(1 + \mathcal{O}(x^{1/2})\big), \tag{135}$$

$$-V'(t,x) = \sqrt{\frac{\varphi(t)}{2}}x^{-1/2}\big(1 + \mathcal{O}(x^{1/2})\big), \tag{136}$$

$$V''(t,x) = \sqrt{\frac{\varphi(t)}{8}}x^{-3/2}\big(1 + \mathcal{O}(x^{1/2})\big), \tag{137}$$

$$V(t,x)V'(t,x) = \varphi(t)\big(1 + \mathcal{O}(x^{1/2})\big). \tag{138}$$

$$\dot{V}(t,x) = \mathcal{O}(x^{1/2}), \tag{139}$$

$$\dot{V}'(t,x) = \mathcal{O}(x^{-1/2}), \tag{140}$$

$$\varphi(t) \asymp 1, \qquad |\varphi(t_1) - \varphi(t_2)| \leq C|t_1 - t_2|. \tag{141}$$

In order to prove (19) we need Example (c) of Theorem 4. of chapter XIII.5 of [24]. With our notations each of the relations

$$-V(t,x) \sim x^{1-1/2}\sqrt{2\varphi(t)} \quad \text{and} \quad \sum_{l=k}^{\infty} v_l(t) \sim \frac{1}{\Gamma(\frac{1}{2})}k^{1/2-1}\sqrt{2\varphi(t)} \tag{142}$$

implies the other.

### 2.4.3 Uniqueness

We are going to prove Theorem 2.1. by proving the uniqueness of (113), (115), (114).

*Proof of Theorem 2.1.* Assume that $V(t,x)$ and $U(t,x)$ are two solutions of the critical Burgers control problem with the same initial conditions and with the control functions $\varphi(t)$ and $\psi(t)$, respectively. Denote

$$S(t,x) := \frac{V(t,x) + U(t,x)}{2}, \quad \sigma(t) := \frac{\varphi(t) + \psi(t)}{2}, \quad \sqrt{\varrho(t)} := \frac{\sqrt{\varphi(t)} + \sqrt{\psi(t)}}{2} \tag{143}$$

$$W(t,x) := \frac{V(t,x) - U(t,x)}{2}, \quad \delta(t) := \frac{\varphi(t) - \psi(t)}{2}. \tag{144}$$

Then, it is easily seen that that (given $S(t,x)$) $W(t,x)$, $\delta(t)$ will solve the linear control problem

$$\dot{W}(t,x) + \big(S(t,x)W(t,x)\big)' = e^{-x}\delta(t), \tag{145}$$

$$W(0,x) \equiv 0, \tag{146}$$

$$W(t,0) \equiv 0. \tag{147}$$

We assume $S(t,x)$ and $\rho(t)$ given, with the regularity properties inherited from Proposition 2.3:

$$-S(t,x) = \sqrt{2\rho(t)}x^{1/2}\big(1+\mathcal{O}(x^{1/2})\big), \tag{148}$$

$$-S'(t,x) = \sqrt{\frac{\rho(t)}{2}}x^{-1/2}\big(1+\mathcal{O}(x^{1/2})\big), \tag{149}$$

$$S''(t,x) = \sqrt{\frac{\rho(t)}{8}}x^{-3/2}\big(1+\mathcal{O}(x^{1/2})\big), \tag{150}$$

$$S(t,x)S'(t,x) = \rho(t)\big(1+\mathcal{O}(x^{1/2})\big). \tag{151}$$

$$\dot{S}(t,x) = \mathcal{O}(x^{1/2}), \tag{152}$$

$$\dot{S}'(t,x) = \mathcal{O}(x^{-1/2}), \tag{153}$$

$$\rho(t) \asymp 1, \qquad |\rho(t_1)-\rho(t_2)| \leq C|t_1-t_2|. \tag{154}$$

We will prove that under these conditions, the unique solution of the problem (145), (146), (147) is $W(t,x)\equiv 0$, $\delta(t)\equiv 0$ which proves the uniqueness of the solutions of (113), (115), (114).

First we define the characteristics of the equation (145): these are the curves $[0,t]\ni s\mapsto \zeta_t(s)$ defined by the ODE

$$\dot{\zeta}_t(s) = S(s,\zeta_t(s)), \qquad \zeta_t(t)=0, \qquad \zeta_t(s)>0 \text{ for } s<t. \tag{155}$$

Next we define the functions $[0,t]\ni s\mapsto \beta_t(s)$

$$\beta_t(s) := S'(s,\zeta_t(s)).$$

The functions $[0,t]\ni s\mapsto \zeta_t(s)$ and $[0,t]\ni s\mapsto \beta_t(s)$ are directly determined by $S(t,x)$ and from (148), (149), (150) and (154) inherit the following regularity properties to be used later:

$$\zeta_t(s) = \frac{\rho(t)}{2}(t-s)^2\big(1+\mathcal{O}(t-s)\big), \tag{156}$$

$$\dot{\zeta}_t(s) = -\rho(t)(t-s)\big(1+\mathcal{O}(t-s)\big), \tag{157}$$

$$\ddot{\zeta}_t(s) = \rho(t)\big(1+\mathcal{O}(t-s)\big), \tag{158}$$

$$\beta_t(s) = -(t-s)^{-1}\big(1+\mathcal{O}(t-s)\big), \tag{159}$$

$$\dot{\beta}_t(s) = -(t-s)^{-2}\big(1+\mathcal{O}(t-s)\big). \tag{160}$$

We define $[0, t] \ni s \mapsto \eta_t(s)$ as

$$\eta_t(s) := W(s, \zeta_t(s)),$$

with $W(t, x)$ given in (144) being solution of (145), (146), (147). Then, for any $t \geq 0$, $\delta(s), \eta_t(s)$, $s \in [0, t]$ solves the ODE (boundary value) control problem

$$\dot{\eta}_t(s) + \beta_t(s)\eta_t(s) = e^{-\zeta_t(s)}\delta(s), \qquad \eta_t(0) = 0 = \eta_t(t) \tag{161}$$

We will prove that this implies $\delta(t) \equiv 0$. Hence it follows that $W(t, x) \equiv 0$.

On the domain $\{(t, s) : 0 \leq s \leq t < \infty\}$ we define the integral kernel

$$\mathcal{K}(t, s) := \exp\Big\{\int_0^s \beta_t(u)\,\mathrm{d}u - \zeta_t(s)\Big\} = \frac{t-s}{t}\mathcal{L}(t, s),$$

defined on the same domain $\{(t, s) : 0 \leq s \leq t < \infty\}$, where

$$\mathcal{L}(t, s) := \exp\Big\{\int_0^s \big(\beta_t(u) + (t-u)^{-1}\big)\,\mathrm{d}u - \zeta_t(s)\Big\}.$$

The ODE control problem (161) is equivalent to

$$\int_0^t \mathcal{K}(t, s)\delta(s)\,\mathrm{d}s = 0. \tag{162}$$

It is handy to introduce the function

$$\gamma(t) := \int_0^t \delta(s)(t-s)\,\mathrm{d}s.$$

Then, after two integrations by parts the identity (162) is transformed into the eigenvalue problem

$$\int_0^t \widehat{\mathcal{K}}(t, s)\gamma(s)\,\mathrm{d}s = \gamma(t), \tag{163}$$

where

$$\widehat{\mathcal{K}}(t, s) := \big(\partial_s \mathcal{K}(t, t)\big)^{-1}\partial^2_{ss}\mathcal{K}(t, s) = \frac{2\partial_s \mathcal{L}(t, s) - (t-s)\partial^2_{ss}\mathcal{L}(t, s)}{\mathcal{L}(t, t)}.$$

Using the regularity properties (156), (157), (158), (159), (160) it follows that

$$\sup_{0 \leq s < t \leq \bar{t}} \Big|\widehat{\mathcal{K}}(t, s)\Big| < \infty. \tag{164}$$

From (163) and (164), by a Grönwall argument we get $\gamma(t) \equiv 0$ and hence $\delta(t) \equiv 0 \equiv W(t, x)$, which proves uniqueness of the solution of (113), (115), (114).

□

# 3 Mean field frozen percolation

## 3.1 Introduction

### 3.1.1 The model

The frozen percolation process on a binary tree was defined by D. J. Aldous in [3]: it is a modification of the dynamical percolation process which makes the following informal description mathematically rigorous: we only occupy an edge if both end-vertices are in a finite cluster. The self-organized critical property of this model manifests in the fact that for $t \geq \frac{1}{2}$, which is the critical time of the corresponding percolation process, a typical finite cluster has the distribution of a critical percolation cluster.

I. Benjamini and O. Schramm showed that it is impossible to define a similar modification of the percolation process on $\mathbb{Z}^2$. An explanation of this non-existence result can be found in Section 3. of [8].

First we give an informal description of the mean field frozen percolation process. The model is similar to the mean field forest fire model, nevertheless our notations will slightly differ from those in the previous Chapter.

The mean field frozen percolation process is also a modification of the Erdős-Rényi random graph process: Initially we have a (not necessarily empty) graph on $\lfloor N \cdot m_0(0) \rfloor$ vertices (one should think about $N$ as being large, but the initial mass $m_0(0)$ is fixed), and between every possible pair of vertices, edges appear with rate $\frac{1}{N}$. Simultaneously lightnings strike vertices with rate $\lambda(t)\mu(N)$ at time $t$ and when a vertex is struck, the fire spreads along the edges and burns the connected component of that vertex: that subgraph is removed from the graph, including vertices. Thus the number of vertices of the random graph decreases with time. The expressions "burnt", "frozen", "deleted" and "removed" are treated as synonyms in the sequel.

If $\mathcal{V}_k^N(t)$ denotes the number of vertices contained in components of size $k$ in the random graph at time $t$, then the vector-valued stochastic process $\underline{\mathcal{V}}(t) = (\mathcal{V}_1^N(t), \mathcal{V}_2^N(t), \dots)$ also has the Markov property (the main advantage of the mean field model is that the graph structure of the connected components has no effect on the evolution of component sizes). We are interested in the model when $1 \ll N$.

Denote by $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

**Definition 3.1.1.** *We fix $m_0(0) \in \mathbb{R}_+$. The mean field frozen percolation process on $N$ vertices is a continuous time Markov process with state space*

$$\Omega_N = \{\underline{\mathcal{V}} \in \mathbb{N}_0^{\mathbb{N}} : \sum_{k \geq 1} \mathcal{V}_k \leq \lfloor N \cdot m_0(0) \rfloor,\ \forall k\ \frac{\mathcal{V}_k}{k} \in \mathbb{N}_0\}$$

*We define the coagulation and deletion operators*

$$\underline{\mathcal{V}}_{k,l}^{+} := \begin{cases} (\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k - k, \dots, \mathcal{V}_l - l, \dots, \mathcal{V}_{k+l} + k + l, \dots) & \text{if } k < l \\ (\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k - 2k, \dots, \mathcal{V}_{2k} + 2k, \dots) & \text{if } k = l \end{cases} \tag{165}$$

$$\underline{\mathcal{V}}_{k}^{-} := (\mathcal{V}_1, \dots, \mathcal{V}_k - k, \dots) \tag{166}$$

*Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive continuous function and $\mu : \mathbb{N} \to \mathbb{R}_+$. The transition rates of the Markov process are*

$$\lambda(\underline{\mathcal{V}} \to \underline{\mathcal{V}}^+_{k,l}) = \begin{cases} \frac{1}{N} \cdot \mathcal{V}_k \cdot \mathcal{V}_l & \text{if } k < l \\ \frac{1}{N} \cdot \frac{\mathcal{V}_k \cdot (\mathcal{V}_k - k)}{2} & \text{if } k = l \end{cases} \tag{167}$$

$$\lambda(\underline{\mathcal{V}} \to \underline{\mathcal{V}}^-_k) = \lambda(t) \cdot \mu(N) \cdot \mathcal{V}_k \tag{168}$$

*Let $v_k^N(t) := \frac{\mathcal{V}_k(t)}{N}$ denote the* mass *of components of size $k$ at time $t$.*

There are some differences between the definition of the forest fire model (see Subsection 2.2.2) and the frozen percolation model:

- $m_0(0) = 1$ in the forest fire model, $m_0(0) \in (0, +\infty)$ in the frozen percolation model (this is a minor difference). Thus $\left(v_k^N(t)\right)_{k \geq 1}$ is a mass distribution rather than a probability distribution. Nevertheless we keep the notation of moments, see (175).
- In the case of the forest fire model, a burnt component of size $k$ is replaced by $k$ isolated vertices, so that the number of vertices in the random graph remains unchanged. In the case of the frozen percolation model, burnt vertices are removed. Thus $m_0^N(t) := \sum_{k \geq 1} v_k^N(t)$ is a non-increasing function of $t$.
- We only considered the forest fire model with a time-homogenous lightning rate in the previous chapter (which would correspond to $\lambda(t) \equiv \lambda$ with the notations of Definition 3.1.1).

The two models both have the self-organized critical property and we believe that they are in the same universality class, which means that the theorems of this thesis have analogous "forest fire" versions.

The reason why we devote an individual chapter to the frozen percolation model is that the the Burgers-type partial differential equations corresponding to the forest fire model are very technical to handle (e.g. no explicit solution is known), but in the case of the frozen percolation model they have an explicit solution which enables us to say more about this model. We discuss another notable case of an explicitly solvable infinite system of differential equations which describes the time evolution of the mass of components of different sizes in a dynamical mean field model (triangle percolation) in [35].

$$\mathbf{V} := \{\underline{\mathbf{v}} = \left(v_k\right)_{k=1}^{\infty} : \; v_k \in \mathbb{R},\, v_k \geq 0 \;\text{ and }\; \sum_{k=1}^{\infty} v_k < \infty\} \tag{169}$$

$$\mathbf{V}^* := \{\underline{\mathbf{v}} : \; \underline{\mathbf{v}} \in \mathbf{V},\; \exists K < +\infty\; \forall k \geq K \quad v_k = 0\} \tag{170}$$

**Definition 3.1.2.** *We consider a sequence of mean field frozen percolation processes with $N \to \infty$, but with the initial state*

$$\underline{\mathbf{v}}(0) = \left(v_1^N(0), v_2^N(0), \dots, v_K^N(0), 0, 0, \dots\right) = \left(\frac{\mathcal{V}_1^N(0)}{N}, \frac{\mathcal{V}_2^N(0)}{N}, \dots, \frac{\mathcal{V}_K^N(0)}{N}, 0, 0, \dots\right) \in \mathbf{V}^*$$

*and the lightning rate function $\lambda(t)$ fixed (independent of $N$). Such a sequence is called*

- *subcritical if* $\mu(N) \equiv 1$
- *critical if* $\frac{1}{N} \ll \mu(N) \ll 1$
- *alternating if* $\mu(N) = \frac{1}{N}$.

*If* $v_k^N(0) = \mathbb{1}[k=1] \cdot m_0(0)$ *then the initial state is called* monodisperse*, otherwise it is* polydisperse*.*

### 3.1.2 Differential equations

We are going to describe the time evolution of the limit object

$$\lim_{N\to\infty} v_k^N(t) = v_k(t). \tag{171}$$

We introduce differential equations to characterize the limiting component size distributions $v_k(t)$ where $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$. They are modifications of the Smoluchowski coagulation equation with multiplicative rate kernel, which look like this in the classical formulation:

$$\dot{c}_k(t) = \frac{1}{2}\sum_{l=1}^{k-1} l \cdot (k-l) \cdot c_l(t) \cdot c_{k-l}(t) - c_k(t)\sum_{l=1}^{\infty} l \cdot c_l(0) \qquad \text{Flory's model} \tag{172}$$

$$\dot{c}_k(t) = \frac{1}{2}\sum_{l=1}^{k-1} l \cdot (k-l) \cdot c_l(t) \cdot c_{k-l}(t) - c_k(t)\sum_{l=1}^{\infty} l \cdot c_l(t) \qquad \text{Stockmayer's model} \tag{173}$$

If we let $v_k(t) = k \cdot c_k(t)$ then (172) becomes

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k \cdot v_k(t) \cdot \sum_{k=1}^{\infty} v_k(0) \tag{174}$$

We are going to use the formulation (174) rather than the classical (172). (174) becomes (10) if we assume $m_0(0) = \sum_{k=1}^{\infty} v_k(0) = 1$, but this restriction is not serious because $(v_k(t))_{k=1}^{\infty}$ is a solution of (174) if and only if $\left(\frac{1}{m_0(0)} v_k\left(\frac{t}{m_0(0))}\right)\right)_{k=1}^{\infty}$ is a solution of (10).

The differential equations (174) describe the time evolution of $(v_k(t))_{k=1}^{\infty}$ defined by (171) for the dynamical Erdős-Rényi random graph process (see Subsection 2.1.2).

**Definition 3.1.3.** *If* $\left(v_k\right)_{k=1}^{\infty} = \underline{\mathbf{v}} \in \mathbf{V}$ *let*

$$m_0 := \sum_{k\geq 1} v_k \qquad m_1 := \sum_{k\geq 1} k v_k \qquad m_2 := \sum_{k\geq 1} k^2 v_k \qquad m_3 := \sum_{k\geq 1} k^3 v_k \tag{175}$$

If we define

$$w_k^N(t) := \sum_{l=1}^{k} v_l^N(t) \qquad \text{and} \qquad \Phi^N(t) := \sum_{l\geq 1} v_l^N(0) - \sum_{l\geq 1} v_l^N(t) = m_0^N(0) - m_0^N(t) \tag{176}$$

then for all $k$ the random function $w_k^N(t)$ is decreasing (since its value decreases if we apply the coagulation and deletion operators) and $\Phi^N(t)$ (the mass of burnt vertices) is increasing.

It might happen (e.g. in the case of the Erdős-Rényi model) that

$$\theta(t) := \lim_{k\to\infty} \lim_{N\to\infty} \left(m_0^N(t) - w_k^N(t)\right) \neq \lim_{N\to\infty} \lim_{k\to\infty} \left(m_0^N(t) - w_k^N(t)\right) = 0.$$

In this case the mass missing from the small components is contained in a giant component of mass $0 < \theta(t)$.

**Definition 3.1.4.** *If $\underline{\mathbf{v}}(t)$ is a solution of* (174)*, we define the gelation time by*

$$T^g := \inf\{t : m_1(t) = +\infty\}.$$

It is well-known from the theory of the Smoluchowski coagulation equation that an alternative characterisation of the gelation time is

$$T^g = \inf\{t : m_0(t) < m_0(0)\}.$$

For the solution of (174) the gelation time is $T^g = \frac{1}{m_1(0)}$ and the mass of the giant component is $\theta(t) = m_0(0) - m_0(t)$. The solution of (174) undergoes phase transition at $t = T^g$, the three phases are described by the three paragraphs between (13) and (14).

**Definition 3.1.5.** *We say that $\underline{\mathbf{v}}(t) = (v_k(t))_{k=1}^{\infty} \in \mathbf{V}$ solves the general frozen percolation equation on $[0,T]$ with initial condition $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, a continuous nonnegative rate function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and control function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ if*

$$\forall\, 0 \leq s \leq t \leq T \ : \quad 0 \leq \Phi(0) \leq \Phi(s) \leq \Phi(t) < m_0(0) \tag{177}$$

*and for all $k = 1, 2, \dots$ the equations*

$$v_k(t) = v_k(0) + \int_0^t \frac{k}{2} \sum_{l=1}^{k-1} v_l(s) v_{k-l}(s) - k v_k(s) \Big( (m_0(0) - \Phi(s)) + \lambda(s) \Big)\, \mathrm{d}s \tag{178}$$

*and the inequality*

$$\forall t \ : \quad 0 \leq \theta(t) := m_0(0) - m_0(t) - \Phi(t) \tag{179}$$

*is satisfied.*

It is easy to see by induction that the absolutely continuous functions $v_1(t), v_2(t), \ldots$ are completely determined by (178), the initial condition $\underline{\mathbf{v}}(0)$ and the functions $\lambda$ and $\Phi$. The only reason why we do not write

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t)\Big((m_0(0) - \Phi(t)) + \lambda(t)\Big) \tag{180}$$

instead of (178) is that the increasing function $\Phi(t)$ might have jumps.

There are three versions of the general frozen percolation equation corresponding to the three regimes of Definition 3.1.2:

- The *subcritical system of integral equations* are (178) with the extra conditions $\forall t\,:\, 0 < \lambda_{inf} \leq \lambda(t)$ and
$$\Phi(t) \equiv m_0(0) - m_0(t). \tag{181}$$
That is $\theta(t) \equiv 0$ by (179) (no giant components appear due to frequent lightnings) and the equations take on the form
$$v_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - k\cdot v_k(s)m_0(s) - \lambda(s)k\cdot v_k(s)\,\mathrm{d}s \tag{182}$$
The term $-\lambda(s)k\cdot v_k(s)$ indicates that in the subcritical regime even small components are burnt with a rate proportional to their sizes and $\lambda(s)$.

- The *critical equations* are (178) with the extra conditions $\lambda(t)\equiv 0$ and (181):
$$v_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - k\cdot v_k(s)m_0(s)\,\mathrm{d}s \tag{183}$$
$\lambda(t)\equiv 0$ indicates that in the critical regime lightnings are not frequent enough to do any harm to small components, but (181) indicates that they are frequent enough to keep the mass of the giant component at zero.

- Let $0 = T_0^b < T_1^b < T_2^b < \ldots$ be a sequence with no accumulation points. Let
$$M(t) := \max\{i : T_i^b < t\} \tag{184}$$
$\underline{\mathbf{v}}(t)$ solves the *alternating equations* with burning times $T_1^b, T_2^b, \ldots$ if
$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - k\cdot v_k(t)m_0(T_{M(t)}^b) \tag{185}$$

Mind the difference between (174) and (183): in the case of the Erdős-Rényi model the small components are allowed to coagulate with the giant component (which is of size $\theta(t) = m_0(0) - m_0(t)$ by $\Phi(t)\equiv 0$ and (179)), but in the case of the critical frozen percolation model the giant components are removed at the time of their birth. Using the terminology of the theory of Smoluchowski coagulation equations we might say that in the case of (174) the gel and the sol do react in the post-gel phase (Flory's model, (172)), but in the case of (183) they do not react (Stockmayer's model, (173)). Nevertheless, for $t \leq T^g$ the solutions of (174) and (183) are identical since $m_0(t) = m_0(0)$ in this regime.

The intuitive meaning of (185) is that giant components are removed from the system at the burning times.

Thus (185) is (178) with $\lambda(T) \equiv 0$ and

$$\theta(t) = m_0(T^b_{M(t)}) - m_0(t) \tag{186}$$

$$\Phi(t) = m_0(0) - m_0(T^b_{M(t)}) = m_0(0) - m_0(t) - \theta(t) = \sum_{j=1}^{M(t)} \theta(T^b_j) \tag{187}$$

Both $\theta(t)$ and $\Phi(t)$ are left-continuous functions of $t$.

Note that in the case of the (sub)critical frozen percolation equations ((182) and (183)) the fact that $\Phi(t)$ is an increasing function automatically follows by (181):

$$\Phi(t) - \Phi(s) = m_0(s) - m_0(t) =$$
$$\sum_{k=1}^{\infty} \int_s^t -\frac{k}{2} \sum_{l=1}^{k-1} v_l(u) v_{k-l}(u) + k \cdot v_k(u) m_0(u) + \lambda(u) \cdot k \cdot v_k(u) \,\mathrm{d}u =$$
$$\lim_{N \to \infty} \int_s^t \sum_{k=1}^{N} \sum_{l=N-k+1}^{\infty} k \cdot v_k(u) v_l(u) + \lambda(u) \cdot k \cdot v_k(u) \,\mathrm{d}u \geq 0$$

**Theorem 3.1.**

- *For any* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *and* $0 < \lambda_{inf} \leq \lambda(t)$ *the equations* (182) *have a unique solution.*
- *For any* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *the equations* (183) *have a unique solution.*
- *For any* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *and any sequence of burning times the equations* (185) *have a unique solution.*

We prove this theorem in Section 3.3. The proofs are much less technical and more constructive than the proof of the uniqueness of the differential equation corresponding to the forest fire model (Subsection 2.4.3).

**Definition 3.1.6.** *The solution of the random alternating equations with rate function* $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *is a* $\mathbf{V}$*-valued continuous-time Markov process:* $\underline{\mathbf{v}}(t)$ *evolves deterministically, driven by the equations* (185)*, but the sequence of burning times* $T^b_1, T^b_2, \dots$ *is random:*

$$\lim_{\mathrm{d}t \to 0} \frac{1}{\mathrm{d}t} \mathbf{P}\big( t \leq T^b_{M(t)+1} \leq t + \mathrm{d}t \,\big|\, \mathcal{F}_t \big) = \lambda(t)\theta(t) \tag{188}$$

*where* $\mathcal{F}_t$ *is the natural filtration generated by the process.*

In plain words: a lightning strikes and burns the giant component with rate proportional to its size and $\lambda(t)$.

### 3.1.3 The differential equations describe the model in the limit

The first sign of the fact that the frozen percolation process is easier to handle than the forest fire model is that the precise formulation of (171) is less technical. In Subsection 2.2.1 we had to introduce forest fire flows in order to find the hidden monotonicity and compactness of the forest fire evolutions arising from the mean field forest fire model. In the case of the mean field frozen percolation model, monotonicity and compactness is "for free":

**Definition 3.1.7.**

$$\mathcal{W} := \{(w_k)_{k=1}^{\infty} \;:\; 0 \leq w_1 \leq w_2 \leq \dots < +\infty\}$$
$$\mathcal{W}^* := \{(w_k)_{k=1}^{\infty} \in \mathcal{W} \;:\; \exists K < +\infty \;\; \forall k \geq K \;\; w_k = w_K\}$$

*If* $\underline{\mathbf{w}} \in \mathcal{W}$ *denote by* $m_0 := \sup_k w_k$.

*We say that* $((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot))$ *is a frozen percolation evolution on* $[0,T]$ *with initial condition* $(w_k(0))_{k=1}^{\infty} = \underline{\mathbf{w}} \in \mathcal{W}^*$*, or briefly*

$$((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot)) \in \mathcal{W}_{\underline{\mathbf{w}}}[0,T]$$

*if*

- *for all* $0 \leq t \leq T$ *we have* $(w_k(t))_{k=1}^{\infty} \in \mathcal{W}$*,*
- *for all* $k$ *the functions* $w_k : [0,T] \to [0, m_0(0)]$ *are left-continuous and decreasing,*
- $\Phi : [0,T] \to [0, m_0(0)]$ *is left continuous and increasing with initial condition* $\Phi(0) = 0$*,*
- *for all* $t \leq T$ *we have* (179)*.*

*We define convergence on the space* $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$*. We say that*

$$((w_k^n(\cdot))_{k=1}^{\infty}, \Phi^n(\cdot)) \to ((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot))$$

*as* $n \to \infty$ *if for all* $k$ *we have* $w_k^n(t) \to w_k(t)$ *for all* $t$ *which is a point of continuity of* $w_k$ *and* $\Phi^n(t) \to \Phi(t)$ *for all* $t$ *which is a point of continuity of* $\Phi$*.*

With this topology the space $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ is metrizable, complete and compact.

From the frozen percolation process of Definition 3.1.1. one gets a random element of $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ by (176). Denote the probability measure on $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ corresponding to the process by $\mathbb{P}_N$.

It is easy to check that $((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot)) \in \mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ where $w_k(t) = \sum_{l=1}^{k} v_l(t)$ and $\underline{\mathbf{v}}(t)$ is a solution of the general frozen percolation equation (177) & (178) & (179).

**Theorem 3.2.** *We consider a sequence of frozen percolation processes (see Definition 3.1.1) with initial state* $\underline{\mathbf{v}}^N(0) = \underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *and* $\lambda(t)$ *positive and continuous. Define* $w_k^N(t)$ *and* $\Phi^N(t)$ *as in* (176)*. Denote the probability measure on* $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ *corresponding to the process by* $\mathbb{P}_N$*.*

*Then* $\mathbb{P}_N$ *converges with respect to the weak convergence of probability measures on the polish space* $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ *to a limiting measure* $\mathbb{P}$*, which depends on the decay rate of* $\mu(N)$ *in the following way:*

- *If* $\mu(N) \equiv 1$ *then* $\mathbb{P}$ *is concentrated on the unique solution of* (182) *with rate function* $\lambda(t)$*.*
- *If* $\frac{1}{N} \ll \mu(N) \ll 1$ *then* $\mathbb{P}$ *is concentrated on the unique solution of* (183)*.*
- *If* $\mu(N) = \frac{1}{N}$ *then* $\mathbb{P}$ *is the law of the solution of the random alternating equation (see Definition 3.1.6) with rate function* $\lambda(t)$*.*

We do not present the proof of this theorem in this thesis.

The proof of the $\mu(N) \equiv 1$ and the $\frac{1}{N} \ll \mu(N) \ll 1$ part of this theorem can be found in [33]. In fact, these proofs are almost identical to the proof of Theorem 2.2 presented in Subsections 2.2.2 and 2.3.3. We believe that these methods can be easily generalized for the $\mu(N) = \frac{1}{N}$ part of Theorem 3.2 as well.

### 3.1.4 Self-organized criticality and self-similarity

If we formally substitute $\lambda(t) \equiv 0$ into (182) or $T^b_{M(t)} \equiv t$ into (185), we get (183). Rigorously:

**Theorem 3.3.** *Let* $(\underline{\mathbf{v}}^n(t))_{n=1}^\infty$ *be a sequence of solutions of* (182) *with the same initial condition* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *where* $\lambda_n(t) \to 0$ *uniformly as* $n \to \infty$*. Then for all* $t$ *and* $k$ $\lim_{n\to\infty} v_k^n(t) = v_k(t)$ *where* $\underline{\mathbf{v}}(t)$ *is the solution of* (183) *with the same initial data.* $\lim_{n\to\infty} \Phi_n(t) = \Phi(t)$ *uniformly on* $[0,\infty)$*.*

In plain words: if the rate of lightning is very small in the subcritical equations, then the solution is similar to that of the critical equation. We prove this theorem in Section 3.5.

**Theorem 3.4.** *Let* $(\underline{\mathbf{v}}^n(t))_{n=1}^\infty$ *be a sequence of solutions of* (185) *with the same initial condition* $\underline{\mathbf{v}}(0)$ *where the sequence of burning times satisfy*

$$\lim_{n\to\infty} \sup_i \{T^b_{i+1}(n) - T^b_i(n)\} = 0.$$

*Then for all* $t$ *and* $k$ $\lim_{n\to\infty} v_k^n(t) = v_k(t)$ *where* $\underline{\mathbf{v}}(t)$ *is the solution of* (183) *with the same initial data.* $\lim_{n\to\infty} \Phi_n(t) = \Phi(t)$ *uniformly on* $[0,\infty)$*.*

In plain words: if the burning times of the alternating equations are very frequent, then the solution is similar to that of the critical equation. We prove this theorem in Section 3.6.

The solution of (183) has the self-organized critical property: for all $T^g \le t$ it has the same power-law decay as the solution of (174) at $t = T^g$:

**Theorem 3.5.** *If* $\underline{\mathbf{v}}(t)$ *is a solution of* (183) *with initial condition* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$*, then* $T^g = \frac{1}{m_1(0)}$*,* $\Phi(t) = \int_{T^g}^t \varphi_{crit}(s)\,\mathrm{d}s$ *where* $\varphi_{crit} : [T^g, +\infty) \to R_+$ *is positive and continuous, and for all* $t \geq T^g$ *we have*

$$\lim_{K\to\infty} K^{\frac{1}{2}} \sum_{k=K}^{\infty} v_k(t) = \sqrt{\frac{2\varphi_{crit}(t)}{\pi}}. \tag{189}$$

**Definition 3.1.8.** *Let* $x^*(t) := \inf\{x : \sum_{k=1}^{\infty} v_k(t) e^{-kx} < +\infty\}$*.*

Note that $x^*(t) = \limsup_{n\to\infty} \frac{1}{n} \ln\big(v_k(t)\big)$ by the Cauchy-Hadamard theorem.
The solutions of our equations have a remarkable rigidity property:

**Theorem 3.6.** *If* $\underline{\mathbf{v}}(t)$ *is the solution of* (182) *or* (185) *and* $\tilde{\underline{\mathbf{v}}}(t)$ *is the solution of* (183) *with the same initial condition, then for all* $t \geq T^g$ *and* $k \in \mathbb{N}$ *we have*

$$\tilde{v}_k(t) = v_k(t) e^{-kx^*(t)}$$

*where* $x^*(t) = \inf\{x : \sum_{k=1}^{\infty} v_k(t) e^{-kx} < +\infty\}$*.*

Thus no matter what lightning rate function $\lambda(t)$ we use in (182) or what sequence of burning times we use in (185), the resulting family of attainable configurations at time $t$ form a one parameter family for each starting configuration.

The solution of (183) with monodisperse initial condition is well-known (see e.g. [39]) and explicit:

**Claim 1.** *If* $\underline{\mathbf{v}}(t)$ *is the solution of* (183) *with* $v_k(0) = \mathbb{1}[k=1] \cdot m_0(0)$ *then for* $t \geq T^g = \frac{1}{m_1(0)} = \frac{1}{m_0(0)}$ *we have*

$$v_k(t) = \frac{1}{t} \frac{k^{k-1}}{k!} e^{-k}. \tag{190}$$

That is, for all $T^g \leq t$ in the $N \to \infty$ limit, the component size of a uniformly chosen (unburnt) vertex in the critical frozen percolation model has Borel distribution with parameter $t = 1$ (see (12)), which is the same as that of a vertex in the Erdős-Rényi graph at $t = T^g$ in the $N \to \infty$ limit.
The same self-similarity phenomenon can be observed in Aldous' frozen percolation model (see [3]) on the binary tree: for $t \geq \frac{1}{2}$ ($t = \frac{1}{2}$ is the critical time of the percolation process on the binary tree), a typical finite cluster has the distribution of a critical percolation cluster on the binary tree.
The solutions started from a polydisperse initial state are asymptotically self-similar:

**Theorem 3.7.** *If* $\underline{\mathbf{v}}(t)$ *is the solution of the critical equation* (183) *with* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$*, and* $v_1(0) > 0$ *then*

$$\lim_{t\to\infty} t \cdot v_k(t) = \frac{k^{k-1}}{k!} e^{-k} \quad \textit{and} \quad \lim_{t\to\infty} t \cdot m_0(t) = 1. \tag{191}$$

Theorems 3.5., 3.6. and 3.7. are proved in Subsection 3.4.2 using the method of Laplace transforms, which is classical for the Smoluchowski equation with multiplicative kernel. The results (191) and $\frac{v_k(t)}{k} = c_k(t) \asymp k^{-5/2}$ (which is a variant of (189)) are already present in [39], but we believe that our approach based on the notion of the *critical core* of $\underline{\mathbf{v}}(t)$ (defined in Section 3.2) gives new insight into these results about the solution of (183).

### 3.1.5 The typical size of a frozen component

In the frozen percolation model on the binary tree, components are frozen (i.e. removed from the system) when their size becomes infinite. The question may arise:

*What is the typical size of a frozen component in the mean field process of Definition 3.1.1?*

In order to precisely formulate this question recall (166) and let

$$\Phi^N([t_1, t_2], k) := \frac{k}{N} \cdot \left|\left\{t \in [t_1, t_2] \ : \ \underline{\mathcal{V}}(t_+) = \underline{\mathcal{V}}_k^-(t_-)\right\}\right|.$$

Thus $\Phi^N([t_1, t_2], k)$ is the mass of burnt components of size $k$ from $t_1$ to $t_2$. We have

$$\sum_{k \geq 1} \Phi^N([t_1, t_2], k) = \Phi^N(t_2) - \Phi^N(t_1) =: \Phi^N([t_1, t_2]).$$

Thus

$$p_k^N[t_1, t_2] := \frac{\Phi^N([t_1, t_2], k)}{\Phi^N([t_1, t_2])}, \qquad k = 1, 2, \dots \tag{192}$$

is a random probability distribution for all $N$ and $t_1 < t_2$.

Denote by $\left|\mathcal{C}_{max}^N(t)\right|$ the size of the largest component at time $t$. Recall the definition of moments in (175).

**Conjecture 3.1.** *If $\mu(N) = N^{-\alpha}$ in a critical sequence of frozen percolation processes (see Definitions 3.1.1 and 3.1.2), where $0 < \alpha < 1$, and if we define*

$$\beta(\alpha) := \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{3} \\ \frac{\alpha+1}{2} & \text{if } \alpha \geq \frac{1}{3} \end{cases} \tag{193}$$

*then for every $T^g < t$ we have*

$$\lim_{N \to \infty} \frac{\log\left(\mathbf{E}\left( m_1^N(t) \right)\right)}{\log(N)} = \alpha \tag{194}$$

$$\lim_{N \to \infty} \frac{\log\left(\mathbf{E}\left( m_2^N(t) \right)\right) - \log\left(\mathbf{E}\left( m_1^N(t) \right)\right)}{\log(N)} = \beta(\alpha) \tag{195}$$

$$\lim_{N \to \infty} \frac{\log\left(\mathbf{E}\left( \left|\mathcal{C}_{max}^N(t)\right| \right)\right)}{\log(N)} = \beta(\alpha) \tag{196}$$

*Moreover for every* $\underline{\mathbf{v}}(0)$, $T^g < t_1 < t_2$ *and* $\alpha$ *there exists a non-defective probability distribution function* $F : (0, \infty) \to (0, 1)$, $\lim_{x\to 0_+} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$ *such that for all* $x \in \mathbb{R}_+$ *we have*

$$\lim_{N\to\infty} \sum_{k\geq 1} \mathbb{1}[\, k \leq xN^{\beta(\alpha)}\,] \cdot p_k^N[t_1, t_2] \;=\; F(x) \tag{197}$$

In plain words we might say that after gelation the typical component size of a frozen vertex and the size of the largest component is of order $N^{\beta(\alpha)}$. This conjecture is supported by heuristic arguments, computer simulations and Theorems 3.8 and 3.9 below. For $0 < \alpha < \frac{1}{3}$ the model is conjectured to behave similarly to the subcritical case described in Theorem 3.8, whereas for $\frac{1}{3} < \alpha < 1$ it is conjectured to behave similarly to the alternating case described in Theorem 3.9. Note that $\beta(\frac{1}{3}) = \frac{2}{3}$ and $N^{\frac{2}{3}}$ is the order of the size of the largest component in the critical Erdős-Rényi random graph.

**Theorem 3.8.** *If* $\underline{\mathbf{v}}^\lambda(t)$ *is the solution of* (182) *with rate function* $\lambda(t) \equiv \lambda$ *and* $\underline{\mathbf{v}}^\lambda(0) = \underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *then there is a constant* $C$ *that depends only on the initial data and* $T$ *such that for all* $0 < \lambda \leq 1$ *and* $\frac{1}{m_1(0)} < t \leq T$ *we have*

$$|\varphi_\lambda(t) - \varphi_{crit}(t)| \leq C\lambda \tag{198}$$

*where*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\lambda(t) = \varphi_\lambda(t) = \lambda m_1^\lambda(t). \tag{199}$$

*Moreover if we define the random variable* $Y_\lambda(t)$ *to have distribution*

$$\mathbf{P}\big(\, Y_\lambda(t) = k \,\big) = \frac{\lambda \cdot k \cdot v_k^\lambda(t)}{\varphi_\lambda(t)} = \frac{k \cdot v_k^\lambda(t)}{m_1^\lambda(t)} \tag{200}$$

*then*

$$\lim_{\lambda\to 0} \mathbf{P}\big(\, \frac{\lambda^2}{2\varphi_{crit}(t)} Y_\lambda(t) < x \,\big) = \int_0^x \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{y}} e^{-y}\, \mathrm{d}y \tag{201}$$

In plain words: for any $t > T^g$ the distribution of the size-biased sample from the component-size distribution $\underline{\mathbf{v}}^\lambda(t)$ rescaled by $\lambda^{-2} \cdot 2\varphi_{crit}(t)$ converges in distribution to a $\Gamma(\frac{1}{2}, 1)$ distribution as $\lambda \to 0$. We prove this theorem in Section 3.6.

The relevance of Theorem 3.8 to Conjecture 3.1 is the following: if we consider a sequence of subcritical frozen percolation models (see Definition 3.1.2) with $\lambda(t) \equiv \lambda$ and denote by $p_k^{N,\lambda}[t_1, t_2]$ the random probability distribution (192) obtained from this model then by Theorem 3.2 we get

$$\lim_{\mathrm{d}t\to 0} \lim_{N\to\infty} p_k^{N,\lambda}[t, t + \mathrm{d}t] = \lim_{\mathrm{d}t\to 0} \frac{\Phi_\lambda([t, t + \mathrm{d}t], k)}{\Phi_\lambda([t, t + \mathrm{d}t])} =$$

$$\lim_{\mathrm{d}t\to 0} \frac{\int_t^{t+\mathrm{d}t} \lambda \cdot k \cdot v_k^\lambda(s)\, \mathrm{d}s}{\int_t^{t+\mathrm{d}t} \sum_{l=1}^\infty \lambda \cdot l \cdot v_l^\lambda(s)\, \mathrm{d}s} = \frac{k \cdot v_k^\lambda(t)}{m_1^\lambda(t)} = \mathbf{P}\big(\, Y_\lambda(t) = k \,\big) \tag{202}$$

If we let $\lambda \to 0$ then by (198) and (199) we get $m_1^\lambda(t) \asymp \lambda^{-1}$ which is a "subcritical" version of (194), and $\frac{m_2^\lambda(t)}{m_1^\lambda(t)} = \mathbf{E}\big(Y_\lambda(t)\big) \asymp \lambda^{-2}$ corresponds to $\beta(\alpha) = 2\alpha$ in (195). Putting (201) together with (202) we get

$$\lim_{\lambda\to 0}\lim_{\mathrm{d}t\to 0}\lim_{N\to\infty}\sum_{k\geq 1}\mathbb{1}[\, k \leq x\cdot 2\varphi_{crit}(t)\cdot\lambda^{-2}\,]\cdot p_k^{N,\lambda}[t,t+\mathrm{d}t] \;=\int_0^x \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{y}}e^{-y}\mathrm{d}y,$$

which is the "subcritical" version of (197).

**Theorem 3.9.** *Let $\underline{\mathbf{v}}^\lambda(t)$ denote the solution of the random alternating equations (see Definition 3.1.6.) with a constant rate function $\lambda(t)\equiv\lambda$.*

*Let $\delta(\lambda)$ be a function satisfying $\lambda^{-\frac{1}{2}} \ll \delta(\lambda) \ll 1$ as $\lambda\to\infty$.*

*Recalling* (184) *and* (186) *let*

$$\Phi_\lambda(t,x) := \sum_{j=1}^{M(t)}\theta^\lambda(T_j^b)\mathbb{1}[\theta^\lambda(T_j^b) > x]$$

*be the random mass of frozen giants of size at least $x$. Then*

$$\lim_{\lambda\to\infty}\frac{\Phi_\lambda\left(t+\delta(\lambda), 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}}x\right)-\Phi_\lambda\left(t, 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}}x\right)}{\delta(\lambda)\varphi_{crit}(t)} = \int_x^\infty \frac{4}{\sqrt{\pi}}y^2e^{-y^2}\,\mathrm{d}y \qquad (203)$$

*in probability.*

We prove this theorem in Section 3.6.

The heuristic meaning of this theorem is the following: if we pick a vertex uniformly from all vertices that were frozen between $t$ and $t+\delta(\lambda)$ and denote the mass of the giant component of that vertex by $Z_\lambda(t)$, then the distribution of $\frac{1}{2}\sqrt{\frac{\lambda}{\varphi_{crit}(t)}}Z_\lambda(t)$ converges to a size-biased Rayleigh distribution (see Definition 3.6.3) as $\lambda\to\infty$. Thus the typical mass of a frozen giant is of order $\lambda^{-\frac{1}{2}}$, which suggests that if $\mu(N)=\frac{N^\varepsilon}{N}$ (that is $\alpha = 1-\varepsilon$ in Conjecture 3.1) then the typical size of a frozen component is of order $(N^\varepsilon)^{-\frac{1}{2}}\cdot N = N^{1-\frac{1}{2}\varepsilon}$, that is $\beta(\alpha)=\frac{\alpha+1}{2}$.

$$\mathbf{E}\big(\,m_1^N(t)\,\big) = \mathbf{E}\Big(\sum_{k=1}^N k\cdot v_k^N(t)\,\Big) \asymp \mathbf{E}\Big(\,|\mathcal{C}_{max}^N(t)|\cdot\frac{|\mathcal{C}_{max}^N(t)|}{N}\,\Big) \asymp N^{1-\frac{1}{2}\varepsilon}\cdot\frac{N^{1-\frac{1}{2}\varepsilon}}{N} = N^{1-\varepsilon} = N^\alpha$$

suggests that (194) holds. If we consider a sequence of alternating frozen percolation models (see Definition 3.1.2) with $\lambda(t)\equiv\lambda$ and denote by $p_k^{N,\lambda}[t_1,t_2]$ the random probability distribution (192) obtained from this model then from Theorem 3.2 and (203) it follows that

$$\lim_{\mathrm{d}t\to 0}\lim_{\lambda\to\infty}\lim_{N\to\infty}\sum_{k\geq 1}\mathbb{1}[k\leq 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}}\cdot x\cdot N]\cdot p_k^{N,\lambda}[t,t+\mathrm{d}t] \;=\int_0^x \frac{4}{\sqrt{\pi}}y^2e^{-y^2}\,\mathrm{d}y$$

which is the "alternating" version of (197).

### 3.1.6 The extremum property of the critical model

The critical frozen percolation model has an extremum property compared to the subcritical and alternating models (see Definition 3.1.2): if each burnt/frozen vertex produces profit at a rate $\$\frac{1}{N}$ per time unit after it has been frozen, but each lightning (even the ones hitting burnt vertices) costs $\$\frac{1}{N \cdot m_0(0)}$, then asymptotically (as $N \to \infty$) the critical model is the best choice if we want to maximize our profit on $[0, T]$. We reformulate this extremum principle in terms of the differential equations (182), (183), (185).

The asymptotic value of our profit produced by burnt vertices as $N \to \infty$ is $\int_0^T \Phi(t)\,\mathrm{d}t$ according to Theorem 3.2. The asymptotic cost of lightnings is $\int_0^T \lambda(t)\,\mathrm{d}t$ for the solution of (182), but it is zero for (183) and (185), since the price we have to pay for the lightnings vanishes in the case of critical and alternating models as $N \to \infty$.

**Theorem 3.10.** *We fix $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$. Let $\underline{\mathbf{v}}^{crit}(t)$ denote the solution of* (183) *with initial condition $\underline{\mathbf{v}}(0)$ and let $\underline{\mathbf{v}}^{sub}$ denote the solution of* (182) *with lightning rate function $\lambda(t)$ and the same initial condition. Then for any $T > 0$*

$$\int_0^T \Phi^{sub}(t)\,\mathrm{d}t - \int_0^T \lambda(t)\,\mathrm{d}t \leq \int_0^T \Phi^{crit}(t)\,\mathrm{d}t - \int_0^T 0\,\mathrm{d}t \tag{204}$$

*If $\underline{\mathbf{v}}^{alt}(t)$ denotes the solution of* (185) *with an arbitrary sequence of burning times and initial condition $\underline{\mathbf{v}}(0)$ then*

$$\int_0^T \Phi^{alt}(t)\,\mathrm{d}t \leq \int_0^T \Phi^{crit}(t)\,\mathrm{d}t \tag{205}$$

**Remark 1.** *Let $T > T^g = \frac{1}{m_1(0)}$ and $\varepsilon > 0$. For a suitable choice of $\lambda(t)$ we have*

$$\int_0^T \Phi^{sub}(t)\,\mathrm{d}t - (1-\varepsilon)\int_0^T \lambda(t)\,\mathrm{d}t > \int_0^T \Phi^{crit}(t)\,\mathrm{d}t - (1-\varepsilon)\int_0^T 0\,\mathrm{d}t \tag{206}$$

*For a suitable choice of burning times*

$$\int_0^T \Phi^{alt}(t)\,\mathrm{d}t + \varepsilon\Phi^{alt}(T) > \int_0^T \Phi^{crit}(t)\,\mathrm{d}t + \varepsilon\Phi^{crit}(T) \tag{207}$$

We prove Theorem 3.10 and Remark 1 in Subsection 3.4.3.

Note that the idea that the critical *forest fire* model solves a variational problem is already present in [19].

## 3.2 Definitions, Transformations

In this section we consider a solution $\underline{\mathbf{v}}(t)$ of the general frozen percolation equation (see Definition 3.1.5.). Taking the Laplace transform we get a similar controlled variant of the Burgers equation as in Subsection 2.2.3. We define the *critical core* of $\underline{\mathbf{v}}(t)$, an auxiliary function that we introduce in order to formulate relations similar to (126).

Denote the Laplace transform (generating function) of $\underline{\mathbf{v}}(t)$ by

$$V(t,x) := \sum_{k=1}^{\infty} v_k(t) e^{-kx} \tag{208}$$

for $x > 0$. Then $V(t,0) = V(t,0_+) = m_0(t)$ and by dominated convergence for $x > 0$ (182) is transformed into

$$V(t,x) = V(0,x) + \int_0^t V'(s,x)\left(-V(s,x) + (m_0(0) - \Phi(s)) + \lambda(s)\right) \mathrm{d}s \tag{209}$$

In the sequel we denote the derivative of functions $f(t,x)$ with respect to the time and space variables by $\dot{f}(t,x)$ and $f'(t,x)$, respectively.

Let

$$U(t,x) := V(t,x) - (m_0(0) - \Phi(t)) \tag{210}$$

Thus (209) is transformed into

$$U(t,x) = U(0,x) + \int_0^t -U(s,x)U'(s,x) + \lambda(s)U'(s,x)\,\mathrm{d}s + \Phi(t) \tag{211}$$

Since $V(t,\cdot)$ is a Laplace transform we have

$$U(t,0) = -\theta(t) \qquad U'(t,0) = -m_1(t) \qquad \lim_{x\to\infty} U(x) = -m_0(0) + \Phi(t) \tag{212}$$

and $U$ is a monotone decreasing convex function of the variable $x$ for every $t$.

**Definition 3.2.1.** *Denote by $X(t,u)$ the inverse function of $U(t,x)$ with respect to $x$, that is $U(t,X(t,u)) = u$.*

*The domain of $X(t,u)$ in the variable $u$ is $(-m_0(t) + \Phi(t), -\theta(t)]$.*

$$X(t,-\theta(t)) = 0 \tag{213}$$

The notion of $X(t,\cdot)$ and a version of the following lemma is already present in [39]. This trick (exchanging the roles of independent and dependent variables) is known as the hodograph transform, see Subsection 4.4.3. of [23].

**Lemma 3.2.1.** *If $X(t,u)$ is defined using a solution of the general frozen percolation equation then the following identity holds:*

$$X(t,u) = X\left(0, u - \Phi(t)\right) + t \cdot (u - \Phi(t)) - \int_0^t \lambda(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}s \tag{214}$$

*Proof.* We fix an $x_{min} > 0$. For any $x \geq x_{min}$ we have

$$|U(t,x)| \leq m_0(0), \quad |U'(t,x)| \leq \frac{m_0(t)}{x_{min}}, \quad |U''(t,x)| \leq \frac{m_0(t)}{x_{min}^2}, \tag{215}$$

moreover $\sup_{0\le t\le T}\lambda(t) < +\infty$. For an $x(0) > x_{min}$ denote by $x(t)$ the solution of the integral equation

$$x(t) = x(0) + \int_0^t U(s, x(s)) - \lambda(s)\,\mathrm{d}s \tag{216}$$

This equation is well-posed on the domain $x(t) \ge x_{min}$, since $U(s,x) - \lambda(s)$ is bounded and Lipschitz-continuous in $x$.

Moreover

$$x(t+\mathrm{d}t) - x(t) = \mathcal{O}(\mathrm{d}t), \quad |U(t, x(t)) - U(t, x(t+\mathrm{d}t))| = \mathcal{O}(\frac{\mathrm{d}t}{x_{min}}).$$

If we differentiate (211) w.r.t. $x$ we get $|U'(t+\mathrm{d}t, x) - U'(t,x)| = \mathcal{O}(\frac{\mathrm{d}t}{x_{min}^2})$.

$$\begin{aligned}
&U(t+\mathrm{d}t, x(t+\mathrm{d}t)) - U(t, x(t)) - (\Phi(t+\mathrm{d}t) - \Phi(t)) = \\
&\quad \big(U(t+\mathrm{d}t, x(t+\mathrm{d}t)) - U(t, x(t+\mathrm{d}t))\big) + \\
&\quad \big(U(t, x(t+\mathrm{d}t)) - U(t, x(t))\big) - (\Phi(t+\mathrm{d}t) - \Phi(t)) = \\
&\int_t^{t+\mathrm{d}t} -U(s, x(t+\mathrm{d}t))U'(s, x(t+\mathrm{d}t)) + \lambda(s)U'(s, x(t+\mathrm{d}t))\,\mathrm{d}s + \\
&\quad U'(t, x(t+\mathrm{d}t)) \int_t^{t+\mathrm{d}t} U(s, x(s)) - \lambda(s)\,\mathrm{d}s + \mathcal{O}(\frac{\mathrm{d}t^2}{x_{min}^2}) = \\
&\int_t^{t+\mathrm{d}t} U(s, x(t+\mathrm{d}t))\big(U'(t, x(t+\mathrm{d}t)) - U'(s, x(t+\mathrm{d}t))\big)\,\mathrm{d}s + \\
&\int_t^{t+\mathrm{d}t} U'(t, x(t+\mathrm{d}t))\big(U(s, x(s)) - U(s, x(t+\mathrm{d}t))\big)\,\mathrm{d}s + \\
&\int_t^{t+\mathrm{d}t} \lambda(s)\big(U'(s, x(t+\mathrm{d}t)) - U'(t, x(t+\mathrm{d}t))\big)\,\mathrm{d}s + \mathcal{O}(\frac{\mathrm{d}t^2}{x_{min}^2}) = \mathcal{O}(\frac{\mathrm{d}t^2}{x_{min}^2})
\end{aligned}$$

Thus $U(t, x(t)) = U(0, x(0)) + \Phi(t)$, and if we substitute this back into (216), we get

$$x(t) = x(0) + tU(0, x(0)) + \int_0^t \Phi(s)\,\mathrm{d}s - \int_0^t \lambda(s)\,\mathrm{d}s$$

By the definition of $X(t,u)$ we have $X(t, U(t, x(t))) = x(t)$, and by substituting

$$u = U(0, x(0)) + \Phi(t)$$

we obtain (214). □

Since $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, $V(0,x)$ is well-defined and analytic for all $x \in \mathbb{R}$, thus $X(0,u)$ can be analytically extended to $(-m_0(0), +\infty)$. (214) makes it possible to extend $X(t,u)$ to $(-m_0(0) + \Phi(t), +\infty)$ analytically. The extended $X(t,u)$ is a strictly convex function of the $u$ variable. If we differentiate (214) w.r.t. $u$, we get

$$X'(t,u) = X'(0, u - \Phi(t)) + t \tag{217}$$

**Definition 3.2.2.** *Define $F(t,w)$ by the identity*

$$F(t, -X'(t,u)) = -u \tag{218}$$

*Thus $-F(t,w)$ is the inverse function of $-X'(t,u)$. If $\hat{X}$ denotes the Legendre-transform of $X$ w.r.t. the variable $u$, then*

$$G(t,w) := \hat{X}(t,-w) = -\min_u \{wu + X(t,u)\} = wF(t,w) - X(t, -F(t,w)) \tag{219}$$

*Let*

$$E(t,w) = G''(t,w) = F'(t,w). \tag{220}$$

*We call $E(t,\cdot)$ the* critical core *of $\underline{\mathbf{v}}(t)$. If we use the extended definition of $X$ then $G(t,w)$ is well-defined and analytic for all $w > -t$.*

We have

$$F(t, -\frac{1}{U'(t,x)}) = -U(t,x) \quad \text{and} \quad E(t, -\frac{1}{U'(t,x)}) = \frac{(-U'(t,x))^3}{U''(t,x)} \tag{221}$$

It follows from the properties of the Legendre-transformation and (214) that

$$G(t,w) = G(0, w+t) - w \cdot \Phi(t) - \int_0^t \Phi(s)\,\mathrm{d}s + \int_0^t \lambda(s)\,\mathrm{d}s \tag{222}$$

$$F(t,w) = F(0, w+t) - \Phi(t) \tag{223}$$

$$E(t,w) = E(0, w+t) \tag{224}$$

$G(t,\cdot)$ is strictly convex and $G$ determines $X$ uniquely since the Legendre-transformation is invertible. Define

$$w^*(t) := -X'(t,0) \iff F(t, w^*(t)) = 0 \iff \mathrm{argmin}_w G(t,w) = w^*(t) \tag{225}$$

With the above definition we have

$$X(t,0) = 0 \implies G(t, w^*(t)) = 0 \implies \forall w \,:\, G(t,w) \geq 0 \tag{226}$$

$$\theta(t) = 0 \quad \implies \quad w^*(t) = \frac{1}{m_1(t)} \geq 0 \tag{227}$$

$$x^*(t) = \inf\{x : \sum_{k=1}^{\infty} v_k(t) e^{-kx} < +\infty\} = \min_u X(t,u) = X(t, -F(t,0)) = -G(t,0) \tag{228}$$

## 3.3 Construction of the solution of the frozen percolation equations

In this section we prove Theorem 3.1: the (sub)critical and alternating equations are well-posed. In Lemma 3.3.1 and Lemma 3.3.2 we prove the uniqueness of solutions.

The existence of solutions of the (sub)critical equations follows from Theorem 3.2, but in this section we explicitly construct the solutions which allows us to perform a more fine analysis of the frozen percolation model compared to the forest fire model.

In Lemma 3.3.3 we give the recipe of the construction of the solutions of the (sub)critical equation by the characterization of the suitable control function $\varphi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)$ in terms of the formula (222).

In Lemma 3.3.4 we construct the solution of the critical equations by identifying the critical core of the initial state as the critical control function: for $t \geq T^g$ we have $\varphi(t) = E(0,t)$.

In Lemma 3.3.5 we give upper and lower bounds on $E(0,w)$ using the first two moments of the initial state.

In Lemma 3.3.6 we construct the solution of the subcritical equation by reducing the problem of finding the control function $\varphi(t)$ to solving a first order ODE.

**Lemma 3.3.1.** *The alternating equation* (185) *is well-posed.*

*Proof.* If we are given the sequence of burning times $0 < T_1^b < T_2^b < \dots$ the solution of (185) can be uniquely constructed by using induction on $i$: if we already have the solution on $[0, T_i^b]$, then we are given $m_0(T_i^b)$, so we can uniquely solve the sequence of ordinary differential equations (185) for $v_1, v_2, \dots$ on $[T_i^b, T_{i+1}^b]$ by repeatedly applying the Picard-Lindelöf theorem, since the equation for $v_k$ only contains $v_1, \dots, v_k$ on its right-hand side. □

The proof of the uniqueness of the differential equations related to the (sub)critical frozen percolation is much less technical than the proof of the uniqueness of those related to the forest fire model (see Subsection 2.4.3):

**Lemma 3.3.2.** *The solution of the integral equations* (182) *is unique for every initial condition* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *if* $\lambda(t)$ *is nonnegative and continuous.*

**Remark 2.** *Choosing* $\lambda(t) \equiv 0$ *implies the uniqueness of the solutions of* (183).

*Proof.* In order to prove the uniqueness of the solution of (182), we only have to prove that given two solutions with the same initial condition, the function $\Phi(t) = m_0(0) - m_0(t)$ of the the two solutions must be the same, because $m_0(t)$ and (182) determines $v_k(t)$ for all $k$ uniquely. For a solution $\underline{\mathbf{v}}(t)$ of (182) we can define $U$ by (210), then $X$ by Definition 3.2.1., which satisfies (214) and the $G$ of Definition 3.2.2. satisfies (222).

Assume that $G_1$ and $G_2$ are obtained this way from two solutions of (182) with the same initial condition $G(0,w)$. Let $\tilde{G} = G_1 - G_2$ and $\tilde{\Phi} = \Phi_1 - \Phi_2$. Then

$$\tilde{G}(t,w) = -w \cdot \tilde{\Phi}(t) - \int_0^t \tilde{\Phi}(s)\,\mathrm{d}s$$

Now by (181) we have $\theta(t) = 0$, thus (213) $\implies$ $X(t,0) = 0$, and (226) $\implies$ $\min_w G_1(t,w) = \min_w G_2(t,w) = 0$ and (227) $\implies$ $w_i^*(t) := \text{argmin}_w G_i(t,w) \geq 0$ for $i = 1,2$, thus we have $\tilde{G}(t, w_1^*(t)) \leq 0$ and $\tilde{G}(t, w_2^*(t)) \geq 0$. Thus the function $\tilde{G}(t,w)$, $w \geq 0$ takes on nonpositive and nonnegative values as well, which implies that $\tilde{\Phi}(t)$ and $\int_0^t \tilde{\Phi}(s)\,\mathrm{d}s$ cannot have the same sign. But if $(t_1, t_2)$ is a maximal interval such that for $t_1 < t < t_2$ we have $\int_0^t \tilde{\Phi}(s)\,\mathrm{d}s > 0$ then $\int_0^{t_1} \tilde{\Phi}(s)\,\mathrm{d}s = 0$ and

$$t \in [t_1, t_2] \implies \int_0^t \tilde{\Phi}(s)\,\mathrm{d}s \geq 0 \implies \tilde{\Phi}(t) \leq 0 \implies \int_{t_1}^t \tilde{\Phi}(s)\,\mathrm{d}s \leq 0$$

which contradicts the definition of $t_1$ and $t_2$. Thus $\int_0^t \tilde{\Phi}(s)\,\mathrm{d}s \leq 0$ for all $t$ and interchanging the roles of $G_1$ and $G_2$ we get $\int_0^t \tilde{\Phi}(s)\,\mathrm{d}s \equiv 0$, so $\Phi_1(t) \equiv \Phi_2(t)$. □

**Lemma 3.3.3.** *If we find a function $\varphi(t)$ such that defining $\Phi(t) := \int_0^t \varphi(s)\,\mathrm{d}s$ and $G(t,w)$ by* (222) *we have*

$$\min_w G(t,w) = 0 \quad \textit{and} \quad w^*(t) = \text{argmin}_w G(t,w) \geq 0 \tag{229}$$

*for all $t$, then the solution of* (178) *with the same $\lambda(\cdot)$, $\Phi(\cdot)$ and initial condition satisfies* (182)*.*

*Proof.* Since the Legendre-transformation is invertible, from (229) we get

$$X(t,0) = 0 \quad \text{and} \quad X'(t,0) \leq 0.$$

$X(t,u)$ is strictly decreasing for $u < 0$, thus it is the inverse function of a $U(t,x)$ satisfying $U(t,0) = 0$. If we plug $\Phi(\cdot)$ into (178) then we get $\theta(t) = -U(t,0) = 0$, therefore (181) is satisfied. □

Now we construct the solutions of the critical equation using the critical core (see Definition 3.2.2).

**Lemma 3.3.4.** *The $\Phi$ of the unique solution of* (183) *is*

$$\Phi(T) = \begin{cases} 0 & \textit{if } t \leq T^g \\ F(0,T) & \textit{if } t \geq T^g \end{cases} \tag{230}$$

*where $T^g = \frac{1}{m_1(0)}$.*

$$\int_0^T \Phi(t)\,\mathrm{d}t = \begin{cases} 0 & \textit{if } T \leq T^g \\ G(0,T) & \textit{if } T \geq T^g \end{cases} \tag{231}$$

*Proof.* The solution is unique according to Lemma 3.3.2. and to prove its existence we only have to find a function $\varphi(t)$ that satisfies the criteria of Lemma 3.3.3 (with $\lambda(t) \equiv 0$). We will show that

$$\varphi(t) = \mathbb{1}[t \geq \frac{1}{m_1(0)}]E(0,t) \tag{232}$$

does the job. For $t \le T^g$ this is trivial by looking at (222): $G(t, w^*(t)) = 0$ and $w^*(t) = \frac{1}{m_1(0)} - t \ge 0$ if $\Phi(t) \equiv 0$.

We will show that for $t \ge T^g$ we have $G(t, 0) \equiv 0$ and $F(t, 0) \equiv 0$, that is $w^*(t) \equiv 0$. $F(0, T^g) = G(0, T^g) = 0$ by (226) and $w^*(0) = \frac{1}{m_1(0)} = T^g$. $F(t, 0) = 0$ follows from (223) and

$$\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s = \int_{T^g}^t E(0, s)\,\mathrm{d}s = F(0, t) - F(0, T^g) = F(0, t)$$

By (222) we have

$$G(t, 0) = G(0, t) - \int_0^t \Phi(s)\,\mathrm{d}s = \int_{T^g}^t F(0, s)\,\mathrm{d}s - \int_{T^g}^t F(0, s)\,\mathrm{d}s = 0$$

□

The well-posedness of the integral equation (183) implies that of the corresponding differential equation, since $m_0(0) - \Phi(t) = m_0(t)$ is a continuous function of $t$, thus $v_k(t)$ are differentiable.

We have shown that the solution of (183) has infinite first moment after the gelation time: $\frac{1}{w^*(t)} = m_1(t) = +\infty$ for all $t \ge T^g$.

**Definition 3.3.1.** *Let $E(0, w)$ denote the critical core of $\underline{\mathbf{v}}(0)$ (see Definition 3.2.2).*

*For $\frac{1}{m_1(0)} \le w_1 \le w_2$ define*

$$E_{inf}(w_1, w_2) := \min_{w_1 \le w \le w_2} E(0, w) \quad \text{and} \quad E_{sup}(w_1, w_2) := \max_{w_1 \le w \le w_2} E(0, w).$$

$$E_{sup} := E_{sup}(\frac{1}{m_1(0)}, +\infty), \quad E_{inf}(w) := E_{inf}(\frac{1}{m_1(0)}, w)$$

**Lemma 3.3.5.** *If $w \ge \frac{1}{m_1(0)}$ then the inequalities*

$$\frac{m_1(0)}{m_2(0)} \frac{1}{w^2} \le E(0, w) \le \frac{1}{w^2} \tag{233}$$

*hold. Thus $E_{sup} \le m_1(0)^2$ and $E_{inf}(w) \ge \frac{m_1(0)}{m_2(0)} \frac{1}{w^2}$.*

*For all $w \ge \frac{1}{m_1(0)}$ we have*

$$|E'(0, w)| \le 4 m_2(0)^2 m_3(0) =: D \tag{234}$$

*which implies*

$$E_{sup}(w_1, w_2) - E_{inf}(w_1, w_2) \le D \cdot (w_2 - w_1) \tag{235}$$

**Remark 3.** *If $m_1(0) = m_2(0)$ then the upper and lower bounds in (233) coincide. This can only happen if $v_k(0) = m_1(0) \cdot \mathbb{1}[k = 1]$, this is the case known as the monodisperse initial condition.*

*Proof.* Let $U(x) := U(0,x)$. Recalling (221) $E\left(0, -\frac{1}{U'(x)}\right) = \frac{(-U'(x))^3}{U''(x)}$ holds. The upper bound of (233) follows from $-U'(x) \leq U''(x)$, and $-U'(x)\frac{m_2(0)}{m_1(0)} \geq U''(x)$ holds because $\log(-U'(x))$ is a convex function, thus $\frac{U''(x)}{U'(x)} \geq \frac{U''(0)}{U'(0)} = \frac{m_2(0)}{-m_1(0)}$. The bound on the Lipschitz constant (234) follows from

$$\left|E'\left(0, -\frac{1}{U'}\right)\right| = \left|\frac{(U')^5 U'''}{(U'')^3} - 3\frac{(U')^4}{U''}\right| \leq \left|(U')^2 U'''\right| + 3\left|(U')^3\right| \leq 4m_2(0)^2 m_3(0)$$

□

Now we construct the solution to the subcritical equation (182).

We assume $\lambda(t) > 0$ for all $t$. If we substitute $x = 0$ into the differential equation (211) and assume $|U'(t,0)| < +\infty$ then (formally) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \varphi(t) = -U'(t,0)\cdot\lambda(t) = m_1(t)\lambda(t) = \frac{\lambda(t)}{w^*(t)},$$

**Definition 3.3.2.** *If $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ and $\lambda(t)$ is a positive continuous function then the subcritical control differential equation for $w^*(t)$ is*

$$\dot{w}^*(t) = \frac{\lambda(t)}{w^*(t)E(0, t + w^*(t))} - 1 \tag{236}$$

*with initial condition $w^*(0) = \frac{1}{m_1(0)} = T^g$.*

**Lemma 3.3.6.** *The subcritical control differential equation is well-posed and the function*

$$\varphi(t) := \frac{\lambda(t)}{w^*(t)}$$

*(where $w^*(t)$ is the solution of (236) with $w^*(0) = \frac{1}{m_1(0)}$) satisfies the criteria of Lemma 3.3.3, which implies the existence of solutions to (182).*

*Proof.* We prove the statement of the lemma on $[0,T]$. The Picard-Lindelöf theorem and the Lipschitz-continouity property (234) gurantee the existence and uniqueness of the solution of (236) before the graph of the solution exits

$$\{(t, w^*) : 0 \leq t \leq T, \quad w^*_{min} \leq w^* \leq w^*_{max}, \quad w^* + t \geq w^*(0)\} \tag{237}$$

for some $0 < w^*_{min} < w^*_{max} < +\infty$.

Let $\lambda_{inf} := \inf_{0 \leq t \leq T} \lambda(t)$, $\lambda_{sup} := \sup_{0 \leq t \leq T} \lambda(t)$. From (236) and a "forbidden region"-type argument we get that $w^*(t) + t \geq w^*(0)$ and $w^*(t) \geq \min\{\frac{\lambda_{inf}}{E_{sup}}, w^*(0)\}$, since

$$w^*(t) > 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}(w^*(t) + t) \geq 0$$

$$w^*(t) + t \geq w^*(0) \implies E(0, t + w^*(t)) \leq E_{sup},$$

thus $w^*(t) < \frac{\lambda_{inf}}{E_{sup}} \implies \dot{w}^*(t) > 0$.

Now we prove that $w^*(t)$ cannot grow too fast using the lower bound of (233). $w^*(t) \leq y(t)$ where $y(0) = w^*(0) = T^g$ and

$$\dot{y}(t) = \lambda_{sup} \frac{m_2(0)}{m_1(0)} \frac{(y(t)+t)^2}{y(t)} \leq \lambda_{sup} \frac{m_2(0)}{m_1(0)} \left( \frac{T^g + t}{T^g} \right) \cdot (y(t)+t)$$

since $y(t)$ is increasing. Thus $\dot{y}(t) \leq a \cdot y(t) + b$ for some $a$ and $b$ depending only on the initial data, the function $\lambda(t)$ and $T$. Thus

$$w^*(t) \leq w^*(0) e^{at} + \frac{b}{a} \cdot (e^{at} - 1).$$

Now we can see that the graph of the solution of (236) indeed doesn't exit (237) until $t = T$ if we define

$$w^*_{min} = \min\{\frac{\lambda_{inf}}{E_{sup}}, T^g\} \quad \text{and} \quad w^*_{max} := (T^g + \frac{b}{a}) e^{aT} \tag{238}$$

Now we prove that $\varphi(t) := \frac{\lambda(t)}{w^*(t)}$ satisfies the criteria of Lemma 3.3.3. by showing that

$$G(t, w^*(t)) \equiv 0 \quad \text{and} \quad F(t, w^*(t)) \equiv 0.$$

This holds for $t = 0$, so it suffices to check $\frac{\mathrm{d}}{\mathrm{d}t} G(t, w^*(t)) \equiv 0$ and $\frac{\mathrm{d}}{\mathrm{d}t} F(t, w^*(t)) \equiv 0$. Using (223)

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t, w^*(t)) = E(0, t + w^*(t)) \cdot \left( 1 + \frac{\lambda(t)}{w^*(t) E(0, t + w^*(t))} - 1 \right) - \frac{\lambda(t)}{w^*(t)} = 0$$

If we combine $F(t, w^*(t)) \equiv 0$ with (223) we get

$$F(0, t + w^*(t)) = \Phi(t) \tag{239}$$

It is straightforward to verify $\frac{\mathrm{d}}{\mathrm{d}t} G(t, w^*(t)) \equiv 0$ by using (222) and (239). □

This completes the proof of the well-posedness of (182).

## 3.4 Properties of the solutions of the frozen percolation equations

In this section we we investigate the properties of the solutions of the frozen percolation equations using the explicit solutions derived in the previous section.

In Subsection 3.4.1 we show how the function $G(t, w)$ (see (222)) corresponding to the solution of the alternating equation evolves and compare it with the functions $G(t, w)$ of the solutions of the (sub)critical equations started from the same initial condition.

In Subsection 3.4.2 and Subsection 3.4.3 we prove the results about the self-organized critical and extremum properties of the critical frozen percolation equation stated in Subsection 3.1.4 and Subsection 3.1.6.

### 3.4.1 Alternating equations

In order to compare the solutions of (185) and (183) we apply the transformations

$$\underline{\mathbf{v}}(t) \to U(t,x) \to X(t,u) \to G(t,w) \tag{240}$$

to the solutions of the alternating equations:

The integral equation

$$U(t,x) = U(0,x) + \int_0^t -U(s,x)U'(s,x)\,\mathrm{d}s + \Phi(t) \tag{241}$$

holds, but $\Phi(t)$ is constant between burning times and jumps by $\theta(T_i^b)$ at $T_i^b$, which means that the giant component is burnt:

$$\lim_{\varepsilon\to 0} -U(T_i^b+\varepsilon, 0) = \lim_{\varepsilon\to 0} \theta(T_i^b+\varepsilon) = \theta(T_{i+}^b) = 0$$

By Lemma 3.2.1. the formulas (214), (222), (223) and (224) are valid (with rate function $\lambda(t) \equiv 0$).

In between the burning times $T_i^b < t \le T_{i+1}^b$ we have

$$X(t,u) = X(T_{i+}^b, u) + (t - T_i^b)u \quad \text{and} \quad G(t,w) = G(T_{i+}^b, w + (t - T_i^b)). \tag{242}$$

If $t - T_i^b > w^*(T_{i+}^b)$ then $\underline{\mathbf{v}}(t)$ is supercritical: if we define $\theta(t)$ by (186) then

$$X'(t,0) > 0, \quad \theta(t) > 0, \quad X(t, -\theta(t)) = 0, \quad X'(t, -\theta(t)) < 0. \tag{243}$$

$\min_w G(t,w) = 0$ still holds, but $\mathrm{argmin}_w G(t,w) = w^*(t) < 0$ in the supercritical phase. Thus $-X'(t,0) = w^*(t)$ is well-defined for all $t \ge 0$ for the solutions of the equations (182), (183) and (185) as well, moreover (228) holds. For the solutions of (185) $w^*(t)$ is left-continuous.

By $G(t, w^*(t)) \equiv 0$, (222) and (239) we get

$$\int_0^t \Phi(s)\,\mathrm{d}s = G(0, t + w^*(t)) - w^*(t)F(0, t + w^*(t)) \tag{244}$$

for the solutions of (185).

Now we take a unified look at the solutions of the subcritical, critical and alternating equations using the transformations (240).

If $\underline{\mathbf{v}}(t)$ is the solution of (182), (183) or (185) started from $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, then (224) holds: the evolution of the critical core does not depend on the rate of lightnings.

Although $E(t,w) = G''(t,w)$ we only need one extra parameter to determine determine $\underline{\mathbf{v}}(t)$ and $\theta(t)$ in addition to $E(t,\cdot)$: if we know $w^*(t)$, then

$$F(t,w) = \int_{w^*(t)}^w E(0, t+y)\,\mathrm{d}y \quad \text{and} \quad G(t,w) = \int_{w^*(t)}^w (w-y)E(0,t+y)\,\mathrm{d}y \tag{245}$$

has all the information about $\underline{\mathbf{v}}(t)$ and $\theta(t)$, since the transformations (240) are invertible (using analytic extensions).

### 3.4.2 Self-organized criticality and self-similarity

In this subsection we prove Theorem 3.5, Claim 1, Theorem 3.6 and Theorem 3.7.

*Proof of Theorem 3.5.* It is clear from (232) and (234) that $\varphi(t)$ is continuous. By (224) we have

$$X''(t,0) = \frac{1}{E(t,0)} = \frac{1}{E(0,t)} = \frac{1}{\varphi(t)}$$

$$X(t,u) = \frac{1}{2\varphi(t)}u^2 + \mathcal{O}(u^3), \quad \lim_{x\to 0} \frac{-U(t,x)}{\sqrt{x}} = \sqrt{2\varphi(t)} \tag{246}$$

To deduce (189) from this we might use the Tauberian theorem (142) similarly to the proof of (19).

□

*Proof of Claim 1.* First assume $m_0(0) = 1$. As a consequence of Remark 3. we can see that

$$E(t,w) = E(0,t+w) = \frac{1}{(w+t)^2} = \frac{1}{t^2}E(1,\frac{w}{t}), \tag{247}$$

but this is the critical core of $\frac{1}{t}\underline{\mathbf{v}}(1)$, and together with $w^*(t) \equiv 0$ for $t \geq T^g = 1$ the identity $v_k(t) = \frac{1}{t}v_k(1)$ follows. We get the explicit formula for $v_k(1)$ in the following way: since $X(1,u) = X(0,u) + u$, the inverse function of $V(1,x)$ is $-\log(v) + v - 1$, thus

$$V(1,x) = -W\left(-e^{-(x+1)}\right) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} e^{-kx}$$

where $W$ is the Lambert W function, the inverse function of $z \mapsto ze^z$.

If $m_0(0) \neq 1$ but we still have a monodisperse initial condition then (247) still holds and for $t \geq \frac{1}{m_0(0)} = T^g$ and we have $w^*(t) = 0$ thus $v_k(t) = \frac{1}{t}\frac{k^{k-1}}{k!}e^{-k}$ must hold. □

*Proof of Theorem 3.7.* Let $H(w) := F(0,w) - m_0(0)$, thus $H\left(-\frac{1}{V'(0,x)}\right) = -V(0,x)$ by (221). Using Lemma 3.3.4. and (223) we get

$$F(t,w) = H(t+w) - H(t) \qquad \text{and} \qquad m_0(t) = F(t,+\infty) = -H(t)$$

for $t \geq T^g$. $v_1(0) > 0$ implies $\lim_{x\to\infty} -\frac{V'(0,x)}{V(0,x)} = 1$, so $\lim_{t\to\infty} t \cdot H(tw) = -\frac{1}{w}$, from which $\lim_{t\to\infty} t m_0(t) = 1$ follows. Moreover

$$1 - \frac{1}{w+1} = \lim_{t\to\infty} t \cdot (H(t\cdot(w+1)) - H(t)) = \lim_{t\to\infty} t \cdot F(t,tw) = \lim_{t\to\infty} \hat{F}(t,w)$$

where $\hat{v}_k(t) = tv_k(t)$. This implies the pointwise convergence of the monotone functions $\hat{X}'(t,u)$, $\hat{X}(t,u)$, $\hat{U}(t,x)$ and $\hat{V}(t,x)$ to the desired limit as $t \to \infty$. The convergence of $\hat{v}_k(t)$ to $\frac{k^{k-1}}{k!}e^{-k}$ follows from the continuity theorem of Laplace transforms. □

*Proof of Theorem 3.6.* It is easy to check that if $\tilde{v}_k(t) = v_k(t)e^{-kx^*(t)}$, then $\tilde{V}(t,x) = V(t,x+x^*(t))$, so $\tilde{x}^*(t) = 0$ and $\tilde{w}^*(t) = 0$, but $\tilde{E}(t,w) = E(0,t+w) = E(t,w)$, so $\underline{\tilde{\mathbf{v}}}(t)$ is identical to the solution of (183) at time $t$. □

### 3.4.3 The extremum property of the critical model

In this Subsection we prove Theorem 3.10 and Remark 1.

*Proof of Theorem 3.10.* If we consider the solution of (182) with given initial data and lightning rate function $\lambda(t) \geq 0, 0 \leq t \leq T$ then (222) provides us with a relation between our cost ($\int_0^T \lambda(t)\,\mathrm{d}t$) and reward ($\int_0^T \Phi(t)\,\mathrm{d}t$).

We prove (204) by considering the cases $T \geq T^g$ and $T \leq T^g$ separately.

According to (231), for $T \geq T^g$ we get

$$0 \leq G^{sub}(T,0) = \int_0^T \Phi^{crit}(t)\,\mathrm{d}t - \int_0^T \Phi^{sub}(t)\,\mathrm{d}t + \int_0^T \lambda(t)\,\mathrm{d}t$$

by substituting $w = 0$ into (222).

For $T \leq T^g$, we want to prove $0 \geq \int_0^T \Phi^{sub}(t)\,\mathrm{d}t - \int_0^T \lambda(t)\,\mathrm{d}t$. Substitute $w = T^g - T$ into (222). Since $G(0,T^g) = 0$ and $(T^g - T)\Phi^{sub}(T) \geq 0$ we get

$$0 \leq G^{sub}(T, T^g - T) \leq -\int_0^T \Phi^{sub}(t)\,\mathrm{d}t + \int_0^T \lambda(t)\,\mathrm{d}t.$$

The proof of the extremum property (205) is equally simple. ☐

If we want to maximize our cost functional for a fixed $T > T^g$, the optimal control is not unique, since the only thing we need for

$$\int_0^T \Phi^{sub}(t)\,\mathrm{d}t - \int_0^T \lambda(t)\,\mathrm{d}t = \int_0^T \Phi^{crit}(t)\,\mathrm{d}t \tag{248}$$

to hold is $G^{sub}(T,0) = 0$: if $\underline{\mathbf{v}}(T)$ is critical at time $T$, then the value of the functional is optimal.

*Proof of Remark 1.* In order to prove (206) first pick an arbitrary $\lambda > 0$ and solve (236) with constant $\lambda(t) = \lambda$. Since $w^*(t) > 0$ and $w^*(0) = T^g$ there is a $0 < t^* \leq T$ such that $w^*(t^*) = T - t^*$, and the lightning rate function $\lambda(t) = \lambda \cdot \mathbb{1}[t \leq t^*]$ makes $T$ a critical time, so (248) holds, thus (206).

Now we prove (207). By using (244) we have to show that

$$G(0, T + w^*(T)) - (w^*(T) - \varepsilon)F(0, T + w^*(T)) > G(0,T) + \varepsilon F(0,T)$$

Using $G(0, T + w^*(T)) - G(0,T) > w^*(T)F(0,T)$ it is easy to see that $0 < w^*(T) \leq \varepsilon$ is sufficient for this to hold. If there is a $T^g < t^* \leq T$ such that $-X'(t^*, -\theta(t^*)) = T - t^* + \varepsilon$, then burning the giant component at time $t^*$ we get $-X'(t^*_+, 0) = T - t^* + \varepsilon$ and $-X'(T,0) = w^*(T) = \varepsilon$. If not, then burning at time $T$ yields $0 < -X'(T, -\theta(T)) = w^*(T_+) < \varepsilon$. ☐

## 3.5 Proof of the subcritical limit theorem

In this section we prove Theorem 3.8. In Lemma 3.5.1 and Lemma 3.5.2 we prove (198), i.e. that if the rate of lightnings $\lambda$ is very small then burning rate $\varphi_\lambda(t)$ is very close to the critical burning rate $\varphi_{crit}(t)$.

In Lemma 3.5.3 we prove the limit theorem (201) using the continuity theorem of Laplace transforms.

The last proof of this section is that of Theorem 3.3.

In order to prove (198), we need to know more about the solution of (236):

**Lemma 3.5.1.** *If $y(t)$ is the solution of the differential equation $\dot{y}(t) = \frac{c}{y(t)} - 1$ with initial condition $y(0) = T^g$ and $t \geq T^g + c\log(\frac{T^g}{c})$ then $y(t) \leq 2c$.*

*Proof.* The solution of this differential equation is

$$y(t) = c \cdot \left(1 + W\left(\exp\left(\frac{T^g - t}{c} - 1\right) \cdot \left(\frac{T^g}{c} - 1\right)\right)\right) \tag{249}$$

where $W$ is the Lambert W function. Thus $W(x) \leq x$ and our claim follows. □

**Lemma 3.5.2.** *If $w^*(t)$ is the solution of (236) with constant $\lambda(t) \equiv \lambda \leq 1$ then there exist $d_1$ and $d_2$ which depend only on $\underline{\mathbf{v}}(0)$ and $T$ such that*

$$T^g + d_1\lambda\log(\frac{1}{\lambda}) \leq t \leq T \quad \Longrightarrow \quad \left|w^*(t) - \frac{\lambda}{E(0,t)}\right| \leq d_2\lambda^2.$$

**Remark 4.** *From this it follows that if we fix $T^g < t \leq T$ and let $\lambda \to 0$ then*

$$\varphi_\lambda(t) - \varphi_{crit}(t) = \lambda m_1^\lambda(t) - E(0,t) = \lambda \cdot \left(\frac{1}{w^*(t)} - \frac{E(0,t)}{\lambda}\right) = \mathcal{O}(\lambda),$$

*which proves (198).*

*Proof.* We have a uniform a priori bound $w^*(t) \leq w^*_{max}$ for all $\lambda \leq 1$ depending only on the initial data and $T$ by (238). Thus by Lemma 3.3.5 we have

$$0 < E_{inf} := E_{inf}(T + w^*_{max}) \leq E(0, t + w^*(t))$$

and substituting this inequality into (236) we get

$$\dot{w}^*(t) \leq \frac{\lambda}{w^*(t)E_{inf}} - 1 \tag{250}$$

Using Lemma 3.5.1. we get

$$\hat{t} := T^g + \frac{\lambda}{E_{inf}}\log(\frac{E_{inf}T^g}{\lambda}) \leq t \leq T \quad \Longrightarrow \quad w^*(t) \leq 2\frac{\lambda}{E_{inf}}.$$

Define

$$z(t) := \frac{w^*(t)E(0, t + w^*(t))}{\lambda} - 1$$

Using (236) we get

$$\dot{z}(t) = -\frac{1}{w^*(t)}z(t) + \frac{E'(0, t + w^*(t))}{E(0, t + w^*(t)} \tag{251}$$

For $\hat{t} \le t \le T$ we have

$$-1 \le z(\hat{t}) \le 2\frac{E_{sup}}{E_{inf}}, \qquad \frac{1}{w^*(t)} \ge \frac{1}{2}\frac{E_{inf}}{\lambda}, \qquad \left|\frac{E'(0, t + w^*(t))}{E(0, t + w^*(t)}\right| \le \frac{D}{E_{inf}} \tag{252}$$

with the $D$ of (234). Solving the linear ODE (251) and using the inequalities (252) we get

$$|z(t)| \le 2\frac{E_{sup}}{E_{inf}} \exp\left(-\frac{1}{2}\frac{E_{inf}}{\lambda}(t - \hat{t})\right) + \lambda\frac{2D}{E_{inf}^2}.$$

Thus for $t \ge \hat{t} + \frac{2}{E_{inf}}\lambda \log\left(\frac{1}{\lambda}\right)$ we have $|z(t)| = \mathcal{O}(\lambda)$, which implies

$$w^*(t) - \frac{\lambda}{E(0, t + w^*(t))} = \mathcal{O}(\lambda^2).$$

If we combine this with

$$\left|\frac{\lambda}{E(0, t + w^*(t))} - \frac{\lambda}{E(0, t)}\right| \le \lambda^2 2\frac{D}{E_{inf}^3}$$

the claim of the Lemma follows. □

Now we prove (201). First we take the Laplace transform of the random variable $Y_\lambda(t)$ defined by (200):

$$\sum_{k=1}^{\infty} \mathbf{P}\big(\, Y_\lambda(t) = k\,\big) \cdot e^{-kx} = \frac{\sum_{k=1}^{\infty} k \cdot v_k^\lambda(t)e^{-kx}}{\sum_{k=1}^{\infty} k \cdot v_k^\lambda(t)} = \frac{U'_\lambda(t, x)}{U'_\lambda(t, 0)} \tag{253}$$

**Lemma 3.5.3.** *Let $U_\lambda(t, x)$ be the solution of (211) with a fixed initial condition $U(0, x)$ obtained from $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ and $\lambda(t) \equiv \lambda$. Then for any $t > T^g$ we have*

$$\lim_{\lambda \to 0} \frac{U'_\lambda\left(t, \frac{\lambda^2}{2E(0,t)}x\right)}{U'_\lambda\,(t, 0)} = \frac{1}{\sqrt{1 + x}} \tag{254}$$

*Proof.* Fix $\lambda > 0$ and denote the solution of (211) with $\lambda(t) \equiv \lambda$ by $U(t, x)$. For all $t \ge 0$ we have

$$X''(t, u) \ge \frac{1}{E_{sup}} \quad \Longrightarrow \quad X(t, u) \ge \frac{1}{2E_{sup}}u^2 \quad \Longrightarrow \quad |U(t, x)| = \mathcal{O}(\sqrt{x}).$$

We use the shorthand notation $E = E(0, t + w^*(t))$.

$$X'(t,u) = -w^*(t) + \frac{u}{E} + \mathcal{O}(u^2), \quad X(t,u) = -uw^*(t) + \frac{u^2}{2E} + \mathcal{O}(u^3)$$

$$U(t,x) = Ew^*(t) - \sqrt{(Ew^*(t))^2 + 2E(x - \mathcal{O}(U(t,x)^3))} = \\ Ew^*(t) - \sqrt{(Ew^*(t))^2 + 2Ex} + \mathcal{O}(x)$$

$$U'(t,x) = \frac{1}{X'(t,U(t,x))} = \frac{1}{-w^*(t) + \frac{U(t,x)}{E}} + \mathcal{O}(1) = \\ \frac{-1}{\sqrt{w^*(t)^2 + \frac{2}{E}x + \mathcal{O}(x)}} + \mathcal{O}(1) = \frac{-1}{\sqrt{w^*(t)^2 + \frac{2}{E}x}} + \mathcal{O}(1)$$

Because of Lemma 3.5.2. we have

$$\lim_{\lambda \to 0} \frac{\lambda^2}{E(0, t + w^*_\lambda(t))E(0,t)w^*_\lambda(t)^2} = 1$$

from which the claim of this lemma follows.

□

The r.h.s. of (254) is the Laplace transform of the $\Gamma(\frac{1}{2}, 1)$ distribution and the r.h.s. of (201) is the distribution function of the $\Gamma(\frac{1}{2}, 1)$ distribution, so (201) follows from (253), (254) and the continuity theorem of Laplace transforms.

*Proof of Theorem 3.3.* First observe that instead of proving uniform convergence of $\Phi_n$ to $\Phi_{crit}$ we only need to show convergence on $[0, T]$ for any $T$, because

$$T \geq T^g \implies m_0(T) = \int_{T+w^*(T)}^{\infty} E(0,w)\,\mathrm{d}w \leq \int_T^{\infty} \frac{1}{w^2}\,\mathrm{d}w = \frac{1}{T}$$

by (233), thus $0 \leq \Phi_n(t) - \Phi_{crit}(t) \leq \frac{1}{T}$ for $t \geq T$. If we prove that $w^*(t)$ is small for $t \geq \frac{1}{m_1(0)}$ then we are done by (239) and Lemma 3.3.4, since

$$0 \leq \Phi_n(t) - \Phi_{crit}(t) = F(0, t + w^*_n(t)) - F(0,t) \leq w^*_n(t)E_{sup} \quad t \geq T^g \tag{255}$$

$$\Phi_n(t) \leq \Phi_n(T^g) = F(0, T^g + w^*_n(T^g)) \leq w^*_n(T^g)E_{sup} \quad t \leq T^g$$

We can give an upper bound on $w^*(t)$ for $t \geq T^g$ if we replace $\lambda(t)$ with $\lambda_{sup}$ in (236): using (250) we get $w^*(t) = \mathcal{O}\left(\lambda \log(\frac{1}{\lambda})\right)$ if we substitute $t \geq T^g$ and $c = \frac{\lambda_{sup}}{E_{inf}}$ into (249), thus $\lim_{n\to\infty} w^*_n(t) = 0$ uniformly for $T^g \leq t \leq T$.

We obtain $\lim_{n\to\infty} v^n_k(t) = v_k(t)$ for $k = 1, 2, \ldots$ by the uniform convergence of $m^n_0(t)$ and $\lambda^n(t)$ to the critical $m_0(t) = m_0(0) - \Phi(t)$ and $\lambda(t) \equiv 0$ in (182). □

## 3.6 Proof of the alternating limit theorem

In this Section we prove Theorem 3.4 and Theorem 3.9.

In Subsection 3.6.1 we introduce the notations needed to describe the sequence of random gelation times and burning times of the solution of the random alternating equations.

In Subsection 3.6.2 we give a heuristic proof of Theorem 3.9 by relating the lengths of time intervals between the gelation times to a sequence of i.i.d. random variables with Rayleigh distribution.

In Subsection 3.6.3 we prove some technical bounds needed for the rigorous proof of Theorem 3.9.

In Subsection 3.6.4 we prove Theorem 3.4 and make the heuristic proof of Theorem 3.9 rigorous using a coupling argument.

Throughout this section we assume $m_0(0) = 1$ but the results generalize easily to the $m_0(0) \neq 1$ case, since if $\underline{\mathbf{v}}(t)$ is the solution of (185) with burning times $T_1^b, T_2^b, \dots$ and $m_0(0) = 1$ then $m \cdot \underline{\mathbf{v}}(m \cdot t)$ is also a solution of (185) with burning times $\frac{T_1^b}{m}, \frac{T_2^b}{m}, \dots$ and $m_0(0) = m$.

### 3.6.1 Notations

First recall some facts related to the subcritical, critical and supercritical phases of the solution of the alternating equation:

The definition of $M(t)$ given $T_1^b, T_2^b, \dots$ and $t$ is (184).

For the solutions of the alternating equations (185) the mass of the giant component $\theta(t)$ is related to $m_0(t)$ by (186).

By (208) we have $V(t,x) := \sum_{k=1}^{\infty} v_k(t) e^{-kx}$.

By (210) we have $U(t,x) := V(t,x) - (m_0(0) - \Phi(t))$.

By (212) we have $U(t,0) = -\theta(t)$ and by (213) we have $X(t, -\theta(t)) = 0$.

The relations between $w^*(t)$ and $X$,$F$,$G$ are formulated in (225).

The relation of $x^*(t)$ to $X$, $F$ and $G$ is formulated in (228).

Note that $x^*(t) = \limsup_{n\to\infty} \frac{1}{n} \ln \big(v_k(t)\big)$ by the Cauchy-Hadamard theorem.

The description of $X(t,u)$ in the supercritical phase is (243).

The behavior of $X$ and $U$ at a critical time is described in (246).

We now summarize the main properties of the solution of the alternating equations (185) started from an initial condition $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ (see (170)).

- If $w^*(t) > 0$ then $\big(\underline{\mathbf{v}}(t), \theta(t)\big)$ is *subcritical*:

  $X'(t,0) < 0$, $\theta(t) = m_0(T^b_{M(t)}) - m_0(t) = 0$, $U(t,0) = 0$, $m_1(t) < +\infty$, $x^*(t) < 0$.

- If $w^*(t) > 0$ then $\big(\underline{\mathbf{v}}(t), \theta(t)\big)$ is *supercritical*:

  $X'(t,0) > 0$, $\theta(t) = m_0(T^b_{M(t)}) - m_0(t) > 0$, $U(t,0) = -\theta(t) < 0$, $X(t, -\theta(t)) = 0$, $X'(t, -\theta(t)) < 0$, $m_1(t) < +\infty$, $x^*(t) < 0$.

- If $w^*(t)=0$ then $(\underline{\mathbf{v}}(t),\theta(t))$ is *critical*:
  $X'(t,0)=0$, $\theta(t)=m_0(T^b_{M(t)})-m_0(t)=0$, $U(t,0)=0$, $m_1(t)=+\infty$, $x^*(t)=0$, $X(t,u)\asymp u^2$, $|U(t,x)|\asymp\sqrt{x}$, $\sum_{l=k}^{\infty}v_l(t)\asymp k^{-1/2}$.

Thus in all the three phases we have

$$\theta(t)=-\min\{u\,:\,X(t,u)=0\}. \tag{256}$$

We assume that $w^*(t)$ and $\theta(t)$ are left-continuous.

**Definition 3.6.1.** *If $\underline{\mathbf{v}}(t)$ is a solution of (185), let $w^*_+(t):=\frac{1}{m_1(t)}$. Thus $w^*_+(t)=-X'(t,-\theta(t))$. Of course if $w^*(t)\geq 0$ then $w^*_+(t)=w^*(t)$.*

If $t$ is a burning time then $w^*(t_+):=\lim_{\varepsilon\to 0}w^*(t+\varepsilon)=w^*_+(t)$.

**Definition 3.6.2.** *If $\underline{\mathbf{v}}(t)$ is the solution of the random alternating equations (see Definition 3.1.6.), denote by $T^b_1<T^b_2<\dots$ the sequence of random burning times and by $T^g=T^g_1<T^g_2<\dots$ the sequence of random gelation times. Indeed $T^g_1<T^b_1<T^g_2<T^b_2<\dots$*

*Let $\tau_i:=T^g_{i+1}-T^g_i$ be the length of the $i$-th critical interval.*

$$N(t):=\max\{i:T^g_i<t\},\quad \tau(t,i):=\tau_{N(t)+i}$$

*$\tau(t,0)$ is the length of the critical interval containing $t$.*

*Let $\theta(t,i):=\theta(T^b_{N(t)+i})$. Let*

$$w^*_-(t,i):=T^b_{N(t)+i}-T^g_{N(t)+i}\overset{(*)}{=}\lim_{\varepsilon\to 0}-w^*(T^b_{N(t)+i}-\varepsilon)=-w^*(T^b_{N(t)+i}) \tag{257}$$

$$w^*_+(t,i):=T^g_{N(t)+i+1}-T^b_{N(t)+i}\overset{(\dagger)}{=}\lim_{\varepsilon\to 0}w^*(T^b_{N(t)+i}+\varepsilon)=w^*_+(T^b_{N(t)+i}) \tag{258}$$

The equation $(*)$ in (257) holds true because if $t$ is not a burning time then

$$\frac{\mathrm{d}}{\mathrm{d}t}w^*(t)=-\frac{\mathrm{d}}{\mathrm{d}t}X'(t,0)\overset{(242)}{=}-1, \tag{259}$$

thus

$$w^*(T^b_{N(t)+i})=w^*(T^g_{N(t)+i})+\int_{T^g_{N(t)+i}}^{T^b_{N(t)+i}}(-1)\mathrm{d}s=0+T^g_{N(t)+i}-T^b_{N(t)+i}$$

Similarly $(\dagger)$ in (258) holds because

$$0=w^*(T^g_{N(t)+i+1})=w^*_+(T^b_{N(t)+i})+\int_{T^b_{N(t)+i}}^{T^g_{N(t)+i+1}}(-1)\mathrm{d}s=w^*_+(T^b_{N(t)+i})-(T^g_{N(t)+i+1}-T^b_{N(t)+i}).$$

**Definition 3.6.3.** *A nonnegative random variable $X$ has Rayleigh distribution with parameter $\sigma$, briefly $X \sim R(\sigma)$, if*

$$\mathbf{P}\big( X > x \big) = \exp(-\frac{1}{2\sigma^2}x^2) =: R(\sigma, x)$$

$\mathbf{E}\big( X \big) = \sigma\sqrt{\frac{\pi}{2}}$. *$Y$ has a size-biased Rayleigh distribution with parameter $\sigma$, briefly $Y \sim R_{sb}(\sigma)$ if*

$$\mathbf{P}\big( Y > y \big) = \frac{\mathbf{E}\big( X \cdot \mathbb{1}[X > y] \big)}{\mathbf{E}\big( X \big)} = R_{sb}(\sigma, y) \tag{260}$$

*The scaling identities*

$$R(\sigma, x) = R(a\sigma, ax) \quad \text{and} \quad R_{sb}(\sigma, x) = R_{sb}(a\sigma, ax) \tag{261}$$

*are valid for $a > 0$.*

### 3.6.2 Sketch proof of Theorem 3.9

First we give the heuristic picture of the evolution of the size of the giant component in the alternating model with frequent lightnings. We use the notations of the statement of Theorem 3.4.

Fix $t \geq T^g = \frac{1}{m_1(0)}$. Since $\lim_{\lambda\to\infty} \delta(\lambda) = 0$ we might assume that $E := E(0,t) = \varphi_{crit}(t) \approx \varphi_{crit}(s)$ for all $t \leq s \leq t + \delta(\lambda)$.

For the solution of the critical equation (183) we have $m_1(t) = +\infty$ for all $t \geq \frac{1}{m_1(0)}$ and later we are going to show that if we consider the solution of the random alternating equations with $1 \ll \lambda$ (i.e. lightnings are very frequent) then the model is close to criticality, i.e. $1 \ll m_1(t)$. Thus $w^*(t) \ll 1$ if $1 \ll \lambda$.

$$X''(t,0) \overset{(218)}{=} \frac{1}{F'(t, w^*(t))} \overset{(220)}{=} \frac{1}{E(t, w^*(t))} \overset{(224)}{=} \frac{1}{E(0, t + w^*(t))} \approx \frac{1}{E}$$

$$X(t,u) \approx X(t,0) + X'(t,0) \cdot u + \frac{1}{2}X''(t,0) \cdot u^2 \approx -w^*(t) \cdot u + \frac{1}{2E}u^2 \tag{262}$$

holds. Note that if $w^*(t) > 0$ then the system is subcritical, if $w^*(t) < 0$ then the system is supercritical and if $w^*(t) = 0$ then the system is critical, as described at the beginning of Subsection 3.6.1.

From (256) and (262) we get

$$\theta(t) \approx -2E \cdot w^*(t) \cdot \mathbb{1}[w^*(t) < 0], \qquad w^*_+(t) \approx |w^*(t)| \tag{263}$$

In fact if $w^*(t) < 0$ and $t$ is a burning time then

$$X(t_-, u) \approx w^*(t) \cdot u + \frac{1}{2E}u^2 \quad \Longrightarrow \quad X(t_+, u) \approx -w^*(t) \cdot u + \frac{1}{2E}u^2$$

This relation is known as the *duality principle* in the theory of random graphs: removing the giant component from a supercritical instance of the model leaves (essentially) a subcritical instance which is of the same distance from criticality. Thus (263) together with (259) characterizes the evolution of the giant component given the burning times on a heuristical level in the alternating model with frequent lightnings.

Now we give a sketch proof of (203). We treat (263) as an equality in the rest of the sketch proof, thus we use $=$ rather than $\approx$.

The Rayleigh distribution emerges in our setting in the following way: if we consider the solution of the random alternating equations with burning times defined by a homogenous Poisson process with rate $1 \ll \lambda$, then by (263) we have $\theta(t) = 2E \cdot (t - T_i^g)$ if $T_i^g < t \le T_i^b$, so

$$\mathbf{P}\big(\, T_i^b - T_i^g > w \,\big) = \exp(-\lambda \int_0^w 2Es \, \mathrm{d}s) = R(\frac{1}{\sqrt{2E\lambda}}, w) \quad \Longrightarrow \quad -w^*(T_i^b) \sim R(\frac{1}{\sqrt{2E\lambda}})$$

From (263) and (261) we get $\theta(T_i^b) \sim R(\sqrt{\frac{2E}{\lambda}})$. From (263) we get $w_+^*(T_i^b) = -w^*(T_i^b)$, thus $-2w^*(T_i^b) = \tau_i \sim R(\sqrt{\frac{2}{E\lambda}})$ holds for the $\tau_i$ of Definition 3.6.2.

Let

$$n(\lambda) := \lfloor \delta(\lambda) \sqrt{\frac{E \cdot \lambda}{\pi}} \rfloor.$$

By our assumption $\lambda^{-\frac{1}{2}} \ll \delta(\lambda)$ we have $1 \ll n(\lambda)$ if $1 \ll \lambda$.

If $1 \ll \lambda$ then the $\theta(t,1), \theta(t,2), \dots, \theta(t, n(\lambda))$ of Definition 3.6.2 are "almost" i.i.d. with distribution $\theta(t,i) \sim R(\sqrt{\frac{2E}{\lambda}})$. Now $\tau(t,i) \sim R(\sqrt{\frac{2}{E\lambda}})$, thus

$$\sum_{i=1}^{n(\lambda)} \tau(t,i) \approx \delta(\lambda) \qquad \text{and} \qquad \sum_{i=1}^{n(\lambda)} \theta(t,i) \approx \delta(\lambda) E \tag{264}$$

by the weak law of large numbers and $\mathbf{E}\big(\, R(\sigma) \,\big) = \sigma \sqrt{\frac{\pi}{2}}$. Let $\hat{x} = 2\sqrt{\frac{E}{\lambda}} x$.

$$\frac{\Phi_\lambda \left( t + \delta(\lambda), 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}} x \right) - \Phi_\lambda \left( t, 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}} x \right)}{\delta(\lambda) \varphi_{crit}(t)} \approx \frac{\Phi\left(t + \delta(\lambda), \hat{x}\right) - \Phi\left(t, \hat{x}\right)}{\delta(\lambda) E} \overset{(264)}{\approx}$$
$$\frac{\sum_{i=1}^{n(\lambda)} \theta(t,i) \cdot \mathbb{1}[\theta(t,i) > \hat{x}]}{\sum_{i=1}^{n(\lambda)} \theta(t,i)} \approx \frac{\mathbf{E}\big(\, \theta(t,1) \mathbb{1}[\theta(t,1) > \hat{x}] \,\big)}{\mathbf{E}\big(\, \theta(t,1) \,\big)} \overset{(260)}{=} R_{sb}(\sqrt{\frac{2E}{\lambda}}, 2\sqrt{\frac{E}{\lambda}} x) \overset{(261)}{=}$$
$$R_{sb}(\frac{1}{\sqrt{2}}, x) = \int_x^\infty \frac{4}{\sqrt{\pi}} y^2 e^{-y^2} \, \mathrm{d}y$$

This is (203).

Our aim in the forthcoming subsections is to make the above argument rigorous.

### 3.6.3 Technical estimates

In this subsection we prove some technical bounds needed for the rigorous proof of Theorem 3.9.

The estimates (268) and (269) below make the formulas of (263) precise.

**Lemma 3.6.1.** *We consider the solution of (185) on* $[0,T]$ *with an arbitrary sequence of burning times. If* $T^g \le t \le T$ *and* $w^*(t) < 0$ *then*

$$\theta(t) \ge \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} |w^*(t)| \tag{265}$$

$$w^*_+(t) \le 4\sqrt{\frac{m_2(0)}{m_1(0)}} \exp\left(\frac{m_2(0)}{m_1(0)} T + 1\right) \cdot |w^*(t)| =: C(T, \underline{\mathbf{v}}(0)) \, |w^*(t)| \tag{266}$$

*If* $w^*(t) < 0$ *and if*

$$w_1 + |w^*(t)| \le t \le w_2 - \sqrt{\frac{E_{sup}(w_1,w_2)}{E_{inf}(w_1,w_2)}} \, |w^*(t)| \tag{267}$$

*holds then*

$$-\sqrt{\frac{E_{inf}(w_1,w_2)}{E_{sup}(w_1,w_2)}} w^*(t) \le w^*_+(t) \le -\sqrt{\frac{E_{sup}(w_1,w_2)}{E_{inf}(w_1,w_2)}} w^*(t) \tag{268}$$

$$-2E_{inf}(w_1,w_2) w^*(t) \le \theta(t) \le -2E_{sup}(w_1,w_2) w^*(t) \tag{269}$$

The following Lemma deals with a technical problem which is similar to the inspection paradox of renewal theory: the length of the critical interval (see Definition 3.6.2) that contains a fixed $t$ tends to be longer than the average length of critical intervals. We give an upper bound on the expected value of related random variables which is of the right order of magnitude.

**Lemma 3.6.2.** *If* $\underline{\mathbf{v}}(t)$ *is the solution of the random alternating equations with constant rate function* $\lambda(t) \equiv \lambda$ *then for every* $T^g \le t \le T$ *we have*

$$\mathbf{E}\big(\, \theta(T^b_{N(t)}) \mathbb{1}[T^b_{N(t)} < T] \,\big) = \mathcal{O}(\lambda^{-\frac{1}{2}}) \tag{270}$$

$$\mathbf{E}\big(\, T^g_{N(t)+1} \wedge (T - t) \,\big) = \mathcal{O}(\lambda^{-\frac{1}{2}}) \tag{271}$$

*as* $\lambda \to \infty$ *where the constant in the* $\mathcal{O}$ *depends only on the initial data and* $T$*.*

The following Lemma is an ingredient of the proof of Lemma 3.6.2.

**Lemma 3.6.3.** *We consider the solution of (185) with initial critical core $E(0,w)$. If $T_1^g < T_2^g$ are two consecutive gelation times, then the unique burning time in between $T_1^g$ and $T_2^g$ is*

$$T^b(T_1^g, T_2^g) = \frac{\int_{T_1^g}^{T_2^g} yE(0,y)\,\mathrm{d}y}{\int_{T_1^g}^{T_2^g} E(0,y)\,\mathrm{d}y} \tag{272}$$

*Moreover*

$$\theta(T^b(T_1^g, T_2^g)) = \int_{T_1^g}^{T_2^g} E(0,y)\,\mathrm{d}y \tag{273}$$

Now we prove the Lemmas stated above.

*Proof of Lemma 3.6.1.* By (218), $w_+^*(t) = -X'(t, -\theta(t))$, (245) and (233) we get

$$\theta(t) = F(t, w_+^*(t)) \geq \int_{w^*(t)}^{0} \frac{m_1(0)}{m_2(0)} \frac{1}{(t+y)^2}\,\mathrm{d}y \geq \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} |w^*(t)|$$

Rearranging (219) and using (218) we get that $w = w_+^*(t)$ is the positive root of the function

$$f(w) := G(t,w) - F(t,w)w = G(t,0) + \left(-\int_0^w yE(t,y)\right)\mathrm{d}y = f(0) + (f(w) - f(0))$$

We prove (266) by considering the cases $\frac{|w^*(t)|}{t} \leq \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$ and $\frac{|w^*(t)|}{t} > \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$ separately.

If $\frac{|w^*(t)|}{t} \leq \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$, then we prove that $w_+^*(t) \leq 2\sqrt{\frac{m_2(0)}{m_1(0)}}\,|w^*(t)|$ by showing that $f(0) \leq |f(w) - f(0)|$ with $w = 2\sqrt{\frac{m_2(0)}{m_1(0)}}\,|w^*(t)|$.

$$f(0) = \int_{w^*(t)}^{0} (-y)E(0,t+y)\,\mathrm{d}y \leq \int_0^{|w^*(t)|} \frac{y}{(t-y)^2}\,\mathrm{d}y$$

by (245) and (233).

$$|f(w) - f(0)| \geq \int_0^w \frac{m_1(0)}{m_2(0)} \frac{y}{(t+y)^2}\,\mathrm{d}y = \int_0^{|w^*(t)|} \frac{m_1(0)}{m_2(0)} \frac{y}{(t\frac{|w^*(t)|}{w} + y)^2}\,\mathrm{d}y \tag{274}$$

It is straightforward to check that

$$0 \leq y \leq |w^*(t)| \;\;\&\;\; \frac{|w^*(t)|}{t} \leq \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}} \quad\Longrightarrow\quad \frac{y}{(t-y)^2} \leq \frac{m_1(0)}{m_2(0)} \frac{y}{(t\frac{|w^*(t)|}{w} + y)^2}$$

which is sufficient for $f(0) \leq \left|f(2\sqrt{\frac{m_2(0)}{m_1(0)}}\,|w^*(t)|) - f(0)\right|$ to hold.

If $\frac{|w^*(t)|}{t} > \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$, then

$$f(0) = G(t,0) = \int_{w^*(t)}^{0} F(t,y)\,\mathrm{d}y \le |w^*(t)| \le T$$

since by (221) we have $F(t,y) \le m_0(t) \le m_0(0) = 1$. Calculating the middle integral in (274) we get that in order to have $f(0) \le |f(w) - f(0)|$

$$\frac{m_1(0)}{m_2(0)}\left(\log(1+\frac{w}{t}) - 1\right) \ge T$$

is sufficient. Rearranging this and using $\frac{|w^*(t)|}{t} > \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$ we obtain (266).

The proof of the upper bound of (268) is similar: using (245) we get that $w_1 \le t - |w^*(t)| \le t + w \le w_2$ implies

$$f(0) \le \frac{1}{2}E_{sup}(w_1,w_2)w^*(t)^2, \quad f(w) - f(0) \le -\frac{1}{2}E_{inf}(w_1,w_2)w^2$$

Using (267) the inequality $f\left(-\sqrt{\frac{E_{sup}(w_1,w_2)}{E_{inf}(w_1,w_2)}}w^*(t)\right) \le 0$ follows. The lower bound of (268) is verified similarly.

If $u \in [-\theta(t), 0]$, then

$$X(t,u) \le -w^*(t)u + \frac{1}{2}\frac{1}{E_{inf}(w_1,w_2)}u^2,$$

since $X''(t,u)$ with $u \in [-\theta(t),0]$ is equal to $\frac{1}{E(0,t+y)}$ for some

$$y \in [w^*(t), w_+^*(t)] \subseteq [w^*(t), -\sqrt{\frac{E_{sup}(w_1,w_2)}{E_{inf}(w_1,w_2)}}w^*(t)],$$

thus $t + y \in [w_1, w_2]$ by (267). This implies the lower bound of (269), and the proof of the upper bound is similar.

□

*Proof of Lemma 3.6.3.* $T^b$ needs to satisfy $T_2^g - T^b = w_+^*(T^b)$, but by the proof of Lemma 3.6.1 $w_+^*(T^b)$ is the unique positive root of $G(T^b,0) - \int_0^w yE(0,T^b+y)\,\mathrm{d}y$.

$G(T^b,0) = -\int_{T_1^g - T^b}^{0} yE(0,T^b+y)\,\mathrm{d}y$ by (245), so $\int_{T_1^g-T^b}^{T_2^g-T^b} yE(0,T^b+y)\,\mathrm{d}y = 0$ must hold, from which (272) easily follows.

By (218), $w_+^*(t) = -X'(t,-\theta(t))$ and (245) we get

$$\theta(T^b) = F(T^b, w_+^*(T^b)) = \int_{w^*(T^b)}^{w_+^*(T^b)} E(0,T^b+y)\,\mathrm{d}y = \int_{T_1^g}^{T_2^g} E(0,y)\,\mathrm{d}y$$

□

*Proof of Lemma 3.6.2.* Let $\gamma(t) := t - T^g_{N(t)}$. Then

$$\lim_{\mathrm{d}t\to 0}\frac{1}{\mathrm{d}t}\mathbf{E}\big(\,\gamma(t+\mathrm{d}t)-\gamma(t)\mid \mathcal{F}_t\,\big) = 1-\gamma(t)\lim_{\mathrm{d}t\to 0}\frac{1}{\mathrm{d}t}\mathbf{P}\big(\,t\le T^g_{N(t)+1}\le t+\mathrm{d}t\mid \mathcal{F}_t\,\big) =$$
$$1-\gamma(t)\lim_{\mathrm{d}t\to 0}\frac{1}{\mathrm{d}t}\mathbf{P}\big(\,T^b(T^g_{N(t)},t)\le T^b_{N(t)}\le T^b(T^g_{N(t)},t+\mathrm{d}t)\mid \gamma(t)\,\big) =$$
$$1-\gamma(t)\theta(T^b(T^g_{N(t)},t))\lambda\frac{\mathrm{d}}{\mathrm{d}s}T^b(T^g_{N(t)},s)\bigg|_{s=t} =$$
$$1-\lambda E(0,t)\gamma(t)\left(t-T^b(t-\gamma(t),t)\right)\le 1-\frac{1}{2}\lambda\frac{E_{inf}(T)^2}{E_{sup}}\gamma(t)^2$$

by Lemma 3.6.3. Taking the expectation of both sides of the above inequality and applying Jensen's inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\big(\,\gamma(t)\,\big)\le 1-\frac{1}{2}\lambda\frac{E_{inf}(T)^2}{E_{sup}}\mathbf{E}\big(\,\gamma(t)\,\big)^2.$$

This differential inequality together with $\gamma(T^g)=0$ implies

$$\mathbf{E}\big(\,\gamma(t)\,\big)\le\frac{1}{\sqrt{\lambda}}\frac{\sqrt{2E_{sup}}}{E_{inf}(T)}=\mathcal{O}(\lambda^{-\frac{1}{2}})\qquad T^g\le t\le T.$$

by a "forbidden region"-type argument. Now we prove

$$\mathbf{E}\big(\,T^g_{N(t)+1}\wedge T-T^g_{N(t)}\,\big)=\mathcal{O}(\lambda^{-\frac{1}{2}}) \tag{275}$$

from which (271) trivially follows. We obtain (270) using (275) and $\theta(T^b_{N(t)})\le 2E_{sup}\cdot(T^b_{N(t)}-T^g_{N(t)})$ by the upper bound of (269).

$$T^g_{N(t)+1}\wedge T-T^g_{N(t)}=\gamma(t)+\Big(T^g_{N(t)+1}\wedge T-t\Big)\,\mathbb{1}[t\ge T^b_{N(t)}]+$$
$$\Big(T^g_{N(t)+1}\wedge T-T^b_{N(t)}\wedge T\Big)\,\mathbb{1}[t<T^b_{N(t)}]+\left(T^b_{N(t)}\wedge T-t\right)\mathbb{1}[t<T^b_{N(t)}]$$

$$\Big(T^g_{N(t)+1}\wedge T-t\Big)\,\mathbb{1}[t\ge T^b_{N(t)}]\;\le\;w^*_+(t,0)\mathbb{1}[t\ge T^b_{N(t)}]\;\le$$
$$C(T,\underline{\mathbf{v}}(0))w^*_-(t,0)\mathbb{1}[t\ge T^b_{N(t)}]\;\le\;C(T,\underline{\mathbf{v}}(0))\gamma(t)$$

where $C(T,\underline{\mathbf{v}}(0))$ is defined in (266).

$$\Big(T^g_{N(t)+1}\wedge T-T^b_{N(t)}\wedge T\Big)\,\mathbb{1}[t<T^b_{N(t)}]\;\le\;w^*_+(t,0)\mathbb{1}[t<T^b_{N(t)}\le T]\;\le$$
$$C(T,\underline{\mathbf{v}}(0))\gamma(t)+C(T,\underline{\mathbf{v}}(0))\left(T^b_{N(t)}\wedge T-t\right)\mathbb{1}[t<T^b_{N(t)}]$$

By (188) and (265) we have

$$\mathbf{E}\big(\, (T^b_{N(t)} \wedge T - t)\, \mathbb{1}[t < T^b_{N(t)}] \,\big) = \mathbf{E}\big(\, (T^b_{N(t)} \wedge T - t) \vee 0 \,\big) =$$
$$\int_0^{T-t} \mathbf{P}\big(\, T^b_{N(t)} - t \ge x \,\big)\, \mathrm{d}x \le \int_0^{T-t} \exp\left( -\lambda \int_0^x \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} y\, \mathrm{d}y \right) = \mathcal{O}(\lambda^{-\frac{1}{2}})$$

$$\mathbf{E}\big(\, T^g_{N(t)+1} \wedge T - T^g_{N(t)} \,\big) = \mathcal{O}(\mathbf{E}\big(\, \gamma(t) \,\big)) + \mathcal{O}\left(\mathbf{E}\big(\, (T^b_{N(t)} \wedge T - t) \vee 0 \,\big)\right) = \mathcal{O}(\lambda^{-\frac{1}{2}})$$

□

### 3.6.4 Proof of Theorem 3.9

In this section we prove Theorem 3.4 and Theorem 3.9 using the notations of Subsection 3.6.1 and the technical estimates proved in Subsection 3.6.3.

*Proof of of Theorem 3.4.* The proof is similar to that of Theorem 3.3.: if $\varepsilon = \sup_i \{T^b_{i+1} - T^b_i\}$ and $T^b_i < t \le T^b_{i+1}$ then $w^*(t) = w^*_+(T^b_i) - (t - T^b_i) \ge -\varepsilon$ and by (266) we have $w^*_+(T^b_i) = \mathcal{O}(\left|w^*(T^b_i)\right|) = \mathcal{O}(\varepsilon)$ on $[0, T]$. □

The main idea in the proof of Theorem 3.9 is the following: in order to make the heuristic argument of Subsection 3.6.2 rigorous we couple the stochastic process of consecutive critical intervals of a solution of the random alternating equations to i.i.d. random variables with Rayleigh distribution. We control the error terms of this coupling by the sandwich/squeeze theorem: in our coupling we simultaneously bound the critical intervals from above and below by random variables with Rayleigh distribution whose parameters are close to each other.

*Proof of Theorem 3.9.* We use the notations of Definitions 3.6.2. and 3.6.3.

$$E := E(t, 0) = E(0, t) = \varphi_{crit}(t)$$

We fix $x \ge 0$ and define

$$\hat{x} := 2\sqrt{\frac{E}{\lambda}} x, \quad \theta(t, i, \hat{x}) := \theta(t, i) \mathbb{1}[\theta(t, i) > \hat{x}], \quad n(\lambda, z) := \lfloor \delta(\lambda) \sqrt{\frac{E\lambda}{\pi}} (1 + z) \rfloor$$

By the assumption $\lambda^{-\frac{1}{2}} \ll \delta(\lambda)$ we have $\lim_{\lambda \to \infty} n(\lambda, z) = +\infty$ for any $-1 < z$.

Let $m(\lambda) := N(t + \delta(\lambda)) - N(t) - 1$.

$$\Phi\left(t + \delta(\lambda), \hat{x}\right) - \Phi\left(t, \hat{x}\right) = \theta(t, 0, \hat{x}) \mathbb{1}[T^b_{N(t)} > t] +$$
$$\sum_{i=1}^{m(\lambda)} \theta(t, i, \hat{x}) + \theta(t + \delta(\lambda), 0, \hat{x}) \mathbb{1}[T^b_{N(t+\delta(\lambda))} < t + \delta(\lambda)] \quad (276)$$

In order to prove (203) we only need to show that we have $\lim_{\lambda\to\infty}\mathbf{P}\big(B(\lambda,\varepsilon)\big)=1$ for every $\varepsilon>0$ where

$$B(\lambda,\varepsilon):=\left\{R_{sb}(\frac{1}{\sqrt{2}},x)-\varepsilon<\frac{\sum_{i=1}^{m(\lambda)}\theta(t,i,\hat{x})}{E\delta(\lambda)}<R_{sb}(\frac{1}{\sqrt{2}},x)+\varepsilon\right\}$$

because the first and the last term on the r.h.s. of (276) divided by $E\delta(\lambda)$ converge to 0 in probability as $\lambda\to\infty$ by (270) and $\lambda^{-\frac{1}{2}}\ll\delta(\lambda)$.

$$E_{sup}(\lambda):=E_{sup}(t,t+2\delta(\lambda)),\quad E_{inf}(\lambda):=E_{inf}(t,t+2\delta(\lambda))$$

By (235) we have

$$E_{sup}(\lambda)\le E+2D\delta(\lambda)\quad\text{and}\quad E-2D\delta(\lambda)\le E_{inf}(\lambda) \tag{277}$$

$$C^u(\lambda):=1+\sqrt{\frac{E_{sup}(\lambda)}{E_{inf}(\lambda)}}\quad C^l(\lambda):=1+\sqrt{\frac{E_{inf}(\lambda)}{E_{sup}(\lambda)}}$$

$\lim_{\lambda\to\infty}C^u(\lambda)=\lim_{\lambda\to\infty}C^l(\lambda)=2$, since $\delta(\lambda)\ll 1$.

We are going to couple the random variables $T^g_{N(t)+1},w^*_-(t,1),w^*_-(t,2),\dots$ to

$$w^l_-(1),w^l_-(2),\dots\qquad\text{and}\qquad w^u_-(1),w^u_-(2),\dots$$

where $w^l_-(i)\sim R(\frac{1}{\sqrt{2E_{sup}(\lambda)\lambda}})$ are i.i.d. and $w^u_-(i)\sim R(\frac{1}{\sqrt{2E_{inf}(\lambda)\lambda}})$ are i.i.d., moreover the auxiliary random variables are independent from $T^g_{N(t)+1}$. If we define the events

$$A^u(\lambda,z,z_2):=\left\{T^g_{N(t)+1}+C^u(\lambda)\cdot\sum_{j=1}^{n(\lambda,z)}w^u_-(j)\le t+\delta(\lambda)\cdot(1+z_2)\right\}$$

$$A^l(\lambda,z,z_2):=\left\{T^g_{N(t)+1}+C^l(\lambda)\cdot\sum_{j=1}^{n(\lambda,z)}w^l_-(j)\ge t+\delta(\lambda)\cdot(1+z_2)\right\}$$

then it is an easy consequence of (271), $\lambda^{-\frac{1}{2}}\ll\delta(\lambda)$, and the weak law of large numbers that

$$-1<z<z_2\implies\lim_{\lambda\to\infty}\mathbf{P}\big(A^u(\lambda,z,z_2)\big)=1$$

$$z>z_2>-1\implies\lim_{\lambda\to\infty}\mathbf{P}\big(A^l(\lambda,z,z_2)\big)=1$$

Our coupling is going to satisfy

$$A^u(\lambda,z,1)\subseteq\bigcap_{i=1}^{n(\lambda,z)}\{w^l_-(i)\le w^*_-(t,i)\le w^u_-(i)\} \tag{278}$$

for any $z$.

The joint construction of $w_-^l(j)$, $w_-^*(t,j)$ and $w_-^u(j)$ for $j = 1, 2, \dots$ is as follows: given $T^g_{N(t)+1}$ and $w_-^*(t,1), \dots, w_-^*(t, j-1)$ we can determine $T^g_{N(t)+j}$ by solving (185). For $s \geq 0$ Let

$$\mu(s) := \lambda\theta(T^g_{N(t)+j} + s), \quad \mu_l(s) := \lambda 2E_{sup}(\lambda)s, \quad \mu_u(s) := \lambda 2E_{inf}(\lambda)s.$$

Let $w_-^*(t,j)$, $w_-^l(j)$ and $w_-^u(j)$ be the horizontal coordinate of the leftmost point below the graphs of $\mu$, $\mu_l$ and $\mu_u$ of the same standard uniform 2-dimensional Poisson process on the first quadrant of the plane. Thus $w_-^l(j) \sim R(\frac{1}{\sqrt{2E_{sup}(\lambda)\lambda}})$, $w_-^u(j) \sim R(\frac{1}{\sqrt{2E_{inf}(\lambda)\lambda}})$ are independent from everything that was constructed earlier and $\mathbf{P}\big(w_-^l(j) \leq w_-^u(j)\big) = 1$. The joint distribution of $T^g_{N(t)+1}, w_-^*(t,1), \dots, w_-^*(t,j)$ agrees with that of the solution of the random alternating equation.

We are going to prove (278) by induction. Assume that $A^u(\lambda, z, 1)$ holds. If

$$\bigcap_{i=1}^{j-1} \{w_-^l(i) \leq w_-^*(t,i) \leq w_-^u(i)\} \cap \bigcap_{i=1}^{j-1} \{\tau(t,i) \leq C^u(\lambda) \cdot w_-^u(i)\} \tag{279}$$

holds for some $j \leq n(\lambda, z)$, then

$$T^g_{N(t)+j} = T^g_{N(t)+1} + \sum_{i=1}^{j-1} \tau(t,i) \leq T^g_{N(t)+1} + C^u(\lambda) \sum_{i=1}^{j-1} w_-^u(i)$$

which implies $\mu_u(s) \leq \mu(s) \leq \mu_l(s)$ for $0 \leq s \leq w_-^u(j)$ by (269) and $A^u(\lambda, z, 1)$. From this $w_-^l(j) \leq w_-^*(t,j) \leq w_-^u(j)$ follows, and (268) can be applied to deduce

$$\tau(t,j) = w_-^*(t,j) + w_+^*(t,j) \leq \left(1 + \sqrt{\frac{E_{sup}(\lambda)}{E_{inf}(\lambda)}}\right) w_-^*(t,j) \leq C^u(\lambda) w_-^u(j)$$

Thus we can replace $j$ with $j+1$ in (279). This completes the proof of (278). Let

$$\theta^u(t,i,\hat{x}) := 2E_{sup}(\lambda) w^u(i) \cdot \mathbb{1}[2E_{sup}(\lambda) w^u(i) > \hat{x}]$$

$$\theta^l(t,i,\hat{x}) := 2E_{inf}(\lambda) w^l(i) \cdot \mathbb{1}[2E_{inf}(\lambda) w^l(i) > \hat{x}]$$

(269) and (278) imply

$$A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda,z)} \{\theta^l(t,i,\hat{x}) \leq \theta(t,i,\hat{x}) \leq \theta^u(t,i,\hat{x})\}$$

$$B^u(\lambda, z, \varepsilon) := \left\{ \frac{\sum_{i=1}^{n(\lambda,z)} \theta^u(t,i,\hat{x})}{E\delta(\lambda)} \leq R_{sb}(\frac{1}{\sqrt{2}}, x) + \varepsilon \right\}$$

$$B^l(\lambda, z, \varepsilon) := \left\{ \frac{\sum_{i=1}^{n(\lambda,z)} \theta^l(t,i,\hat{x})}{E\delta(\lambda)} \geq R_{sb}(\frac{1}{\sqrt{2}}, x) - \varepsilon \right\}$$

The law of large numbers, (261) and (277) imply that

$$z < \varepsilon \implies \lim_{\lambda\to\infty} \mathbf{P}\big(\, B^u(\lambda, z, \varepsilon)\,\big) = 1 \quad \text{and} \quad -\varepsilon < z \implies \lim_{\lambda\to\infty} \mathbf{P}\big(\, B^l(\lambda, z, \varepsilon)\,\big) = 1.$$

We can use (278) and (268) to show

$$A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda,z)} \{C^l(\lambda) w^l_-(i) \le \tau(t,i) \le C^u(\lambda) w^u_-(i),\}$$

Since

$$m(\lambda) = \max\{j : T^g_{N(t)+1} + \sum_{i=1}^{j} \tau(t,i) < t + \delta(\lambda)\}$$

and $A^u(\lambda, z, 0) \subseteq A^u(\lambda, z, 1)$ by definition,

$$A^u(\lambda, z, 0) \subseteq \{m(\lambda) \ge n(\lambda, z)\}, \quad A^l(\lambda, z, 0) \cap A^u(\lambda, z, 1) \subseteq \{m(\lambda) \le n(\lambda, z)\},$$

$$A^l(\lambda, \frac{\varepsilon}{2}, 0) \;\cap\; A^u(\lambda, \frac{\varepsilon}{2}, 1) \;\cap\; B^u(\lambda, \frac{\varepsilon}{2}, \varepsilon) \;\cap\; B^l(\lambda, -\frac{\varepsilon}{2}, \varepsilon) \;\cap\; A^u(\lambda, -\frac{\varepsilon}{2}, 0) \;\;\subseteq\;\; B(\lambda, \varepsilon)$$

This completes the proof of $\lim_{\lambda\to\infty} \mathbf{P}\big(\, B(\lambda, \varepsilon)\,\big) = 1$.

□

# 4 The edge reconnecting model

## 4.1 Introduction

We introduce the *edge reconnecting model*, a random multigraph (undirected graph with multiple and loop edges) evolving in time. Denote the multigraph at time $T$ by $\mathcal{G}_n(T)$, where $T = 0, 1, 2, \dots$ and $n = |V(\mathcal{G}_n(T))|$ is the number of vertices. We denote by $m = |E(\mathcal{G}_n(T))|$ the number of edges (the number of vertices and edges does not change over time).

Given the multigraph $\mathcal{G}_n(T)$ we get $\mathcal{G}_n(T+1)$ by uniformly choosing an edge in $E(\mathcal{G}_n(T))$, choosing one of the endpoints of that edge with a coin flip and reconnecting the edge to a new endpoint which is chosen using the rule of linear preferential attachment: a vertex $v$ is chosen with probability $\frac{d(v)+\kappa}{2m+n\kappa}$, where $d(v)$ is the degree of vertex $v$ and $\kappa \in (0, +\infty)$ is a parameter. We give the formal definition of the edge reconnecting model in **Section 4.2**. A similar dynamical random graph model is studied in [11].

Our aim is to describe the time evolution of edge reconnecting model $\mathcal{G}_n(T)$ when $1 \ll n$ using the terminology of *dense graph limits*. The notion of convergence of simple graph sequences was defined in [30] (with notations slightly different from what we are using now): we say that a sequence of simple graphs $\left(G_n\right)_{n=1}^{\infty}$ is convergent if for every simple graph $F$ the limit $\lim_{n\to\infty} t_=(F, G_n)$ exists, where

$$t_=(F,G) = \frac{1}{n^{|V(F)|}} \sum_{\varphi: V(F)\to V(G)} \mathbb{1}[\forall\, v, w \in V(F):\ E(v,w) = E(\varphi(v), \varphi(w))] \qquad (280)$$

and $E(v, w)$ denotes the number of edges between $v$ and $w$. If $(\xi_v)_{v\in V(F)}$ are independently and uniformly distributed elements of $V(G)$ then

$$t_=(F,G) = \mathbf{P}\big(\forall\, v, w \in V(F):\ E(v,w) = E(\xi_v, \xi_v)\big). \qquad (281)$$

$t_=(F, G)$ may be interpreted as the density of copies of $F$ in $G$, thus the sequence $\left(G_n\right)_{n=1}^{\infty}$ is convergent if the density of every fixed graph $F$ in $G_n$ converges as $n \to \infty$.

In [30] several equivalent characterizations of *graphons* (limit objects of convergent graph sequences) are given. In [28] we give a natural generalization of the theory of dense graph limits to multigraphs: a sequence of multigraphs $(G_n)_{n=1}^{\infty}$ is convergent if for every multigraph $F$ the limit $g(F) = \lim_{n\to\infty} t_=(F, G_n)$ exists (moreover $g(\cdot)$ is a "non-defective probability distribution", see Definition 4.3.2 for details) where $t_=(F, G)$ is defined by (280). The limit object of a convergent multigraph sequence is a function $W : [0, 1] \times [0, 1] \times \mathbb{N}_0 \to [0, 1]$ satisfying

$$W(x,y,k) \equiv W(y,x,k), \quad \sum_{k=0}^{\infty} W(x,y,k) \equiv 1, \quad W(x,x,2k+1) \equiv 0.$$

Such functions are called *multigraphons*. Denote by $\mathcal{M}_k$ the set of multigraphs on $k$ vertices. We say that $G_n \to W$ if for every $k \in \mathbb{N}$ and every $F \in \mathcal{M}_k$ we have $\lim_{n\to\infty} t_=(F, G_n) = t_=(F, W)$ where

$$t_=(F, W) := \int_{[0,1]^k} \prod_{v \le w \le k} W(x_v, x_w, E(v, w))\, \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_k. \tag{282}$$

The probabilistic meaning of $t_=(F, W)$ is similar to (281): given $F \in \mathcal{M}_k$ and $W$ we can generate a random multigraph $\mathcal{G}_W^k$ for which $V(\mathcal{G}_W^k) = V(F)$ and $\mathbf{P}\big(F = \mathcal{G}_W^k\big) = t_=(F, W)$ using the following procedure: if $(U_v)_{v\in V(F)}$ are independent and uniformly distributed on the unit interval $[0, 1]$ then given these *background variables* let us choose the number of multiple edges between the vertices $v$ and $w$ in $\mathcal{G}_W^k$ according to the probability distribution $\big(W(U_v, U_w, k)\big)_{k=1}^{\infty}$ and choose the number of loop edges at vertex $v$ according to the probability distribution $\big(W(U_v, U_v, 2k)\big)_{k=1}^{\infty}$. See Definition 4.5.1 for the precise formulation.

We give a short survey of the theory of multigraph limits in **Section 4.3**.

The main results of this Chapter are stated in **Section 4.4**: if we consider a sequence of edge reconnecting models with a convergent sequence of initial graphs $\mathcal{G}_n(0) \to W$ (satisfying some extra technical conditions), then for every $t \in (0, +\infty)$ we have

$$\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{d}} \breve{W}_t \qquad \text{and} \qquad \mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{d}} \tilde{W}_t \tag{283}$$

(the precise definition of $\xrightarrow{\mathrm{d}}$ is given in Section 4.3) where the multigraphons $\breve{W}_t$ and $\tilde{W}_t$ are explicit functions of $t$, $W$ and the linear preferential attachment parameter $\kappa$. Moreover we have

$$\lim_{t\to 0_+} \breve{W}_t = W, \qquad \lim_{t\to\infty} \breve{W}_t = \lim_{t\to 0_+} \tilde{W}_t, \qquad \lim_{t\to\infty} \tilde{W}_t = W_\infty \tag{284}$$

where $W_\infty$ is the limiting multigraphon of the stationary states of the edge reconnecting models: $\mathcal{G}_n(\infty) \xrightarrow{\mathrm{d}} W_\infty$. Thus by (284) the convergence theorems (283) give the *full characterization* of the time evolution of the multigraphons arising as the graph limits of the edge reconnecting model.

Although the theorems in Section 4.4 are stated using the "multigraphon" formalism, in their proofs we use the correspondence between the theory of graph limits and that of exchangeable arrays, a connection first observed in [18]. In **Section 4.5** we provide the dictionary that we use to translate results about random exchangeable arrays to results about mutigraphs and multigraphons. The basic idea here is the following: by (281) one can see that that for a given multigraph $G$ the knowledge of the collection of induced homomorphism densities $\big(t_=(F, G)\big)_{F\in\mathcal{M}_k}$ is the same as knowing the distribution of the random subgraph $\mathcal{G}^k$ of $G$ spanned by the random sample $\xi_1, \ldots, \xi_k$. The sequence $\big(G_n\big)_{n=1}^{\infty}$ is convergent if and only if for each $k \in \mathbb{N}$ the the random subgraph $\mathcal{G}_n^k$ of

$G_n$ spanned by a random sample of $k$ vertices converges in distribution as $n \to \infty$. It follows from Aldous' representation theorem (Theorem 4.4) that in this case there exists a multigraphon $W$ such that for every $k \in \mathbb{N}$ we have $\mathcal{G}_n^k \xrightarrow{\mathrm{d}} \mathcal{G}_W^k$ as $n \to \infty$ where the random multigraph $\mathcal{G}_W^k$ is defined in the paragraph after (282). From this it follows that $G_n \to W$.

In **Section 4.6** we give the intuitive explanation of the results stated in Section 4.4 using exchangeable arrays. These sketch proofs also serve as an outline of the rigorous proofs. The basic idea here is to relate the time evolution of the edge reconnecting model to certain continuous-time stochastic processes using an appropriate rescaling of time:

- If we fix a vertex $v \in V(\mathcal{G}_n(0))$ and denote by $d(T, v)$ the degree of $v$ in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{R}_+$-valued continuous-time stochastic process $\frac{1}{n} d(n^3 \cdot t, v)$ "almost looks like" that of a Cox-Ingersoll-Ross process (a diffusion process that is commonly used in financial mathemathics to model the evolution of interest rates). This fact is rigorously proved using the theory of stochastic differential equations and is used in the proof of $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{d}} \tilde{W}_t$.
- If we fix two vertices $v, w \in V(\mathcal{G}_n(0))$ and denote by $E(T, v, w)$ the number of parallel/loop edges connecting $v$ and $w$ in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{N}$-valued continuous-time stochastic process $E(n^2 \cdot t, v, w)$ "almost looks like" that of the queue length of an M/M/$\infty$-queue. This fact is rigorously proved using a coupling argument and is used in the proof of $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{d}} \breve{W}_t$.

In **Section 4.7** we explicitly describe the stationary distribution $\mathcal{G}_n(\infty)$ of the edge reconnecting model by relating it to the Pólya urn model and prove $\mathcal{G}_n(\infty) \xrightarrow{\mathrm{d}} W_\infty$. As an intermediate step we describe the multigraph limits of random multigraphs which are uniformly chosen from the set of multigraphs with a given degree sequence (this construction is known as the *configuration model* in the theory of random graphs).

In **Section 4.8** we state and prove technical lemmas needed for the proof of $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{d}} \breve{W}_t$ (Theorem 4.2) in **Section 4.9** and the proof of $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{d}} \tilde{W}_t$ (Theorem 4.3) in **Section 4.10**.

The most interesting property of the edge reconnecting model is the separation of *two different timescales* in (283) and (284): the degrees of the vertices only change significantly on the $n^3$ timescale, whereas the number of parallel (or loop) edges between two vertices evolves on the much faster $n^2$ timescale. The arrival rate of the M/M/$\infty$-queue describing the evolution of $E(n^2 \cdot t, v, w)$ depends on the current degrees of $v$ and $w$, but since the degrees evolve on the much slower $n^3$ timescale, they may be treated as constant background parameters on the $n^2$ timescale. If we fix $k$ vertices in the model and denote by $\mathcal{G}_n^k(T)$ the random subgraph of $\mathcal{G}_n(T)$ spanned by these vertices then the

$\mathcal{M}_k$-valued stochastic process $\mathcal{G}_n^k(n^3 \cdot t + n^2 \cdot s)$ looks stationary in the time variable $s \in \mathbb{R}$ if $t \in (0, +\infty)$ is fixed and $1 \ll n$, but different values of $t$ yield distinct pseudo-stationary distributions since $n^3 \cdot (t_2 - t_1)$ steps are enough for the background variables (degrees) to significantly change, see Subsection 4.10.3 for the precise formulation.

This phenomenon is called *aging* in statistical physics, see [4] and [9].

If $n^2 \ll T \ll n^3$ then the degrees of vertices in $\mathcal{G}_n(0)$ and $\mathcal{G}_n(T)$ are more or less the same, whereas $T$ steps are enough for the model to rearrange and mix the edges, so $\mathcal{G}_n(T)$ looks like a configuration model: a uniformly chosen element of the set of multigraphs with the degree sequence of $\mathcal{G}_n(0)$. Similarly, for any $t > 0$, the distribution of the random multigraph $\mathcal{G}_n(n^3 \cdot t)$ looks uniform given its degree sequence.

## 4.2 The edge reconnecting model

In this section we precisely define the edge reconnecting model and introduce some notations.

Denote by $\mathcal{M}$ the set of undirected multigraphs (graphs with multiple and loop edges) and by $\mathcal{M}_n$ the set of multigraphs on $n$ vertices. Let $G \in \mathcal{M}_n$. The adjacency matrix of a labeling of the multigraph $G$ with $[n] = \{1, 2, \dots, n\}$ is denoted by $(B(i,j))_{i,j=1}^n$, where $B(i,j) \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is the number of edges connecting the vertices labeled by $i$ and $j$. $B(i,j) = B(j,i)$ since the graph is undirected and $B(i,i)$ is two times the number of loop edges at vertex $i$ (thus $B(i,i)$ is an even number).

An unlabeled multigraph is the equivalence class of labeled multigraphs where two labeled graphs are equivalent if one can be obtained by relabeling the other. Thus $\mathcal{M}$ is the set of these equivalence classes of labeled multigraphs, which are also called isomorphism types.

We denote the set of adjacency matrices of multigraphs on $n$ nodes by $\mathcal{A}_n$, thus

$$\mathcal{A}_n = \left\{ B \in \mathbb{N}_0^{n \times n} \,:\, B^T = B,\, \forall\, i \in [n] \;\; 2 \,|\, B(i,i) \right\}.$$

The degree of the vertex labeled by $i$ in $G$ with adjacency matrix $B \in \mathcal{A}_n$ is

$$d(B,i) := \sum_{j=1}^{n} B(i,j), \tag{285}$$

thus $d(B,i)$ is the number of edge-endpoints at $i$ (loop edges count twice).

Let

$$m = \frac{1}{2} \sum_{i,j=1}^{n} B(i,j) = \frac{1}{2} \sum_{i=1}^{n} d(B,i) \tag{286}$$

denote the number of edges. Denote by $\mathcal{A}_n^m$ the set of adjacency matrices on $n$ vertices with $m$ edges.

Now we describe the dynamics of the edge reconnecting model, which is a discrete time Markov chain with state space $\mathcal{A}_n^m$: neither the number of vertices, nor the number of edges is changed by the dynamics. $(X(T,i,j))_{i,j=1}^n = \mathbf{X}(T)$ is the state of our Markov chain at time $T$.

Given the adjacency matrix $\mathbf{X}(T)$ we get $\mathbf{X}(T+1)$ in the following way: let $\kappa \in (0,+\infty)$. We choose a random vertex $\mathcal{V}_{old}(T)$ with distribution

$$\mathbf{P}\big(\, \mathcal{V}_{old}(T) = i \,\big|\, \mathbf{X}(T) \,\big) = \frac{d(\mathbf{X}(T), i)}{2m} \tag{287}$$

Then we choose a uniform edge $\mathcal{E}_{old}(T) = \{\mathcal{V}_{old}(T), \mathcal{W}(T)\}$ going out of $\mathcal{V}_{old}(T)$:

$$\mathbf{P}\big(\, \mathcal{W}(T) = i \,\big|\, \mathbf{X}(T), \mathcal{V}_{old}(T) \,\big) = \frac{X(T, \mathcal{V}_{old}(T), i)}{d(\mathbf{X}(T), \mathcal{V}_{old})} \tag{288}$$

Note that $\mathcal{E}_{old}(T)$ is uniformly distributed over all edges of the graph at time $T$ and given $\mathcal{E}_{old}(T)$, $\mathcal{V}_{old}(T)$ is uniformly chosen from the endvertices of $\mathcal{E}_{old}(T)$, moreover

$$\mathbf{P}\big(\, \mathcal{W}(T) = i \,\big|\, \mathbf{X}(T) \,\big) = \frac{d(\mathbf{X}(T), i)}{2m} \tag{289}$$

Given $\mathbf{X}(T)$, choose $\mathcal{V}_{new}(T)$ according to the rules of linear preferential attachment:

$$\mathbf{P}\big(\, \mathcal{V}_{new}(T) = i \,\big|\, \mathbf{X}(T), \mathcal{V}_{old}(T), \mathcal{W}(T) \,\big) = \frac{d(\mathbf{X}(T), i) + \kappa}{2m + n\kappa} \tag{290}$$

Thus $\mathcal{V}_{new}$ is conditionally independent from $\mathcal{V}_{old}$ and $\mathcal{W}$ given $\mathbf{X}(T)$.

Let $\mathcal{E}_{new}(T) := \{\mathcal{V}_{new}(T), \mathcal{W}(T)\}$.

One step of the Markov chain consists of replacing the edge $\mathcal{E}_{old}(T)$ with $\mathcal{E}_{new}(T)$:

$$\begin{aligned} X(T+1,i,j) = X(T,i,j) - \mathbb{1}[\mathcal{V}_{old}(T) = i, \mathcal{W}(T) = j] - \mathbb{1}[\mathcal{V}_{old}(T) = j, \mathcal{W}(T) = i] + \\ \mathbb{1}[\mathcal{V}_{new}(T) = i, \mathcal{W}(T) = j] + \mathbb{1}[\mathcal{V}_{new}(T) = j, \mathcal{W}(T) = i] \end{aligned} \tag{291}$$

This Markov chain is irreducible and aperiodic on $\mathcal{A}_n^m$. Note that if we "erase" the labeling of $\mathbf{X}(T)$ and consider the time evolution of the isomorphism class, then the resulting $\mathcal{M}_n$-valued stochastic process $\mathcal{G}_n(T)$ itself has the Markov property.

## 4.3 Homomorphism densities, multigraphons

In this section we define the notion of induced homomorphism density $t_{=}(F,G)$ of $F, G \in \mathcal{M}$: this definition is a natural multigraph counterpart of the notions of graph homomorphism densities for simple graphs, see [30]. We define multigraphons in Definition 4.3.1 and convergence of multigraph sequences in Definition 4.3.2.

We state Theorem 4.1 in which we identify multigraphons as the limit objects of convergent multigraph sequences. This theorem is a natural generalization of Theorem 2.2 of [30]. We only state Theorem 4.1 here, the full proof can be found in [28].

At the end of the section we define when a sequence of random multigraphs converges to a multigraphon.

Suppose $F \in \mathcal{M}_k$, $G \in \mathcal{M}_n$ and denote by $A \in \mathcal{A}_k$ and $B \in \mathcal{A}_n$ the adjacency matrices of $F$ and $G$.

If $g : \mathcal{M} \to \mathbb{R}$ then we say that $g$ is a multigraph parameter. Let $g(A) := g(F)$. Conversely, if $g : \bigcup_{k=1}^{\infty} \mathcal{A}_k \to \mathbb{R}$ is constant on isomorphism classes, then $g$ defines a multigraph parameter.

We define the *induced homomorphism density* of $F$ into $G$ by

$$t_{=}(F,G) := t_{=}(A,B) := \frac{1}{n^k} \sum_{\varphi:[k]\to[n]} \mathbb{1}\left[\forall i,j \in [k] :\ A(i,j) = B(\varphi(i),\varphi(j))\right]. \tag{292}$$

Even though we used a particular labeled member of the isomorphism class $F$ and $G$ to define $t_{=}(F,G)$, this quantity is well-defined for unlabeled graphs $F$ and $G$, since relabeling $F$ and $G$ does not change its value. Thus for every fixed multigraph $G$, $t_{=}(\cdot,G) : \mathcal{M} \to [0,1]$ is a multigraph parameter.

**Definition 4.3.1.** *A* multigraphon *is a measurable* $W : [0,1]\times[0,1]\times\mathbb{N}_0 \to [0,1]$ *function satisfying*

$$W(x,y,k) \equiv W(y,x,k), \tag{293}$$

$$\sum_{k=0}^{\infty} W(x,y,k) \equiv 1, \tag{294}$$

$$W(x,x,2k+1) \equiv 0. \tag{295}$$

For every multigraphon $W$ and multigraph $F \in \mathcal{M}_k$ with adjacency matrix $A \in \mathcal{A}_k$ we define

$$t_{=}(F,W) := t_{=}(A,W) := \int_{[0,1]^k} \prod_{i\leq j\leq k} W(x_i,x_j,A(i,j))\, \mathrm{d}x_1\, \mathrm{d}x_2 \,\ldots\, \mathrm{d}x_k \tag{296}$$

The function $t_{=}(\cdot,W)$ is indeed a multigraph parameter: its value is invariant under relabeling of $F$. Let

$$W_G(x,y,k) := W_B(x,y,k) := \mathbb{1}\left[B(\lceil nx\rceil,\lceil ny\rceil) = k\right] \tag{297}$$

be the multigraphon generated by $G$. It is easy to see that $t_{=}(\cdot,G) \equiv t_{=}(\cdot,W_G)$.

If $A, A' \in \mathcal{A}_k$ then we say that $A \leq A'$ if $\forall\, i,j \in [k] :\ A(i,j) \leq A'(i,j)$.

The *inverse Möbius transform* of a non-negative function $g : \mathcal{A}_k \to \mathbb{R}$ is defined by

$$g^{-\dagger}(A) = \sum_{A'\geq A} g(A'). \tag{298}$$

If $g$ is a multigraph parameter (i.e. its value is invariant under relabeling of vertices) then $g^{-\dagger}$ is also a multigraph parameter.

If a multigraph parameter $f$ satisfies $f(F_1F_2) = f(F_1)f(F_2)$, where $F_1F_2$ denotes the disjoint union of $F_1$ and $F_2$, then we say that $f$ is *multiplicative.*

It is straightforward to derive from the above definitions that for any multigraph $G$ and any multigraphon $W$ the multigraph parameters $t_=^{-\dagger}(\cdot, G)$ and $t_=^{-\dagger}(\cdot, W)$ are multiplicative, moreover for all $k$ we have

$$\sum_{A\in\mathcal{A}_k} t_=(A,B) = 1 \quad \text{and} \quad \sum_{A\in\mathcal{A}_k} t_=(A,W) = 1,$$

but these properties will become trivial as soon as we give probabilistic meaning to $t_=(\cdot, G)$ and $t_=(\cdot, W)$ in Section 4.5.

Note that if $F \in \mathcal{M}$ and $\mathcal{G}$ is a random element of $\mathcal{M}$ then $t_=(F, \mathcal{G})$ is a random variable, but $t_=^{-\dagger}(F, \mathcal{G})$ is still multiplicative, that is

$$\mathbf{P}\big(\, t_=^{-\dagger}(F_1F_2, \mathcal{G}) = t_=^{-\dagger}(F_1, \mathcal{G}) \cdot t_=^{-\dagger}(F_2, \mathcal{G}) \,\big) = 1.$$

**Definition 4.3.2.** *We say that a sequence $(W_n)_{n=1}^\infty$ of multigraphons is* convergent *if $g(F) = \lim_{n\to\infty} t_=(F, W_n)$ exists for every multigraph $F$, moreover $\forall k : \sum_{A\in\mathcal{A}_k} g(A) = 1$.*

*A sequence $(G_n)_{n=1}^\infty$ of multigraphs is convergent if $(W_{G_n})_{n=1}^\infty$ is convergent.*

*Let $\mathcal{T}_=$ denote the set of graph parameters $g$ arising as limits of induced homomorphism densities of convergent multigraph sequences:*

$$g \in \mathcal{T}_= \quad \iff \quad \exists\, (G_n)_{n=1}^\infty \; \forall F \in \mathcal{M} : \; g(F) = \lim_{n\to\infty} t_=(F, G_n) \;\; \textit{and} \;\; \forall k : \sum_{A\in\mathcal{A}_k} g(A) = 1 \tag{299}$$

This definition is a natural generalization of the definition of convergence of simple graph sequences, see [30]. $t_=(F, G)$ may be interpreted as the density of copies of $F$ in $G$, thus the sequence $\big(G_n\big)_{n=1}^\infty$ is convergent if the density of every fixed multigraph $F$ converges as $n \to \infty$.

If $G_n$ is the multigraph with one vertex and $n$ loop edges then $\lim_{n\to\infty} t_=(F, G_n) = 0$ for every $F \in \mathcal{M}$, but $g(F) \equiv 0$ does not satisfy $\forall k : \sum_{A\in\mathcal{A}_k} g(A) = 1$, so the sequence $(G_n)_{n=1}^\infty$ is not convergent in this case.

The following theorem identifying the limit objects of convergent multigraph sequences as multigraphons is a corollary of Theorem 1 of [28]:

**Theorem 4.1.** *For a multigraph parameter $g$ the following are equivalent:*

*(a)* $g \in \mathcal{T}_=$.

*(b) There exists a multigraphon $W$ for which $g(\,\cdot\,) = t_=(\cdot, W)$.*

*(c) $g^{-\dagger}$ is multiplicative, $g \geq 0$ and $\forall k : \sum_{A\in\mathcal{A}_k} g(A) = 1$.*

If a sequence of multigraphs $(G_n)_{n=1}^\infty$ converges according to Definition 4.3.2 and if we denote the limiting multigraphon by $W$, then we write $G_n \to W$. If $B_n$ is the adjacency matrix of $G_n$, we also write $B_n \to W$.

For a multigraphon $W$ and $x \in [0,1]$ we define the *average degree* of $W$ at $x$ and the edge density of $W$ by

$$D(W,x) := \int_0^1 \sum_{k=0}^\infty k \cdot W(x,y,k)\,\mathrm{d}y \tag{300}$$

$$\rho(W) := \int_0^1 \int_0^1 \sum_{k=0}^\infty k \cdot W(x,y,k)\,\mathrm{d}y\,\mathrm{d}x \tag{301}$$

If $\rho(W) < +\infty$ then $D(W,x) < +\infty$ for almost all $x$. Recalling (285), (286) and (297) we get

$$D(W_B,x) = \frac{1}{n} d(B, \lceil n \cdot x \rceil), \qquad \rho(W_B) = \frac{2m}{n^2}. \tag{302}$$

These formulas (302) justify the names of the quantities defined by (300) and (301).

Now we define when a sequence of *random* multigraphs converges to a multigraphon. We formulate the definition in terms of random adjacency matrices:

Let $g$ denote a nonnegative multigraph parameter satisfying $\sum_{A \in \mathcal{A}_k} g(A) = 1$ for all $k$. If $\mathbf{X}_n = (X_n(i,j))_{i,j=1}^n$ is a random element of $\mathcal{A}_n$ for each $n$ and $\forall\, k\ \forall\, A \in \mathcal{A}_k$ we have $t_=(A, \mathbf{X}_n) \xrightarrow{\mathrm{d}} g(A)$ as $n \to \infty$, that is

$$\forall\, k\ \forall\, A \in \mathcal{A}_k\ \forall\, \varepsilon > 0 : \ \lim_{n \to \infty} \mathbf{P}\Big( \, |t_=(A, \mathbf{X}_n) - g(A)| > \varepsilon \, \Big) = 0 \tag{303}$$

then using $\sum_{A \in \mathcal{A}_k} g(A) = 1$ it is easy to check (the proof is essentially the same as that of Scheffé's theorem, see page 214 of [10]) that

$$\forall\, k\ \forall\, A \in \mathcal{A}_k : \ \ t_=(A, \mathbf{X}_n) \xrightarrow{\mathrm{d}} g(A) \quad \iff \quad \forall\, k\ \forall\, A \in \mathcal{A}_k : \ \ t_=^{-\dagger}(A, \mathbf{X}_n) \xrightarrow{\mathrm{d}} g^{-\dagger}(A) \tag{304}$$

and that $g^{-\dagger}$ inherits the multiplicative property of the (random) multigraph parameter $t_=^{-\dagger}(\cdot, \mathbf{X}_n)$, thus by Theorem 4.1 there is a multigraphon $W$ such that $g(\cdot) = t_=(\cdot, W)$. In this case we say that $\mathbf{X}_n \xrightarrow{\mathrm{d}} W$.

## 4.4 Statements of Theorem 4.2 and Theorem 4.3

In this section we formulate the main results of this Chapter describing the time evolution of the limiting multigraphons of a sequence of edge reconnecting models $\mathbf{X}_n(\cdot)$, $n \to \infty$.

In Lemma 4.4.1 we give an explicit formula for the stationary distribution $\mathbf{X}_n(\infty)$ of the edge reconnecting Markov chain.

In Proposition 4.1 we precisely formulate $\mathbf{X}_n(\infty) \xrightarrow{\mathrm{d}} W_\infty$.

In Theorem 4.2 we precisely formulate $\mathbf{X}_n(t \cdot n^2) \xrightarrow{\mathrm{d}} \breve{W}_t$.

In Theorem 4.3 we precisely formulate $\mathbf{X}_n(t \cdot n^3) \xrightarrow{\mathrm{d}} \tilde{W}_t$.

Note that in Section 4.1 and above the we used the notations $W_\infty$, $\breve{W}_t$ and $\tilde{W}_t$ in order to give the most simple formulations of these results, nevertheless our real notations are going to be slightly different. Also note that the aim of this section is to precisely state the results of the Chapter: the probabilistic meaning of the results is only going to be explained in Section 4.6.

Recall the formulas defining the Poisson, Binomial and Gamma distributions:

$$\mathbf{p}(k, \lambda) := e^{-\lambda} \frac{\lambda^k}{k!} \tag{305}$$

$$\mathbf{b}(k, n, p) := \binom{n}{k} p^k (1-p)^{n-k} \tag{306}$$

$$\mathbf{g}(x, \alpha, \beta) := x^{\alpha-1} \frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)} \mathbb{1}[x > 0] \tag{307}$$

First we describe the graph limit of the stationary state of the edge reconnecting model (that is, the Markov chain $\mathbf{X}(T)$ defined in Section 4.2).

For an adjacency matrix $B \in \mathcal{A}_n$ denote by

$$e'(B) = \sum_{i=1}^{n} \sum_{j=1}^{i-1} B(i,j) \tag{308}$$

the number of non-loop edges of the corresponding graph.

**Lemma 4.4.1.** *The unique stationary (reversible) distribution of the irreducible, aperiodic Markov chain* $\mathbf{X}(T)$ *with state space* $\mathcal{A}_n^m$ *is*

$$\lim_{T\to\infty} \mathbf{P}\big(\mathbf{X}(T) = B\big) = \frac{\prod_{i=1}^n \prod_{j=1}^{d(B,i)} (\kappa + j - 1)}{\prod_{j=1}^{2m} (\kappa n + j - 1)} \frac{m! 2^{e'(B)}}{\left(\prod_{i=1}^n \prod_{j=1}^{i-1} B(i,j)!\right) \left(\prod_{i=1}^n \frac{B(i,i)}{2}!\right)} \tag{309}$$

We prove this lemma in Section 4.7.

In Section 3.4 of [13] a random multigraph called *preferential attachment graph with* $n$ *nodes and* $m$ *edges* (briefly $\mathrm{PAG}(n,m)$) is defined. The distribution of $\mathrm{PAG}(n,m)$ is the same as the distribution of the isomorphism class of the random adjacency matrix with distribution (309) with $\kappa = 1$. We explain this coincidence in Section 4.7.

At the end of Section 3.4 of [13] the following theorem is stated:

Let $\mathrm{SPAG}(n,m)$ denote the simple graph obtained from $\mathrm{PAG}(n,m)$ by deleting loops and keeping only one copy of the parallel edges. Then

$$\mathrm{SPAG}\left(n, \frac{n^2}{2} \cdot (\rho + o(1))\right) \xrightarrow{\mathrm{d}} W_s, \qquad W_s(x,y) := 1 - \exp(-\rho \ln(x) \ln(y)),$$

where (analogously to (303)) the symbol $\xrightarrow{\mathrm{d}}$ denotes convergence in probability of a sequence of random simple graphs to a (simple) graphon.

It is easy to see that this theorem is a corollary of the following proposition:

**Proposition 4.1.** *Let us fix $\kappa, \rho \in (0, +\infty)$. If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n^{m(n)}$ with distribution* (309) *for $n = 1, 2, \dots$, moreover the asymptotic edge density is*

$$\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho,$$

*then $\mathbf{X}_n \xrightarrow{d} W$ where*

$$W(x, y, k) = \begin{cases} \mathbf{p}(k, \frac{F^{-1}(x)F^{-1}(y)}{\rho}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F^{-1}(x)F^{-1}(y)}{2\rho}\right) & \text{if } x = y \end{cases} \tag{310}$$

*and $F^{-1}$ is the inverse function of $F(x) = \int_0^x \mathbf{g}(y, \kappa, \frac{\kappa}{\rho})\mathrm{d}y$.*

We prove this proposition in Section 4.7.

Now we describe the evolution of the edge reconnecting model by describing the evolution of the limiting multigraphons. We consider a sequence of initial multigraphs $(G_n)_{n=1}^{\infty}$ which converge to a multigraphon $W$. We assume $|V(G_n)| = n$. We denote the adjacency matrix of $G_n$ by $B_n \in \mathcal{A}_n$. We assume that the technical condition

$$\exists\, \lambda > 0,\ C < +\infty\ \ \forall\, n: \quad \frac{1}{\binom{n}{2}} \sum_{i \le j \le n} e^{\lambda B_n(i,j)} \le C, \quad \frac{1}{n} \sum_{i=1}^{n} e^{\lambda B_n(i,i)} \le C \tag{311}$$

holds. The probabilistic meaning of (311) is given by (380) in Section 4.8.

First we state Theorem 4.2 about the evolution of the edge reconnecting model on the $T = \mathcal{O}(n^2)$ timescale. In the Introduction Theorem 4.2 was referred to as $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{d}} \breve{W}_t$ in order to make the notation as simple as possible. In fact we are going to prove $\mathcal{G}_n(t \cdot \frac{\rho(W)}{2} \cdot n^2) \xrightarrow{\mathrm{d}} W_t$, thus $W_t = \breve{W}_{\frac{\varrho}{2}t}$. The notation $\breve{W}_t$ will no longer be used.

**Theorem 4.2.** *Let us fix $\kappa \in (0, +\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0, 1, \dots$ on the state space $\mathcal{A}_n^{m(n)}$ and initial state $\mathbf{X}_n(0) = B_n \in \mathcal{A}_n^{m(n)}$ for $n = 1, 2, \dots$. We assume $B_n \to W$ for some multigraphon $W$ and that* (311) *holds.*

*Then for all $t \in [0, +\infty)$ we have*

$$\mathbf{X}_n\left(\lfloor t \cdot \frac{\rho(W) \cdot n^2}{2} \rfloor\right) \xrightarrow{d} W_t \qquad \text{as} \qquad n \to \infty \tag{312}$$

*where*

$$W_t(x, y, k) = \\ \sum_{h=0}^{\infty} W(x, y, h) \sum_{l=0}^{k} \mathbf{b}(l, h, e^{-t})) \cdot \mathbf{p}\left(k - l, \frac{D(W, x) \cdot D(W, y)}{\rho(W)}(1 - e^{-t}))\right) \\ \text{if } x \neq y \tag{313}$$

$W_t(x,y,k) =$

$$\mathbb{1}[2|k] \cdot \sum_{h=0}^{\infty} W(x,y,h) \sum_{l=0}^{\frac{k}{2}} \mathbf{b}(l, \frac{h}{2}, e^{-t})) \cdot \mathbf{p}\left(\frac{k}{2} - l, \frac{D(W,x)\cdot D(W,y)}{2\rho(W)}(1-e^{-t}))\right) \quad \textit{if } x = y \quad (314)$$

The reader should not worry about the meaning of the (rather complicated) formulas (313) & (314) at this point: the message is that there is an explicit expression for $W_t$. We are going to give an intuitive explanation of Theorem 4.2 in Section 4.6 by relating the evolution of the number of parallel/loop edges between two vertices to the evolution of the queue length of an M/M/$\infty$-queue. We rigorously prove Theorem 4.2 in Section 4.9.

Now we look at the evolution of the edge reconnecting model on the $T = \mathcal{O}(n^3)$ timescale. In the Introduction Theorem 4.3 was referred to as $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{d}} \tilde{W}_t$ in order to make the notation as simple as possible. In fact we are going to prove $\mathcal{G}_n(t \cdot \rho(W) \cdot n^3) \xrightarrow{\mathrm{d}} \hat{W}_t$, thus $\hat{W}_t = \tilde{W}_{\rho t}$. The notation $\tilde{W}_t$ will no longer be used.

**Theorem 4.3.** *Let us fix $\kappa \in (0,+\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0,1,\dots$ on the state space $\mathcal{A}_n^{m(n)}$ and initial state $\mathbf{X}_n(0) = B_n \in \mathcal{A}_n^{m(n)}$ for $n = 1,2,\dots$. We assume $B_n \to W$ for some multigraphon $W$ and that* (311) *holds.*

*Then for all $t \in (0,+\infty)$ (but not for t=0) we have*

$$\mathbf{X}_n\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor\right) \xrightarrow{d} \hat{W}_t \qquad \textit{as} \qquad n \to \infty \tag{315}$$

*where*

$$\hat{W}_t(x,y,k) = \begin{cases} \mathbf{p}(k, \frac{F_t^{-1}(x)F_t^{-1}(y)}{\rho(W)}) & \textit{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F_t^{-1}(x)F_t^{-1}(y)}{2\rho(W)}\right) & \textit{if } x = y \end{cases} \tag{316}$$

*and $F_t^{-1}$ is the inverse function of $F_t(x) = \int_0^x f(t,y)\,\mathrm{d}y$ where*

$$f(t,x) = \int_0^\infty \sum_{i=0}^{\infty} \mathbf{p}(i, z\cdot\tau(\alpha,t))\mathbf{g}(x, \kappa+i, \tau(\alpha,t)+\alpha)\,\mathrm{d}F_0(z), \tag{317}$$

$\alpha = \frac{\kappa}{\rho(W)}$, $\tau(\alpha,t) = \frac{\alpha}{\exp(\alpha t)-1}$ *and* $F_0(x) = \int_0^1 \mathbb{1}[D(W,y) \le x]\,\mathrm{d}y$, $x \in [0,+\infty)$.

The reader should not worry about the meaning of the formula (317) at this point: the message is that there is an explicit expression for $F_t$ and $\hat{W}_t$.

We prove Theorem 4.3 in Section 4.10, but first we give an intuitive explanation in Section 4.6 by relating the evolution of the degree of a vertex to a diffusion process (the C.I.R. process).

In Section 4.7 we describe the *edge-stationary* states of the edge reconnecting model, a family of random multigraphs whose limiting multigraphons are of form (310) where

$F$ denotes an arbitrary probability distribution function. The construction of edge-stationary random graphs is closely related to that of the *configuration model*, well-known in the theory of random graphs.

In Subsection 4.10.3 we give a precise formulation of the fact that the edge reconnecting model exhibits aging (see page 97 for the heuristic description), but before that we need to introduce some more notation.

## 4.5 Vertex exchangeable arrays

In this section we give probabilistic meaning to homomorphism densities in Subsection 4.5.1 and relate the theory of graph limits to that of vertex exchangeable infinite arrays in Subsection 4.5.2.

The notion of the $W$-random graph (see Definition 4.5.1) is already present in [30], but the connection between Theorem 4.1 and Aldous' representation theorem (Theorem 4.4) was first observed in [18].

### 4.5.1 Convergence of exchangeable random adjacency matrices

In this subsection we define the notion of vertex exchangeability of random adjacency matrices and show that vertex exchangeability is preserved by the edge reconnecting dynamics.

In Lemma 4.5.1 we give a useful probabilistic reformulation of convergence of random multigraph sequences (see (303)).

Let $F \in \mathcal{M}_k$, $G \in \mathcal{M}_n$ with adjacency matrices $A \in \mathcal{A}_k$, $B \in \mathcal{A}_n$. In (292) we gave the definition of $t_=(F,G)$. If $k \le n$ we also define the *induced injective homomorphism density* of $F$ into $G$ by

$$t^0_=(F,G) := t^0_=(A,B) := \frac{1}{n(n-1)\dots(n-k+1)} \sum_{\varphi:[k]\hookrightarrow[n]} \mathbb{1}\left[\forall i,j \in [k]:\ A(i,j) = B(\varphi(i),\varphi(j))\right], \quad (318)$$

where $\varphi : [k] \hookrightarrow [n]$ is an injective mapping from $[k]$ to $[n]$.

$t^0_=(F,G)$ is well-defined for unlabeled graphs $F$ and $G$, since relabeling $F$ and $G$ does not change their value. Thus for every fixed multigraph $G$ the function $t^0_=(\cdot,G) : \bigcup_{k\le n}\mathcal{M}_k \to [0,1]$ is a multigraph parameter.

It is well-known (e.g. the proof Lemma 2.1 of [30] works for multigraphs as well) that

$$\left|t_=(F,G) - t^0_=(F,G)\right| \le \frac{1}{n}\binom{k}{2}. \quad (319)$$

Let $\mathbf{X} = (X(i,j))_{i,j=1}^n$ denote a random element of $\mathcal{A}_n$. We say that the distribution $\mathbf{X}$ is *vertex exchangeable* if for all permutations $\tau : [n] \to [n]$ and $B \in \mathcal{A}_n$

$$\mathbf{P}\big(\forall\, i,j \in [n]:\ B(i,j) = X(i,j)\big) = \mathbf{P}\big(\forall\, i,j \in [n]:\ B(i,j) = X(\tau(i),\tau(j))\big),$$

that is $(X(i,j))_{i,j=1}^n$ and $(X(\tau(i),\tau(j)))_{i,j=1}^n$ have the same distribution:

$$(X(i,j))_{i,j=1}^n \sim (X(\tau(i),\tau(j)))_{i,j=1}^n \,. \tag{320}$$

Vertex exchangeable probability distributions on $\mathcal{A}_n$ are in one-to-one correspondence with probability distributions on the set of isomorphism types $\mathcal{M}_n$, and the extremal points of the convex set of vertex exchangeable probability distributions on $\mathcal{A}_n$ are the ones that are uniformly distributed on the set of adjacency matrices that arise as labelings of a multigraph $G \in \mathcal{M}_n$.

It is easy to check that the edge reconnecting dynamics defined in Section 4.2 preserves exchangeability: if the distribution of $\mathbf{X}(T)$ is vertex exchangeable then the distribution of $\mathbf{X}(T+1)$ is also vertex exchangeable. The stationary distribution (309) is vertex exchangeable.

In the statements of Theorem 4.2 and Theorem 4.3 the initial state of the Markov chain $\mathbf{X}_n(T) = (X_n(T,i,j))_{i,j=1}^n$ was the deterministic adjacency matrix $\mathbf{X}_n(0) = B_n$, but if we define

$$\hat{X}_n(0,i,j) := B_n\left(\pi(i),\pi(j)\right). \tag{321}$$

where $\pi$ denotes a uniformly chosen random permutation of $[n]$ and denote the edge reconnecting Markov chain with this initial distribution by $\hat{\mathbf{X}}_n(T)$, $T = 1,2,\dots$, then

$$\forall\, T \in \mathbb{N}: \quad \left(\hat{X}_n(T,i,j)\right)_{i,j=1}^n \sim \left(X_n(T,\pi(i),\pi(j))\right)_{i,j=1}^n, \quad t_=(A,\mathbf{X}_n) \sim t_=(A,\hat{\mathbf{X}}_n), \tag{322}$$

thus by (303) we get that the assertion of Theorem 4.2 and Theorem 4.3 holds for $\mathbf{X}_n(T)$ if and only if it holds for $\hat{\mathbf{X}}_n(T)$. From now on we are going to replace $\mathbf{X}_n(T)$ by $\hat{\mathbf{X}}_n(T)$ and assume that the distribution of $\mathbf{X}_n(T)$ is vertex exchangeable.

If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n$ then

$$\mathbf{X}_n^{[k]} := (X_n(i,j))_{i,j=1}^k$$

is a random element of $\mathcal{A}_k$.

**Lemma 4.5.1.** *Let $\mathbf{X}_n = (X_n(i,j))_{i,j=1}^n$ be a random, vertex exchangeable element of $\mathcal{A}_n$ for all $n \in \mathbb{N}$. The following statements are equivalent:*

*(a)* $\mathbf{X}_n \xrightarrow{d} W$*, that is* $\forall\, k\; \forall\, A \in \mathcal{A}_k: \;\; t_=(A,\mathbf{X}_n) \xrightarrow{d} t_=(A,W)$

*(b)* $\forall\, k\; \forall\, A \in \mathcal{A}_k: \;\; \lim_{n\to\infty} \mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big) = t_=(A,W)$

*Proof.*

$(a) \implies (b)$: By exchangeability, (318) and the linearity of expectation we have

$$\mathbf{E}\big(t_=^0(A,\mathbf{X}_n)\big) = \mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big),$$

so $t_=(A,\mathbf{X}_n) \xrightarrow{\text{d}} t_=(A,W)$ and (319) imply $\mathbf{P}\big(\mathbf{X}_n^{[k]} = A\big) \to t_=(A,W)$.

$(b) \implies (a)$: The idea of this proof is the same as that of Lemma 2.4 of [30].

Recalling the definition of the inverse Möbius transform (298) denote

$$t_{\leq}(A, \mathbf{X}_n) := t_{=}^{-\dagger}(A, \mathbf{X}_n), \quad t_{\leq}(A, W) := t_{=}^{-\dagger}(A, W).$$

By (304) we only need to check $\forall\, k\; \forall\, A \in \mathcal{A}_k :\; t_{\leq}(A, \mathbf{X}_n) \xrightarrow{\mathrm{d}} t_{\leq}(A, W)$ to obtain (a).

From $(b)$ we get $\mathbf{E}\big(\, t_{\leq}(A, \mathbf{X}_n)\,\big) \to t_{\leq}(A, W)$ for all $A$ by the argument used in the proof of $(a) \implies (b)$.

In order to have $t_{\leq}(A, \mathbf{X}_n) \xrightarrow{\mathrm{d}} t_{\leq}(A, W)$ we only need to show

$$\mathbf{D}^2\left(t_{\leq}(A, \mathbf{X}_n)\right) \to 0.$$

If $F$ is the multigraph with adjacency matrix $A$, denote by $AA$ the adjacency matrix of $FF$. The multigraph parameter $t_{\leq}(\cdot, W)$ and the (random) multigraph parameter $t_{\leq}(\cdot, \mathbf{X}_n)$ are multiplicative, thus we have $\mathbf{P}\big(\, t_{\leq}(AA, \mathbf{X}_n) = t_{\leq}(A, \mathbf{X}_n)^2\,\big) = 1$ and

$$\lim_{n\to\infty} \mathbf{D}^2\left(t_{\leq}(A, \mathbf{X}_n)\right) = \lim_{n\to\infty} \mathbf{E}\big(\, t_{\leq}(AA, \mathbf{X}_n)\,\big) - \mathbf{E}\big(\, t_{\leq}(A, \mathbf{X}_n)\,\big)^2 = \\ t_{\leq}(AA, W) - t_{\leq}(A, W)^2 = t_{\leq}(A, W)^2 - t_{\leq}(A, W)^2 = 0 \quad (323)$$

□

### 4.5.2 Infinite exchangeable arrays

In this subsection we introduce random infinite arrays $\mathbf{X} = \big(X(i,j)\big)_{i,j=1}^{\infty}$ that arise as the adjacency matrices of random infinite labeled multigraphs.

In Definition 4.5.1 we introduce $\mathbf{X}_W$, the infinite random matrix corresponding to the multigraphon $W$.

Theorem 4.4 (Aldous' representation theorem) states that if $\mathbf{X}$ is the adjacency matrix of an infinite random multigraph whose distribution is invariant under the relabeling of vertices and has the property that subgraphs spanned by disjoint vertex sets are independent, then $\mathbf{X} \sim \mathbf{X}_W$ for some multigraphon $W$.

We conclude this subsection by relating Theorem 4.4 to Lemma 4.5.1 and Theorem 4.1.

Let $\mathcal{A}_{\mathbb{N}}$ denote the set of adjacency matrices $(A(i,j))_{i,j=1}^{\infty}$of countable multigraphs:

$$\mathcal{A}_{\mathbb{N}} = \left\{ A \in \mathbb{N}_0^{\mathbb{N}\times\mathbb{N}} \,:\, \forall\, i,j\; A(i,j) \equiv A(j,i),\, \forall\, i\;\, 2\,|\, A(i,i) \right\}. \quad (324)$$

We consider the probability space $(\mathcal{A}_{\mathbb{N}}, \mathcal{F}, \mathbf{P})$ where $\mathcal{F}$ is the coarsest sigma-algebra with respect to which $A(i,j)$ is measurable for all $i,j$ and $\mathbf{P}$ is a probability measure on the measurable space $(\mathcal{A}_{\mathbb{N}}, \mathcal{F})$. We are going to denote the infinite random array with distribution $\mathbf{P}$ by $\mathbf{X} = (X(i,j))_{i,j=1}^{\infty}$. We use the standard notation $\mathbf{X} \sim \mathbf{Y}$ if $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed (i.e., their distribution $\mathbf{P}$ is identical on $(\mathcal{A}_{\mathbb{N}}, \mathcal{F})$).

**Definition 4.5.1.** *Let $U_i, i \in \mathbb{N}$ be independent random variables uniformly distributed in $[0,1]$. For a multigraphon $W$ we define the random countable adjacency matrix $\mathbf{X}_W = (X_W(i,j))_{i,j=1}^\infty$ as follows: Given the background variables $\mathbf{U} := (U_i)_{i=1}^\infty$ the random variables $(X_W(i,j))_{i\le j\in\mathbb{N}}$ are conditionally independent and*

$$\mathbf{P}\big(\, X_W(i,j) = m \,\big|\, \mathbf{U}\,\big) = W(U_i, U_j, m),$$

*that is if $A \in \mathcal{A}_k$ then we have*

$$\mathbf{P}\big(\, \mathbf{X}_W^{[k]} = A \,\big|\, \mathbf{U}\,\big) := \prod_{i\le j\le k} W(U_i, U_j, A(i,j)). \tag{325}$$

In plain words: if $i \neq j$ and $U_i = x$, $U_j = y$ then the number of multiple edges between the vertices labeled by $i$ and $j$ in $\mathbf{X}_W$ has distribution $\big(W(x,y,k)\big)_{k=1}^\infty$ and the number of loop edges at vertex $i$ has distribution $\big(W(x,x,2k)\big)_{k=1}^\infty$ (these are indeed proper probability distributions by (294) and (295)).

For every multigraphon $W$ and multigraph $F \in \mathcal{M}_k$ with adjacency matrix $A \in \mathcal{A}_k$ we have

$$t_=(F,W) = \mathbf{P}\big(\, \mathbf{X}_W^{[k]} = A\,\big), \tag{326}$$

from which $\sum_{A\in\mathcal{A}_k} t_=(A,W) = 1$ trivially follows for all $k \in \mathbb{N}$.

Recalling (300) and (301) we have

$$D(W,x) = \mathbf{E}\big(\, X_W(1,2) \,\big|\, U_1 = x\,\big), \qquad \rho(W) = \mathbf{E}\big(\, X_W(1,2)\,\big). \tag{327}$$

If $\rho(W) < +\infty$ then $D(W,U_1) < +\infty$ almost surely. It follows from the law of large numbers that

$$D(\mathbf{X}_W, i) := \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n X_W(i,j) \overset{\text{a.s.}}{=} D(W, U_i) \tag{328}$$

**Definition 4.5.2.**

- *A random infinite array $\mathbf{X} = (X(i,j))_{i,j=1}^\infty$ is* vertex exchangeable *if*

$$(X(\tau(i),\tau(j)))_{i,j=1}^\infty \sim (X(i,j))_{i,j=1}^\infty \tag{329}$$

  *for all finitely supported permutations $\tau : \mathbb{N} \to \mathbb{N}$.*

- *We call $\mathbf{X} = (X(i,j))_{i,j=1}^\infty$* dissociated *if for all $m, n \in \mathbb{N}$ the $\mathcal{A}_n$-valued random variable $(X(i,j))_{i,j=1}^n$ is independent of the $\mathcal{A}_m$-valued random variable $(X(i,j))_{i,j=n+1}^{n+m}$.*

In our case an infinite exchangeable array can be taught of as the adjacency matrix of a random multigraph with vertex set $\mathbb{N}$: the adjacency matrix of this random infinite multigraph is vertex exchangeable iff the distribution of the random graph is invariant under the relabeling of the vertices and dissociated iff subgraphs spanned by disjoint vertex sets are independent.

It follows from Definition 4.5.1 that $\mathbf{X}_W$ is dissociated which in turn implies the multiplicativity of the multigraph parameter $t_{=}^{-\dagger}(\cdot, W)$.

The following theorem is a reformulation of Aldous' representation theorem (see Theorem 1.4, Proposition 3.3 and Theorem 5.1 in [1]). The connection with the theory of graph limits was observed in [18]. See also Theorem 3.1, Theorem 3.2, Proposition 3.4 of [31].

**Theorem 4.4.**

*(i) A random element $\mathbf{X}$ of $\mathcal{A}_{\mathbb{N}}$ is vertex exchangeable and dissociated if and only if $\mathbf{X} \sim \mathbf{X}_W$ for some multigraphon $W$.*

*(ii) The extremal points of the convex set of vertex exchangeable probability measures on $(\mathcal{A}_{\mathbb{N}}, \mathcal{F})$ are of form $\mathbf{X}_W$.*

Proving (i) is essentially the same as proving the equivalence of (b) and (c) in Theorem 4.1 (for details, see the proofs of Theorem 1 and Theorem 2 in [28]). For the proof of (ii), see Theorem 5.5 of [18].

We say that a sequence of infinite arrays $(\mathbf{X}_n)_{n=1}^{\infty}$ *converges in distribution* if

$$\forall\, k\ \forall\, A \in \mathcal{A}_k : \lim_{n\to\infty} \mathbf{P}\big(\forall i,j \le k : A(i,j) = X_n(i,j)\big) = g(A), \tag{330}$$

$$\forall\, k : \sum_{A \in \mathcal{A}_k} g(A) = 1 \tag{331}$$

for some $g : \bigcup_{k=1}^{\infty} \mathcal{A}_k \to [0,1]$.

Alternatively we might say that $(\mathbf{X}_n)_{n=1}^{\infty}$ converges in distribution if and only if $\mathbf{X}_n^{[k]}$ converges in distribution to some random element of $\mathcal{A}_k$ for all $k$.

By Kolmogorov's extension theorem ([29], Section 1.4, Theorem 1.) there exists a random infinite array $\mathbf{X} = (X(i,j))_{i,j=1}^{\infty}$ such that for all $k$ and $A \in \mathcal{A}_k$

$$\mathbf{P}\big(\forall i,j \le k :\ A(i,j) = X_n(i,j)\big) \overset{n\to\infty}{\longrightarrow} \mathbf{P}\big(\forall i,j \le k :\ A(i,j) = X(i,j)\big) \tag{332}$$

In this case we say that $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}$. If $\mathbf{X}_n$ is vertex exchangeable for all $n$, then $\mathbf{X}$ is also vertex exchangeable. If $\mathbf{X}_n$ is dissociated for all $n$, then $\mathbf{X}$ is also dissociated. It follows from Theorem 4.4 (i) that if $\mathbf{X}_{W_n} \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}$ then $\mathbf{X} \sim \mathbf{X}_W$ for some multigraphon $W$. It is straightforward to see that $W_n \to W$ if and only if $\mathbf{X}_{W_n} \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}_W$.

If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n$ then $\mathbf{X}_n^{[k]}$ is well-defined for $k \le n$, thus we might define $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}$ by (332), and by Lemma 4.5.1 we have that if $\mathbf{X}_n$ is vertex exchangeable then $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} W$ if and only if $\mathbf{X}_n \overset{\mathrm{d}}{\longrightarrow} \mathbf{X}_W$. We are going to take advantage of this approach in our proofs.

## 4.6 Sketch proofs

In this section we recall the definition of two stochastic processes (the M/M/$\infty$-queue and the C.I.R. process) and give sketch proofs of Theorem 4.2 in Subsection 4.6.1 and Theorem 4.3 in Subsection 4.6.2 using these stochastic processes.

The non-rigorous arguments presented in this section also serve as an outline of the rigorous proofs of Theorems 4.2 and 4.3 that we are going to present in Sections 4.8, 4.9 and 4.10.

We also discuss conservation of the quantities $D(W_t, x)$ and $\rho(\hat{W}_t)$ and show

$$\lim_{t\to 0_+} W_t = W, \qquad \lim_{t\to\infty} W_t = \lim_{t\to 0_+} \hat{W}_t, \qquad \lim_{t\to\infty} \hat{W}_t = \hat{W}_\infty$$

where $\hat{W}_\infty$ is the multigraphon defined by the r.h.s. of (310).

First we recall the definition of two stochastic processes:

- The M/M/$\infty$-queue with arrival rate $\mu$ and service rate 1 is an $\mathbb{N}_0$-valued continuous-time Markov chain $Y_t$, $t \in [0, +\infty)$ with infinitesimal jump rates

$$\mathbf{P}\big(\, Y_{t+\mathrm{d}t} = k+1 \,\big|\, Y_t = k \,\big) = \mu \,\mathrm{d}t + o(\mathrm{d}t) \tag{333}$$
$$\mathbf{P}\big(\, Y_{t+\mathrm{d}t} = k-1 \,\big|\, Y_t = k \,\big) = k \,\mathrm{d}t + o(\mathrm{d}t) \tag{334}$$
$$\mathbf{P}\big(\, Y_{t+\mathrm{d}t} = k \,\big|\, Y_t = k \,\big) = 1 - (\mu + k)\mathrm{d}t + o(\mathrm{d}t) \tag{335}$$

  It is well-known (see Exercise 5.8 of [27]) that if $Y_0 = h \in \mathbb{N}_0$ then

$$Y_t \sim \mathrm{BIN}(h, e^{-t}) * \mathrm{POI}\left((1-e^{-t})\mu\right). \tag{336}$$

- Fix $\kappa, \rho \in (0, +\infty)$. The Cox–Ingersoll–Ross (C.I.R.) process is a diffusion process with stochastic differential equation

$$\mathrm{d}Z_t = \left(\kappa - \frac{\kappa}{\rho} Z_t\right) \mathrm{d}t + \sqrt{2 Z_t}\, \mathrm{d}B_t, \tag{337}$$

  where $B_t$ denotes the standard Brownian motion.

  It is well-known (see Chapter 4.6 of [17]) that if we denote

$$\alpha := \frac{\kappa}{\rho} \qquad \text{and} \qquad \tau(\alpha, t) := \frac{\alpha}{\exp(\alpha t) - 1}$$

  and if we assume $Z_0 = z$ then $2(\tau(\alpha,t) + \alpha) \cdot Z_t$ follows a noncentral chi-square distribution with $2\kappa$ degrees of freedom and non-centrality parameter $2z \cdot \tau(\alpha, t)$, thus (recall the formulas (305) and (307)) we have $\mathbf{P}\big(\, Z_t \le x \,\big|\, Z_0 = z \,\big) = \int_0^x f(t, z, y)\,\mathrm{d}y$ where

$$f(t, z, x) = \sum_{i=0}^{\infty} \mathbf{p}(i, z \cdot \tau(\alpha, t)) \mathbf{g}(x, \kappa + i, \tau(\alpha, t) + \alpha). \tag{338}$$

When dealing with $n \to \infty$ limits we are going to use non-rigorous arguments in our sketch proofs: we forget about error terms and omit technical details, we also replace convergent sequences with their limit objects. We are going to use the notation $a_n \overset{\infty}{\cong} a$ to replace $\lim_{n\to\infty} a_n = a$, $a_n \overset{\infty}{\cong} b_n$ to replace $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\mathbf{X}_n \overset{\infty}{\approx} \mathbf{X}$ to replace $\mathbf{X}_n \xrightarrow{\mathrm{d}} \mathbf{X}$.

Assume given a sequence of vertex exchangeable initial states $\mathbf{X}_n(0) \xrightarrow{\mathrm{d}} \mathbf{X}_W$. Denote by $\mathcal{D}_n(T,i) := \frac{1}{n} d(\mathbf{X}_n(T), i)$.

We make the following non-rigorous assumptions (see (302), (328) and (332)):

$$X_n(0,i,j) \overset{\infty}{\cong} X_W(i,j) \qquad \mathcal{D}_n(0,i) \overset{\infty}{\cong} D(\mathbf{X}_W, i) \qquad \frac{2m}{n^2} \overset{\infty}{\cong} \rho(W) =: \rho \tag{339}$$

It follows from (287) and (290) that

$$\mathbf{E}\big(\, \mathcal{D}_n(T+1,i) - \mathcal{D}_n(T,i) \,\big|\, \mathbf{X}_n(T) \,\big) = \frac{\mathcal{D}_n(T,i) + \frac{\kappa}{n}}{2m + n\kappa} - \frac{\mathcal{D}_n(T,i)}{2m} \overset{\infty}{\cong} \frac{1}{2mn}\left(\kappa - \frac{\kappa}{\rho}\mathcal{D}_n(T,i)\right) \tag{340}$$

$$\mathbf{D}^2\left(\mathcal{D}_n(T+1,i) - \mathcal{D}_n(T,i) \,\big|\, \mathbf{X}_n(T)\right) \overset{\infty}{\cong} \frac{\mathcal{D}_n(T,i) + \frac{\kappa}{n}}{2mn + n^2\kappa} + \frac{\mathcal{D}_n(T,i)}{2mn} \overset{\infty}{\cong} \frac{1}{2mn} 2\mathcal{D}_n(T,i) \tag{341}$$

### 4.6.1 Sketch proof of Theorem 4.2

Assuming $\mathcal{D}_n(T,i) = \mathcal{O}(1)$ for all $T$ we get that if $T = \mathcal{O}(n^2)$ then (340) and (341) implies

$$\mathbf{E}\big(\, \mathcal{D}_n(T,i) - \mathcal{D}_n(0,i) \,\big) \overset{\infty}{\cong} \mathcal{O}(n^2 \cdot \frac{1}{n^3}) \quad \mathbf{D}^2\left(\mathcal{D}_n(T,i) - \mathcal{D}_n(0,i)\right) \overset{\infty}{\cong} \mathcal{O}(n^2 \cdot \frac{1}{n^3}), \tag{342}$$

thus similarly to (339) we might assume $\mathcal{D}_n(T,i) \overset{\infty}{\cong} \mathcal{D}_n(0,i) \overset{\infty}{\cong} D(\mathbf{X}_W, i)$.

We look at the evolution of $X_n(T,i,j)$ for some $i \neq j$. From (289) and (290) it follows that

$$\mathbf{P}\big(\, X_n(T+1,i,j) = X_n(T,i,j) + 1 \,\big|\, \mathbf{X}_n \,\big) \overset{\infty}{\cong} \frac{\mathcal{D}_n(T,i)\cdot n}{2m} \cdot \frac{\mathcal{D}_n(T,j)\cdot n + \kappa}{2m + n\kappa} + \frac{\mathcal{D}_n(T,j)\cdot n}{2m} \cdot \frac{\mathcal{D}_n(T,i)\cdot n + \kappa}{2m + n\kappa} \overset{\infty}{\cong} \frac{1}{m} \frac{D(\mathbf{X}_W, i) D(\mathbf{X}_W, j)}{\rho} \tag{343}$$

$$\mathbf{P}\big(\, X_n(T+1,i,j) = X_n(T,i,j) - 1 \,\big|\, \mathbf{X}_n \,\big) \overset{\infty}{\cong} \frac{1}{m} X_n(T,i,j) \tag{344}$$

In the statement of Theorem 4.2 we used the time scaling $T = \lfloor t \cdot \frac{\rho \cdot n^2}{2} \rfloor \overset{\infty}{\cong} t \cdot m$, thus if $\mathrm{d}t := \frac{1}{m}$ then $T+1$ corresponds to $t + \mathrm{d}t$. If we define $Y_t := X(t\cdot m, i, j)$ then the time evolution of $Y_t$ approximates that of an M/M/$\infty$-queue with arrival rate $\mu = \frac{D(\mathbf{X}_W,i)D(\mathbf{X}_W,j)}{\rho}$ and service rate 1. Thus by (336) if $Y_0 = h$ then

$$Y_t \sim \mathrm{BIN}(h, e^{-t}) * \mathrm{POI}\left(\frac{D(\mathbf{X}_W, i) D(\mathbf{X}_W, j)}{\rho}(1 - e^{-t})\right).$$

Since $Y_0 \overset{\infty}{\cong} X_W(i,j)$ we have $\mathbf{P}\big(\, Y_0 = h \,\big|\, \mathbf{U} \,\big) = W(U_i, U_j, h)$. Thus by (336) and (328) we get $\mathbf{P}\big(\, Y_t = h \,\big|\, \mathbf{U} \,\big) = W_t(U_i, U_j, h)$ where $W_t$ is defined by (313).

One can similarly show that the distribution of the number of loop edges at vertex $i$ at time $T = t \cdot m$ can be approximated by

$$\frac{X_n(t\cdot m, i, i)}{2} \overset{\infty}{\approx} \mathrm{BIN}(h, e^{-t}) * \mathrm{POI}\left( \frac{D(\mathbf{X}_W, i)^2}{2\rho}(1-e^{-t}) \right) \tag{345}$$

Thus $\mathbf{P}\big(\, X_n(t\cdot m, i, i) = k \,\big|\, \mathbf{U} \,\big) \overset{\infty}{\cong} W_t(U_i, U_i, k)$, see (314). By Lemma 4.5.1 we need to show that for all $k \in \mathbb{N}$

$$\mathbf{X}_n^{[k]}(t \cdot \frac{\rho \cdot n^2}{2}) \overset{\infty}{\approx} \mathbf{X}_{W_t}^{[k]}$$

in order to show that (313) and (314) holds. We have already seen that for all $i, j$ the conditional distributions of $X_n(t \cdot \frac{\rho \cdot n^2}{2}, i, j)$ and $X_{W_t}(i,j)$ are approximately the same given $\mathbf{U}$. By Definition 4.5.1 we only need to show that the random variables $\left( X_n(t \cdot \frac{\rho \cdot n^2}{2}, i, j) \right)_{i \leq j \leq k}$ are approximately conditionally independent given $\mathbf{U}$. It is easy to convince ourselves of this fact, since conditional independence holds for $t = 0$ and the evolutions of the entries of the adjacency matrix do not have much effect on each other.

If we substitute $t \to 0_+$ into (336) and (345) we indeed get $W_t \to W$. It is also reassuring to see that $D(W_t, x) \equiv D(W, x)$ (in agreement with our heuristic assumption $\mathcal{D}_n(T, i) \overset{\infty}{\cong} \mathcal{D}_n(0, i)$): if $Y_0 = X_W(1,2)$ then

$$\mathbf{E}\big(\, Y_t \,\big|\, U_1 = x, U_2 = y \,\big) \overset{(336)}{=} e^{-t} \sum_{h=0}^{\infty} h \cdot W(x,y,h) + (1-e^{-t}) \frac{D(W,x)D(W,y)}{\rho}$$

$$D(W_t, x) = \int_0^1 \mathbf{E}\big(\, Y_t \,\big|\, U_1 = x, U_2 = y \,\big) \,\mathrm{d}y = e^{-t} D(W,x) + (1-e^{-t}) \frac{D(W,x)\cdot\rho}{\rho}$$

If we let $t \to \infty$ in (336) and (345) (or in (314) and (313)) we get that

$$\lim_{t\to\infty} W_t(x,y,k) = \begin{cases} \mathbf{p}(k, \frac{D(W,x)D(W,y)}{\rho(W)}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left( \frac{k}{2}, \frac{D(W,x)D(W,y)}{2\rho(W)} \right) & \text{if } x = y \end{cases} \tag{346}$$

Thus on the $T \asymp n^2$ timescale the distribution of the number of parallel edges of $\mathbf{X}_{W_t}$ reach an equilibrium state given the distribution of the degrees of the vertices. On the $T \asymp n^3$ timescale we can see how the distribution of the degrees reach their equilibrium state:

### 4.6.2 Sketch proof of Theorem 4.3

We first pick a vertex $i$ and look at the time evolution of the stochastic process $Z_t := \mathcal{D}_n(\lfloor t \cdot 2nm \rfloor, i)$. From (339) we get $T = t \cdot 2mn \overset{\infty}{\cong} t \cdot \rho \cdot n^3$. If we let $\mathrm{d}t = \frac{1}{2nm}$ then $T+1$ corresponds to $t + \mathrm{d}t$. Let $\mathrm{d}Z_t := Z_{t+\mathrm{d}t} - Z_t$. From (340) and (341) we get

$$\mathbf{E}\big(\, \mathrm{d}Z_t \,\big) \overset{\infty}{\cong} \left( \kappa - \frac{\kappa}{\rho} Z_t \right) \mathrm{d}t \qquad \mathbf{D}^2\left(\mathrm{d}Z_t\right) \overset{\infty}{\cong} 2 Z_t \mathrm{d}t$$

Thus the process $Z_t$ approximates the solution of the S.D.E. of the C.I.R. process (337).

By (339) and (328) we have $Z_0 \stackrel{\infty}{\cong} D(W, U_i)$, thus

$$\mathbf{P}\big(\, Z_0 \leq x \,\big) = F_0(x) = \int_0^1 \mathbb{1}[D(W, y) \leq x]\, \mathrm{d}y$$

and the probability density function $f(t, x)$ of $Z_t$ is given by (317) for $t > 0$.

The joint distribution of the initial state $(\mathcal{D}(0, i))_{i=1}^k \stackrel{\infty}{\cong} (D(W, U_i))_{i=1}^k$ is i.i.d. by our assumption (339) and it is easy to convince ourselves that the evolution of the coordinates of $(\mathcal{D}(t \cdot 2nm, i))_{i=1}^k$ have little effect on each other, thus we might assume

$$(\mathcal{D}(t \cdot 2nm, i))_{i=1}^k \stackrel{\infty}{\approx} \left(F_t^{-1}(\hat{U}_i)\right)_{i=1}^k$$

where $\left(\hat{U}_i\right)_{i=1}^k$ are i.i.d. and uniformly distributed on $[0, 1]$ and $F_t^{-1}$ is the inverse function of $F_t(x) = \int_0^x f(t, y)\, \mathrm{d}y$.

The distribution of the number of parallel edges between the vertices $i$ and $j$ reaches its M/M/$\infty$−Poisson equilibrium given the degrees of the vertices $i$ and $j$ on the $n^2$ timescale, whereas $\mathcal{D}(T, i)$ and $\mathcal{D}(T, j)$ change only on the slower $n^3$ timescale, thus similarly to (346) we get

$$\mathbf{P}\big(\, \mathbf{X}_n^{[k]}(t \cdot \rho \cdot n^3, i, j) = A \,\big) \stackrel{\infty}{\cong} t_=(A, \hat{W}_t)$$

for $t > 0$ with the $\hat{W}_t$ of Theorem 4.3. By Lemma 4.5.1 this implies $\mathbf{X}_n\left(\lfloor t \cdot \rho \cdot n^3 \rfloor\right) \xrightarrow{\mathrm{d}} \hat{W}_t$.

Although $\lim_{t\to\infty} W_t$ and $\lim_{t\to 0_+} \hat{W}_t$ have different forms as multigraphons, they are "the same" in the following sense:

$$\forall\, F \in \mathcal{M} : \lim_{t\to\infty} t_=(F, W_t) = \lim_{t\to 0_+} t_=(F, \hat{W}_t) \tag{347}$$

The above identity holds because $D(W, U_i) \sim F_0^{-1}(\hat{U}_i)$ where

$$F_0^{-1}(u) := \min\{x : F_0(x) \geq u\}.$$

The fact that our edge reconnecting dynamics preserves the number of edges corresponds to $\rho \equiv \rho(\hat{W}_t)$:

$$\rho(\hat{W}_t) \stackrel{(327)}{=} \mathbf{E}\big(\, D(\mathbf{X}_{\hat{W}_t}, i) \,\big) = \mathbf{E}\big(\, Z_t \,\big) \stackrel{(317)}{=} \int_0^\infty \sum_{i=0}^\infty \mathbf{p}(i, z \cdot \tau(\alpha, t)) \frac{\kappa + i}{\tau(\alpha, t) + \alpha}\, \mathrm{d}F_0(z) =$$
$$\int_0^\infty \frac{\kappa + z \cdot \tau(\alpha, t)}{\tau(\alpha, t) + \alpha}\, \mathrm{d}F_0(z) = \frac{\kappa + \rho \cdot \tau(\alpha, t)}{\tau(\alpha, t) + \alpha} = \rho$$

If we let $t \to \infty$ in (338) we can see that the stationary distribution of the C.I.R. process solving the SDE (337) has density function $\mathbf{g}(x, \kappa, \frac{\kappa}{\rho})$ (see (307)), thus $\lim_{t\to\infty} \hat{W}_t$ is the multigraphon defined in (310): the "mixing time" of the edge reconnecting Markov chain on $n$ vertices is of order $T \asymp n^3$.

## 4.7 (Edge)-stationarity

In this section we prove Lemma 4.4.1 and Proposition 4.1 using the methods of Section 3.4 of [13]: we relate random adjacency matrices to random urn configurations. We also introduce the notion of *edge-stationarity* of the edge reconnecting model and make some remarks about the relation of edge-stationarity to (346).

### 4.7.1 Random urn configurations

A random labeled multigraph is edge-stationary if its distribution is uniform given the degree sequence (this construction is known as the *configuration model* in the theory of random graphs). In this subsection we define a way of constructing random adjacency matrices using random urn configurations which allows us to show that edge-stationarity is preserved by the edge reconnecting dynamics and construct the stationary distribution of the edge reconnecting model.

Let $n, m \in \mathbb{N}$. An random urn configuration with $2m$ balls of $n$ different colors is a probability distribution on $[n]^{[2m]}$, that is a random function $\Psi : [2m] \to [n]$. If $l \in [2m]$ we say that the $l$'th ball has color $\Psi(l)$. Let $d(\Psi, i) := \sum_{l=1}^{2m} \mathbb{1}[\Psi(l) = i]$ for $i \in [n]$ denote the multiplicity of color $i$ in $\Psi$.

We say that a random urn configuration $\Psi$ is *ball exchangeable* if for all permutations $\nu : [2m] \to [2m]$ we have

$$(\Psi(l))_{l=1}^{2m} \sim (\Psi(\nu(l)))_{l=1}^{2m}.$$

$\Psi$ is ball exchangeable if and only if the following property holds: conditioned on the value of the *type vector* $(d(\Psi, i))_{i=1}^{n}$, the distribution of $\Psi$ is uniform on the elements of $[n]^{[2m]}$ with this particular type vector, more precisely if $\psi \in [n]^{[2m]}$ then

$$\mathbf{P}\big(\, \Psi = \psi \,\big) = \frac{\mathbf{P}\big(\, (d(\Psi, i))_{i=1}^{n} = (d(\psi, i))_{i=1}^{n} \,\big)}{\left( \frac{(2m)!}{\prod_{i=1}^{n} d(\psi, i)!} \right)} \tag{348}$$

We say that $\Psi$ is *color exchangeable* if for all permutations $\tau : [n] \to [n]$ we have

$$(\Psi(l))_{l=1}^{2m} \sim (\tau(\Psi(l)))_{l=1}^{2m}.$$

To a random urn configuration $\Psi$ we assign a random element $\mathbf{X}$ of $\mathcal{A}_n^m$ by defining

$$X(i, j) := \sum_{e=1}^{m} \mathbb{1}[\Psi(2e-1) = i, \Psi(2e) = j] + \mathbb{1}[\Psi(2e-1) = j, \Psi(2e) = i] \tag{349}$$

for all $i, j \in [n]$. In plain words: the colors of the balls correspond to the labels of the vertices and if for any $1 \leq e \leq m$ we see a ball of color $i$ at position $2e - 1$ and a ball of color $j$ at position $2e$ in the urn then we draw an edge between the vertices $i$ and $j$ in the corresponding labeled multigraph (and if $i = j$ then we draw a loop edge at vertex $i$).

With the definition (349) we have $\mathbf{P}\big( d(\mathbf{X}, i) = d(\Psi, i) \big) = 1$. It is easy to see that all probability measures on $\mathcal{A}_n^m$ arise this way.

If $\Psi$ is color exchangeable then $\mathbf{X}$ is vertex exchangeable. All vertex exchangeable probability measures on $\mathcal{A}_n^m$ arise this way.

If $\Psi$ is ball exchangeable then for all $B \in \mathcal{A}_n^m$ we have

$$\mathbf{P}\big( \mathbf{X} = B \big) = \frac{\mathbf{P}\big( \, (d(\mathbf{X}, i))_{i=1}^n = (d(B, i))_{i=1}^n \, \big)}{\left( \frac{(2m)!}{\prod_{i=1}^n d(B,i)!} \right)} \frac{m! 2^{e'(B)}}{\left( \prod_{i<j} B(i,j)! \right) \left( \prod_{i=1}^n \frac{B(i,i)}{2}! \right)} \tag{350}$$

where $e'(B)$ is defined by (308). The first term in (350) is $\mathbf{P}\big( \Psi = \psi \big)$ for some $\psi$ that produces $B$ via (349), the second term is the number of elements of $[n]^{[2m]}$ that produce $B$ via (349).

If (350) holds for a random element $\mathbf{X}$ of $\mathcal{A}_n^m$ then we say that the distribution of $\mathbf{X}$ is *edge stationary*. It is easy to see that all edge-stationary probability distributions on $\mathcal{A}_n^m$ arise from ball exchangeable distributions on $[n]^{[2m]}$ via (349).

Now we define two different dynamics on random urn configurations:

- *The Pólya urn model:* Fix $\kappa \in (0, +\infty)$. Let $\Psi_L$ be a random element of $[n]^{[L]}$. Given $\Psi_L$ we generate a random element of $[n]^{[L+1]}$ which we denote by $\Psi_{L+1}$ in the following way: let $\Psi_{L+1}(l) := \Psi_L(l)$ for all $l \in [L]$ and

$$\forall\, i \in [n] : \quad \mathbf{P}\big( \Psi_{L+1}(L+1) = i \,\big|\, \Psi_L \big) = \frac{d(\Psi_L, i) + \kappa}{L + n\kappa}$$

- *The ball replacement model:* Fix $\kappa \in (0, +\infty)$. Let $\Psi_T$ be a random element of $[n]^{[2m]}$. Given $\Psi_T$ we generate a random element of $[n]^{[2m]}$ which we denote by $\Psi_{T+1}$ in the following way: let $\xi_T$ denote a uniformly chosen element of $[2m]$. For all $l \in [2m] \setminus \xi_T$ let $\Psi_{T+1}(l) := \Psi_T(l)$ and

$$\forall\, i \in [n] : \quad \mathbf{P}\big( \Psi_{T+1}(\xi_T) = i \,\big|\, \Psi_T, \xi_T \big) = \frac{d(\Psi_T, i) + \kappa}{2m + n\kappa} \tag{351}$$

It is well-known that if we start with an empty urn $\Psi_0$ and repeatedly apply the Pólya urn scheme to get $\Psi_L$ for $L = 1, 2, \ldots, 2m$ then the distribution of $\Psi_{2m}$ is of the following form:

$$\forall\, \psi \in [n]^{[2m]} : \quad \mathbf{P}\big( \Psi_{2m} = \psi \big) = \frac{\prod_{i=1}^n \prod_{j=1}^{d(\psi,i)} (\kappa + j - 1)}{\prod_{j=1}^{2m} (\kappa n + j - 1)} \tag{352}$$

Thus the distribution of $\Psi_{2m}$ is ball and color exchangeable.

The ball replacement model is in fact an $[n]^{[2m]}$-valued Markov chain, which is irreducible and aperiodic with unique stationary distribution (352): if we delete the $\xi_T$'th ball from $\Psi_{2m}$, then by ball exchangeability the distribution of the resulting $[n]^{[2m-1]}$-valued random variable is the same as deleting the $2m$'th ball: Pólya-$\Psi_{2m-1}$. Thus replacing the removed $\xi_T$'th ball with a new one according to (351) we get a $[n]^{[2m]}$-valued random variable with Pólya-$\Psi_{2m}$ distribution again by ball exchangeability. It is easy to check that the ball replacement Markov chain is reversible with respect to the Pólya-$\Psi_{2m}$ distribution.

Now consider the ball replacement Markov chain $\Psi_T$, $T = 0, 1, \dots$ with $\Psi_0$ being an arbitrary $[n]^{[2m]}$-valued random variable. If we use the mapping (349) to create $\mathbf{X}(T)$ from $\Psi_T$, then it is easily seen that the resulting $\mathcal{A}_n^m$-valued stochastic process $\mathbf{X}(T)$, $T = 0, 1, \dots$ evolves according to the rules of the edge reconnecting Markov chain defined in Section 4.2. Some consequences of this fact:

- Let $k \leq n$. The $\mathbb{N}_0^{[k]}$-valued stochastic process $(d(\mathbf{X}(T), i))_{i=1}^k$, $T = 0, 1, \dots$ is itself a Markov chain.
- If the distribution of $\Psi_0$ is color exchangeable then $\Psi_T$ is also color exchangeable for all $T$, thus if $\mathbf{X}(0)$ is vertex exchangeable then $\mathbf{X}(T)$ is also vertex exchangeable for all $T$ (we have already seen this in Section 4.5).
- If the distribution of $\Psi_0$ is ball exchangeable then $\Psi_T$ is also ball exchangeable for all $T$, thus if $\mathbf{X}(0)$ is edge stationary then $\mathbf{X}(T)$ is also edge stationary for all $T$ (hence the name "edge stationarity").
- The distribution (352) is stationary for the ball replacement model, thus the image of this distribution under the mapping (349) is the unique stationary distribution of the edge reconnecting model. Lemma 4.4.1 follows from (352) and (350).

Note that the PAG$(n, m)$ (defined in Section 3.4 of [13]) is the random multigraph obtained as the isomorphism class of the image of the random urn configuration (352) under the mapping (349).

### 4.7.2 Limits of (edge)-stationary multigraph sequences

In this subsection we prove Proposition 4.1 in two stages:

First we state and prove Lemma 4.7.1 in which we characterize the limiting multigraphons of convergent edge stationary random multigraph sequences. Given this result the proof of Proposition 4.1 reduces to a limit theorem about the joint distribution of the number of balls with color $1, 2, \dots, k$ in the Pólya urn model.

**Lemma 4.7.1.** *Let $F : [0, +\infty) \to [0, 1]$ denote the distribution function of a positive random variable $Z$:*

$$\lim_{x \to 0_+} F(x) = 0 \qquad \lim_{x \to \infty} F(x) = 1 \tag{353}$$

*Let $F^{-1}(u) := \min\{x : F(x) \geq u\}$. Let $Z_1, Z_2, \dots$ be i.i.d. random variables with $Z_i \sim Z \sim F^{-1}(U_i)$.*

*If $\mathbf{X}_n$ is an $\mathcal{A}_n^{m(n)}$-valued random variable for $n = 1, 2, \dots$, moreover the distribution of $\mathbf{X}_n$ is vertex exchangeable and edge stationary, the asymptotic edge density is*

$$\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho \tag{354}$$

*and for all $k \in \mathbb{N}$ we have*

$$\left(\frac{1}{n}d(\mathbf{X}_n, i)\right)_{i=1}^{k} \xrightarrow{d} (Z_i)_{i=1}^{k} \tag{355}$$

*then $\mathbf{X}_n \xrightarrow{d} W$ where*

$$W(x,y,k) = \begin{cases} \mathbf{p}(k, \frac{F^{-1}(x)F^{-1}(y)}{\rho}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F^{-1}(x)F^{-1}(y)}{2\rho}\right) & \text{if } x = y \end{cases} \tag{356}$$

*Proof.* The infinite random array $\mathbf{X}_W$ (see Definition 4.5.1) can be alternatively defined in the following way: Let $(X_W(i,j))_{i\leq j}$ be conditionally independent given $(Z_i)_{i\in\mathbb{N}}$ with conditional distribution $X_W(i,j) \sim \mathrm{POI}\left(\frac{Z_iZ_j}{\rho}\right)$ if $i < j$ and $\frac{X_W(i,i)}{2} \sim \mathrm{POI}\left(\frac{Z_iZ_i}{2\rho}\right)$.

If $A \in \mathcal{A}_k$ let $A^*$ denote the following modified matrix: $A^*(i,j) := A(i,j)$ if $i \neq j$ but $A^*(i,i) := \frac{A(i,i)}{2}$. Thus $A^*(i,i)$ is the number of loop edges at vertex $i$. Let

$$m_{[k]} := \frac{1}{2}\sum_{i,j} A(i,j).$$

Define

$$\mathbf{p}(A, (z_i)_{i=1}^{k})) := \exp\left(\frac{-1}{2\rho}\left(\sum_{i=1}^{k} z_i\right)^2\right) \cdot \prod_{i\leq j} \frac{1}{A^*(i,j)!} \cdot \prod_{i=1}^{k} (z_i)^{d(A,i)} \cdot \rho^{-m_{[k]}} \cdot 2^{-\sum_{i=1}^{k} A^*(i,i)} \tag{357}$$

By Definition 4.5.1, (356) and (325) we have

$$\mathbf{P}\left(\mathbf{X}_W^{[k]} = A \,\middle|\, (Z_i)_{i=1}^{k}\right) = \prod_{i=1}^{k}\prod_{j=i}^{k} \mathbf{p}\left(A^*(i,j), \frac{Z_i \cdot Z_j}{\rho \cdot (1 + \mathbb{1}[i=j])}\right) = \mathbf{p}(A, (Z_i)_{i=1}^{k})). \tag{358}$$

By Lemma 4.5.1 we only need to show that for all $k \in \mathbb{N}$ we have

$$\forall\, A \in \mathcal{A}_k : \lim_{n\to\infty} \mathbf{P}\left(\mathbf{X}_n^{[k]} = A\right) = \mathbf{P}\left(\mathbf{X}_W^{[k]} = A\right) \tag{359}$$

in order to prove $\mathbf{X}_n \xrightarrow{\mathrm{d}} W$.

Let $\Psi_n$ denote the ball and color exchangeable $[n]^{[2m]}$-valued random variable from which $\mathbf{X}_n$ can be generated via (349). To determine the distribution of $\mathbf{X}_n^{[k]}$ we only need to know the positions of the balls of color $i \in [k]$. We paint the rest of the balls "grey". Conditioned on $(d(\mathbf{X}_n,i))_{i=1}^k = (d_i)_{i=1}^k$, $\Psi_n$ is uniformly distributed on the urn configurations with this type vector. Let

$$d_{[k]} := \sum_{i=1}^k d_i, \qquad z_i := \frac{d_i}{n}.$$

Fix $\varepsilon > 0$ and $A \in \mathcal{A}_k$. We are going to prove that

$$\mathbf{P}\big(\, \mathbf{X}_n^{[k]} = A \,\big|\, (d(\mathbf{X}_n,i))_{i=1}^k = (d_i)_{i=1}^k \,\big) = \tag{360}$$

$$\mathbf{p}(A,(z_i)_{i=1}^k)) + \mathrm{Err}(n,A,(z_i)_{i=1}^k) \tag{361}$$

with $\lim_{n\to\infty} \sup\{\mathrm{Err}(n,A,(z_i)_{i=1}^k) \;:\; i \in [k],\ \varepsilon \le z_i \le \varepsilon^{-1}\} = 0$.

In order to prove this we first give an explicit formula for (360).

The number of grey balls is $2m - d_{[k]}$. The number of all urn configurations with type vector $(d_1,\dots,d_k,2m-d_{[k]})$ is

$$\frac{(2m)!}{\left(\prod_{i=1}^k d_i!\right)\cdot (2m-d_{[k]})!} \tag{362}$$

Let $m_g := m - d_{[k]} + m_{[k]}$ denote the number of edges of the multigraph spanned by grey vertices. The number of urn configurations with type vector $(d_1,\dots,d_k,2m-d_{[k]})$ for which $\mathbf{X}_n^{[k]} = A$ is

$$\frac{m!\cdot 2^{m-m_g-\sum_{i=1}^k A^*(i,i)}}{\left(\prod_{i\le j} A^*(i,j)!\right)\cdot\left(\prod_{i=1}^k (d_i - d(A,i))!\right)\cdot m_g!} \tag{363}$$

Thus (360) $= \frac{(363)}{(362)}$. Our aim is to prove $\frac{(363)}{(362)} =$ (361). After dividing both sides by $\prod_{i\le j}\frac{1}{A^*(i,j)!}\cdot 2^{-\sum_i A^*(i,i)}$ we only need to prove

$$\frac{m!\cdot\left(\prod_{i=1}^k d_i!\right)\cdot\left(2m-d_{[k]}\right)!\cdot 2^{m-m_g}}{\prod_{i=1}^k (d_i-d(A,i))!\cdot m_g!\cdot(2m)!} = \tag{364}$$

$$\exp\left(\frac{1}{2\rho}\left(\sum_{i=1}^k z_i\right)^2\right)\cdot\prod_{i=1}^k (z_i)^{d(A,i)}\cdot\rho^{-m_{[k]}} + \mathrm{Err}_2(n,A,\varepsilon) \tag{365}$$

$$(364) = \left(\prod_{i=1}^{k}\prod_{l=1}^{d(A,i)}(d_i - d(A,i)+l)\right)\cdot\left(\prod_{l=1}^{m-m_g}(m_g+l)\right)\cdot\frac{2^{m-m_g}}{\prod_{l=1}^{d_{[k]}}(2m-d_{[k]}+l)} =$$

$$\left(\prod_{l=1}^{m-m_g}\frac{2m_g+2l}{2m-d_{[k]}+l}\right)\cdot\frac{\prod_{i=1}^{k}\prod_{l=1}^{d(A,i)}(d_i-d(A,i)+l)}{\prod_{l=1}^{m_{[k]}}(2m-m_{[k]}+l)} =$$

$$\left(\prod_{l=1}^{d_{[k]}}\frac{2m-2l}{2m-l}\right)\cdot\frac{\prod_{i=1}^{k}(d_i)^{d(A,i)}}{(2m)^{m_{[k]}}}\cdot\left(1+\frac{1}{n}\mathrm{Err}_3(A,\varepsilon)\right)\overset{(354)}{=}$$

$$\left(\prod_{l=1}^{d_{[k]}}\frac{1-\frac{2l}{\rho n^2}}{1-\frac{l}{\rho n^2}}\right)\cdot\frac{\prod_{i=1}^{k}(n\cdot z_i)^{d(A,i)}}{(n^2\cdot\rho)^{m_{[k]}}}\cdot(1+\mathrm{Err}_4(n,A,\varepsilon)) = (365)$$

Having established (360) = (361) it is easy to prove (359):

$$\left|\mathbf{P}\big(\mathbf{X}_n^{[k]}=A\big)-\mathbf{P}\big(\mathbf{X}_W^{[k]}=A\big)\right| \le \tag{366}$$

$$\left|\mathbf{E}\big(\mathbf{p}\left(A,\left(\frac{1}{n}d(\mathbf{X}_n,i)\right)_{i=1}^{k}\right)\big)-\mathbf{E}\big(\mathbf{p}\left(A,(Z_i)_{i=1}^{k}\right)\big)\right|+ \tag{367}$$

$$\mathrm{Err}(\varepsilon,A,n)+1-\mathbf{P}\big(\forall\, i\in[k]:\ \varepsilon<\frac{1}{n}d(\mathbf{X}_n,i)<\varepsilon^{-1}\big) \tag{368}$$

By the assumption (355) we have

$$\lim_{n\to\infty}(367)=0 \quad\text{ and }\quad \limsup_{n\to\infty}(368)\le 1-(F(\varepsilon^{-1})-F(\varepsilon))^k.$$

Now by (353) the last upper bound goes to 0 as $\varepsilon\to 0$, form which (359) and the statement of the lemma follows. □

Note that the multigraph limit (316) is of form (356), moreover by (346) and (347) we can see that the multigraphon $\lim_{t\to\infty}W_t$ also has a representation of form (356). We might interpret this result as saying that the edge reconnecting model becomes essentially edge stationary on the $\mathcal{O}(n^2)$ timescale.

*Proof of Proposition 4.1.* The distribution (309) arises from the Pólya-$\Psi^n_{2m}$ urn model (352) with $2m$ balls and $n$ colors via (349). The distribution (352) is ball and color exchangeable, so $\mathbf{X}_n$ is vertex exchangeable and edge stationary. If we want to prove Proposition 4.1 then by Lemma 4.7.1 we only need to show that (355) holds for all $k\in\mathbb{N}$ where $(Z_i)_{i\in\mathbb{N}}$ are i.i.d. with density function $\mathbf{g}(x,\kappa,\frac{\kappa}{\rho})$ (see (307)). We may use the method of moments to prove convergence in distribution, since the Gamma distribution is uniquely determined by its moments. Thus we need to show that if $\nu_1,\dots,\nu_k\in\mathbb{N}$ then

$$\lim_{n\to\infty}\mathbf{E}\big(\prod_{i=1}^{k}\left(\frac{1}{n}d(\Psi^n_{2m(n)},i)\right)^{\nu_i}\big)=\mathbf{E}\big(\prod_{i=1}^{k}Z_i^{\nu_i}\big)=\prod_{i=1}^{k}\left(\frac{\rho}{\kappa}\right)^{\nu_i}\cdot\prod_{j=1}^{\nu_i}(\kappa+j-1)\,. \tag{369}$$

Fix $k$ and $\nu_i$, $i \in [k]$. Let $\nu = \sum_{i=1}^{k} \nu_i$ and denote by $\psi$ a particular element of $[k]^{[\nu]}$ with type vector $(\nu_1, \dots, \nu_k)$. By the construction of the Pólya-$\Psi_{2m}^n$ distribution we have

$$\mathbf{P}\big(\forall\, l \in [\nu] : \; \Psi_{2m}^n(l) = \psi(l)\,\big) = \frac{\prod_{i=1}^{k} \prod_{j=1}^{\nu_i} (\kappa + j - 1)}{\prod_{j=1}^{\nu} (\kappa n + j - 1)} = \mathcal{O}(n^{-\nu}) \tag{370}$$

Denote by $\underline{\nu} := \{(i,j) \,:\, i \in [k],\, j \in [\nu_i]\}$. The number of functions $f : \underline{\nu} \to [2m]$ with $|\mathcal{R}(f)| = N$ is $\mathcal{O}((2m(n))^N) = \mathcal{O}(n^{2N})$ if $1 \le N \le \nu$.

$$\lim_{n\to\infty} \mathbf{E}\Big( \prod_{i=1}^{k} \left( \frac{1}{n} d(\Psi_{2m(n)}^n, i) \right)^{\nu_i} \Big) =$$

$$\lim_{n\to\infty} \frac{1}{n^\nu} \sum_{f : \underline{\nu} \to [2m]} \mathbf{P}\big(\forall\, (i,j) \in \underline{\nu} : \; \Psi_{2m(n)}^n(f(i,j)) = i\,\big) =$$

$$\lim_{n\to\infty} \frac{1}{n^\nu} \sum_{f : \underline{\nu} \hookrightarrow [2m]} \mathbf{P}\big(\forall\, (i,j) \in \underline{\nu} : \; \Psi_{2m(n)}^n(f(i,j)) = i\,\big) + \lim_{n\to\infty} \frac{1}{n^\nu} \sum_{N=1}^{\nu-1} \mathcal{O}(n^{2N})\mathcal{O}((n^{-N}) \overset{(*)}{=}$$

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{\nu}(2m(n) - k + 1)}{n^\nu} \mathbf{P}\big(\forall\, l \in [\nu] : \; \Psi_{2m}^n(l) = \psi(l)\,\big) \overset{(354)}{=} \prod_{i=1}^{k} \left(\frac{\rho}{\kappa}\right)^{\nu_i} \cdot \prod_{j=1}^{\nu_i} (\kappa + j - 1)\,.$$

The equation $(*)$ holds true by ball exchangeability. $\square$

## 4.8 Technical lemmas

In this section we state and prove technical lemmas needed for the proof of Theorem 4.2 and Theorem 4.3: Lemma 4.8.1 is a rigorous version of (339), Lemma 4.8.2 and Lemma 4.8.3 make the heuristic bounds $\frac{1}{n} d(\mathbf{X}_n(T), i) = \mathcal{O}(1)$ and $X_n(T, i, j) = \mathcal{O}(1)$ precise. Lemma 4.8.4 is a rigorous version of (342).

### 4.8.1 Convergence of multigraph degrees

For a real-valued nonnegative random variable $X$ define $\mathbf{E}\big(\, X; m\,\big) := \mathbf{E}\big(\, X \cdot \mathbb{1}[X \ge m]\,\big)$. A sequence of real-valued nonnegative random variables $(X_n)_{n=1}^\infty$ is uniformly integrable (see Chapter 13 of [38]) if

$$\lim_{m\to\infty} \max_n \mathbf{E}\big(\, X_n; m\,\big) = 0. \tag{371}$$

We say that a sequence of multigraphons $(W_n)_{n=1}^\infty$ is uniformly integrable if $(X_{W_n}(1,2))_{n=1}^\infty$ is uniformly integrable, or more explicitly

$$\lim_{m\to\infty} \max_n \int_0^1 \int_0^1 \sum_{k=m}^{\infty} k \cdot W_n(x, y, k) \, \mathrm{d}x \, \mathrm{d}y = 0.$$

The following lemma 4.8.1 is a rigorous version of (339).

**Lemma 4.8.1.**

*(i) If $(W_n)_{n=1}^\infty$ is a uniformly integrable sequence of multigraphons and $W_n \to W$ (which is equivalent to $\mathbf{X}_{W_n} \xrightarrow{d} \mathbf{X}_W$) then $\rho(W_n) \to \rho(W)$ and for all $k \in \mathbb{N}$*

$$\left(\mathbf{X}_{W_n}^{[k]}, (D(\mathbf{X}_{W_n}, i))_{i=1}^k\right) \xrightarrow{d} \left(\mathbf{X}_W^{[k]}, (D(\mathbf{X}_W, i))_{i=1}^k\right) \tag{372}$$

*(ii) If $B_n \in \mathcal{A}_n^{m(n)}$ for each $n$, and if we define the vertex exchangeable array $\mathbf{X}_n$ by (321), assume $B_n \to W$ (which is equivalent to $\mathbf{X}_n \xrightarrow{d} \mathbf{X}_W$ by Lemma 4.5.1 and the last paragraph of Section 4.5) and that the sequences $(X_n(1,1))_{n=1}^\infty$ and $(X_n(1,2))_{n=1}^\infty$ are uniformly integrable then $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho(W)$ and for all $k \in \mathbb{N}$*

$$\left(\mathbf{X}_n^{[k]}, \left(\frac{1}{n} d(\mathbf{X}_n, i)\right)_{i=1}^k\right) \xrightarrow{d} \left(\mathbf{X}_W^{[k]}, (D(\mathbf{X}_W, i))_{i=1}^k\right). \tag{373}$$

*Proof.*
(i): By the definition of convergence of countable random adjacency matrices (see Chapter 4.5)

$$\mathbf{X}_{W_n} \xrightarrow{\mathrm{d}} \mathbf{X}_W \quad \Longleftrightarrow \quad \forall\, k: \; \mathbf{X}_{W_n}^{[k]} \xrightarrow{\mathrm{d}} \mathbf{X}_W^{[k]}.$$

We first prove that if $\mathbf{X}_{W_n} \xrightarrow{\mathrm{d}} \mathbf{X}_W$ and $\mathbf{P}\big( X_{W_n}(i,j) \le m \big) \equiv 1$ for some $m \in \mathbb{N}$ then (372) holds. To make our notation simpler we denote

$$X_n(i,j) := X_{W_n}(i,j), \quad D_n(i) := D(\mathbf{X}_{W_n}, i), \quad X(i,j) := X_W(i,j), \quad D(i) := D(\mathbf{X}_W, i)$$

By the method of moments we only need to show that for all $\mu_{i,j} \in \mathbb{N}_0$, $1 \le i \le j \le k$ and $\nu_i \in \mathbb{N}_0$, $1 \le i \le k$ we have

$$\lim_{n\to\infty} \mathbf{E}\big( \prod_{i \le j \le k} X_n(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k D_n(i)^{\nu_i} \big) = \mathbf{E}\big( \prod_{i \le j \le k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k D(i)^{\nu_i} \big). \tag{374}$$

For every $i \in [k]$ let $J(i) \subseteq \mathbb{N}$ such that $|J(i)| = \nu_i$, $J(i) \cap J(i') = \emptyset$ if $i \ne i'$ and $J(i) \cap [k] = \emptyset$. In order to prove (374) we first show

$$\mathbf{E}\big( \prod_{i \le j \le k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k D(i)^{\nu_i} \big) \overset{(327),(328)}{=} \mathbf{E}\big( \prod_{i \le j \le k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k \prod_{j \in J(i)} \mathbf{E}\big( X(i,j) \,|\, U_i \big) \big)$$

$$\overset{(*)}{=} \mathbf{E}\big( \prod_{i \le j \le k} X(i,j)^{\mu_{i,j}} \cdot \prod_{i=1}^k \prod_{j \in J(i)} X(i,j) \big) \tag{375}$$

It is straightforward to derive equation $(*)$ from the fact that $(U_i)_{i=1}^\infty$ are i.i.d and that $(X(i,j))_{i \le j \in \mathbb{N}}$ are conditionally independent given $(U_i)_{i=1}^\infty$ (see Definition 4.5.1). Now

(374) easily follows from $\mathbf{X}_n \xrightarrow{\text{d}} \mathbf{X}$ and $\mathbf{P}\big( X_n(i,j) \le m \big) \equiv 1$ if we rewrite both the l.h.s and the r.h.s. of (374) in the form corresponding to the r.h.s. of (375).

Having established (372) under the condition $\mathbf{P}\big( X_n(i,j) \le m \big) \equiv 1$ we now prove the statement of the lemma. If we define $X^m(i,j) := \min\{X_W(i,j), m\}$, then for each $m \in \mathbb{N}$ we have $\mathbf{X}_n^m \xrightarrow{\text{d}} \mathbf{X}^m$ from which

$$\left(\mathbf{X}_n^{m,[k]}, (D(\mathbf{X}_n^m, i))_{i=1}^k\right) \xrightarrow{\text{d}} \left(\mathbf{X}^{m,[k]}, (D(\mathbf{X}^m, i))_{i=1}^k\right)$$

follows by the previous argument. By uniform integrability for every $\varepsilon > 0$ there is an $m$ such that for all $n$ we have

$$\mathbf{E}\big( D(\mathbf{X}_n, i) - D(\mathbf{X}_n^m, i) \big) = \mathbf{E}\big( X(i,i+1) - \min\{X(i,i+1), m\} \big) \le \frac{\varepsilon}{3}.$$

It follows from Fatou's lemma that $\mathbf{E}\big( D(\mathbf{X}, i) - D(\mathbf{X}^m, i) \big) \le \frac{\varepsilon}{3}$ also holds.

In order to prove (372) we only need to check

$$\lim_{n\to\infty} f\left(\mathbf{X}_n^{[k]}, (D(\mathbf{X}_n, i))_{i=1}^k\right) = f\left(\mathbf{X}^{[k]}, (D(\mathbf{X}, i))_{i=1}^k\right) \tag{376}$$

for any bounded and continuous $f : \mathcal{A}_k \times [0, +\infty)^k \to \mathbb{R}$. This can be easily proved using the $\varepsilon/3$-argument.

$\rho(W_n) \to \rho(W)$ follows from (327) and the fact that $X_n(1,2) \xrightarrow{\text{d}} X(1,2)$ and the uniform integrability of $(X_n(1,2))_{n=1}^\infty$ together imply $\mathbf{E}\big( X_n(1,2) \big) \to \mathbf{E}\big( X(1,2) \big)$.

Now we prove (ii). Given $B_n$ we define $\mathbf{X}_n$ by (321), thus $d(\mathbf{X}_n, i) = d(B_n, \pi(i))$. Define $W_n$ by (297), thus $W_n \to W$ and by (302) and (328) we have

$$X_{W_n}(i,j) = B\left(\lceil nU_i \rceil, \lceil nU_j \rceil\right), \qquad D(\mathbf{X}_{W_n}, i) = \frac{1}{n} d(B_n, \lceil nU_i \rceil).$$

The uniform integrability of $(X_{W_n}(1,2))$ easily follows from the uniform integrability of $(X_n(1,2))_{n=1}^\infty$ and $(X_n(1,1))_{n=1}^\infty$ (which is assumed in the statement of (ii)): if we define the random variable $\xi_n$ by $\mathbf{P}\big( \xi_n = 1 \big) = \frac{1}{n}$ and $\mathbf{P}\big( \xi_n = 2 \big) = \frac{n-1}{n}$ then $X_{W_n}(1,2) \sim X_n(1, \xi_n)$. From (i) it follows that

$$\left( \left(B\left(\lceil nU_i \rceil, \lceil nU_j \rceil\right)\right)_{i,j=1}^k, \left(\frac{1}{n} d(B_n, \lceil nU_i \rceil)\right)_{i=1}^k \right) \xrightarrow{\text{d}} \left(\mathbf{X}_W^{[k]}, (D(\mathbf{X}_W, i))_{i=1}^k\right). \tag{377}$$

The distribution of $\left( (B(\pi(i), \pi(j)))_{i,j=1}^k, \left(\frac{1}{n} d(B_n, \pi(i))\right)_{i=1}^k \right)$ is the same as the conditional distribution of $\left( \left(B\left(\lceil nU_i \rceil, \lceil nU_j \rceil\right)\right)_{i,j=1}^k, \left(\frac{1}{n} d(B_n, \lceil nU_i \rceil)\right)_{i=1}^k \right)$ under the condition $\{\forall\, i \neq j \in [k] : \lceil nU_i \rceil \neq \lceil nU_j \rceil\}$.

It is easy to check that if $A$ is an event and $Y$ is a bounded random variable on the same probability space then

$$\big| \mathbf{E}\big( Y \big) - \mathbf{E}\big( Y \mid A \big) \big| \le 2 \, ||Y||_\infty \, (1 - \mathbf{P}\big( A \big)). \tag{378}$$

In order to prove (373) we only need to check that for all bounded and continuous functions $f : \mathcal{A}_k \times [0,+\infty)^k \to \mathbb{R}$ we have

$$\lim_{n\to\infty} f\left( (B(\pi(i),\pi(j)))_{i,j=1}^k , \left( \frac{1}{n} d(B_n,\pi(i)) \right)_{i=1}^k \right) = f\left( \mathbf{X}_W^{[k]}, (D(\mathbf{X}_W,i))_{i=1}^k \right). \quad (379)$$

From (377) we get (379) if we choose $A = \{\forall\, i \neq j \in [k] : \lceil nU_i \rceil \neq \lceil nU_j \rceil\}$ in (378). □

### 4.8.2 Bounds on multiple edges and degrees

In this subsection we state and prove the Lemmas 4.8.2, 4.8.3, 4.8.4 and 4.8.5.

If we replace the initial matrix $B_n$ with its vertex exchangeable version (321) then the technical condition (311) becomes

$$\exists\, \lambda > 0,\ C < +\infty\ \ \forall n\, \forall i,j \in [n] : \quad \mathbf{E}\big( e^{\lambda X_n(0,i,j)} \big) \leq C \quad (380)$$

It is easy to see that (380) implies that the sequences $(X_n(0,i,j))_{n=1}^\infty$ are uniformly integrable, thus by Lemma 4.8.1 (ii) we have $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho(W) =: \rho$.

The following lemma makes heuristic bound $\frac{1}{n} d(\mathbf{X}_n(T),i) = \mathcal{O}(1)$ precise.

**Lemma 4.8.2.** *Let us fix $\kappa, \rho \in (0,+\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0,1,\dots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1,2,\dots$ satisfying $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho$ and (380) for some $\lambda < \frac{\kappa}{\rho}$.*

*Then there exists an $n' \in \mathbb{N}$ such that for every $t, z \in [0,+\infty)$ and $n \geq n'$ we have*

$$\mathbf{P}\big( \max_{0 \leq T \leq 2mnt} \frac{1}{n} d(\mathbf{X}_n(T),i) \geq z \big) \leq C \cdot e^{2\lambda\kappa t} \cdot e^{-\lambda z} \quad (381)$$

*with the $C$ of (380).*

The following lemma is makes the heuristic bound $X_n(T,i,j) = \mathcal{O}(1)$ rigorous.

**Lemma 4.8.3.** *Let us fix $\kappa, \rho \in (0,+\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0,1,\dots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1,2,\dots$ satisfying $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho$ and (380) for some $\lambda < \frac{\kappa}{\rho}$.*

*Then for every $p > 1$ and $t \in [0,+\infty)$ there exists a $C' = C'(\kappa,\rho,\lambda,C,p,t)$ such that for all $n \in \mathbb{N}$, $i,j \in [n]$ and $T \leq 2mnt$ we have $\mathbf{E}\big( X_n(T,i,j)^p \big) \leq C'$.*

Now we formulate a rigorous version of (342).

**Lemma 4.8.4.** *Let us fix $\kappa, \rho \in (0,+\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0,1,\dots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1,2,\dots$ satisfying $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho$ and (380) for some $\lambda < \frac{\kappa}{\rho}$.*

*There exists a constant $C'' = C''(\kappa, \rho, \lambda, C, t)$ such that for all $n \in \mathbb{N}$, all $T_1 \leq T_2 \leq t \cdot n^3$ and $i \in [n]$ we have*

$$\mathbf{E}\Big( \left(\frac{1}{n} d(\mathbf{X}_n(T_2), i) - \frac{1}{n} d(\mathbf{X}_n(T_1), i)\right)^2 \Big) \leq \frac{C'' \cdot (T_2 - T_1)}{n^3} \tag{382}$$

We state a lemma about the speed of convergence of the M/M/$\infty$-queue to its stationary distribution.

**Lemma 4.8.5.** *Let $Y_t$ be an $\mathbb{N}_0$-valued continuous-time Markov chain with infinitesimal jump rates* (333),(334),(335) *and initial state $h \in \mathbb{N}_0$. Then for all $t \geq 0$ and $l \in \mathbb{N}_0$ we have*

$$\left| \mathbf{P}\big( Y_t = l \big) - \lim_{s \to \infty} \mathbf{P}\big( Y_s = l \big) \right| \leq e^{-t} \cdot (h + \mu) \tag{383}$$

Now we prove the lemmas stated above.

*Proof of Lemma 4.8.2.* Fix $i \in [n]$ and denote

$$d(T) := d(\mathbf{X}_n(T), i) \qquad D(T) := \frac{1}{n} d(\mathbf{X}_n(T), i).$$

Denote by $(\mathcal{F}_T)_{0 \leq T}$ the natural filtration generated by the process.

If $a(T) := \mathbf{E}\big( e^{\lambda \cdot (D(T+1) - D(T))} - 1 \,\big|\, \mathcal{F}_T \big)$ then $M(T) := e^{\lambda D(T)} \prod_{l=0}^{T-1} (1 + a(l))^{-1}$ is a nonnegative martingale. By the submartingale inequality we have

$$\mathbf{P}\Big( \max_{0 \leq T \leq T'} M(T) \geq x \Big) \leq \frac{\mathbf{E}\big( e^{\lambda D(0)} \big)}{x} \tag{384}$$

$$\max_{0 \leq T \leq T'} M(T) < x \qquad \Longrightarrow \qquad \forall\, T \leq T' :\; e^{\lambda D(T)} \leq x \exp\left( \sum_{l=0}^{T-1} a(l) \right) \tag{385}$$

Now we give an upper bound on $a(T)$. Using

$$D(T+1) = D(T) + \frac{1}{n} \mathbb{1}[\mathcal{V}_{new}(T) = i] - \frac{1}{n} \mathbb{1}[\mathcal{V}_{old}(T) = i], \tag{386}$$

(287), (290) and the fact that $\mathcal{V}_{new}(T)$ and $\mathcal{V}_{old}(T)$ are conditionally independent given $\mathcal{F}_T$ we get

$$\begin{aligned} a(T) = \left( 1 + \frac{d(T) + \kappa}{2m + n\kappa} (e^{\frac{\lambda}{n}} - 1) \right) \left( 1 + \frac{d(T)}{2m} (e^{-\frac{\lambda}{n}} - 1) \right) - 1 \leq \\ \frac{d(T) + \kappa}{2m + n\kappa} \left( \frac{\lambda}{n} + \frac{1}{2} e^{\frac{\lambda}{n}} \frac{\lambda^2}{n^2} \right) + \frac{d(T)}{2m} \left( -\frac{\lambda}{n} + \frac{1}{2} e^{\frac{\lambda}{n}} \frac{\lambda^2}{n^2} \right) = \\ \frac{\lambda}{n} \frac{1}{4m^2 + n2m\kappa} \left( d(T) \cdot \left( e^{\frac{\lambda}{n}} \lambda \cdot \left( \frac{2m}{n} + \frac{1}{2}\kappa \right) - n\kappa \right) + 2m\kappa \cdot \left( 1 + \frac{1}{2} e^{\frac{\lambda}{n}} \frac{\lambda}{n} \right) \right) \end{aligned} \tag{387}$$

Now $\lambda < \frac{\kappa}{\rho}$ and $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho$, thus if $n$ is big enough then $\lambda < e^{-\frac{\lambda}{n}} \cdot \frac{\kappa}{\rho + \frac{1}{2}\frac{\kappa}{n}}$, which implies that the coefficient of $d(T)$ is negative in the r.h.s. of (387), thus

$$a(T) \leq \frac{1}{2mn} \lambda\kappa \frac{1 + \frac{1}{2}\frac{\lambda}{n} e^{\frac{\lambda}{n}}}{1 + \frac{n\kappa}{2m}} \leq \frac{1}{2mn} 2\lambda\kappa. \tag{388}$$

From (384), (385) and (388) it follows that

$$\mathbf{P}\big( \max_{0 \leq T \leq 2mnt} e^{\lambda D(T)} \geq x \exp(2\lambda\kappa t) \big) \leq \frac{\mathbf{E}\big( e^{\lambda D(0)} \big)}{x}$$

Substituting $x = \exp(-2\kappa\lambda t)\exp(\lambda z)$ and using

$$\mathbf{E}\big( e^{\lambda D(0)} \big) = \mathbf{E}\big( \exp\left( \frac{1}{n} \sum_{j=1}^{n} \lambda X(0,i,j) \right) \big) \leq \mathbf{E}\big( \frac{1}{n} \sum_{j=1}^{n} \exp(\lambda X(0,i,j)) \big) \leq C$$

we arrive at (381). □

*Proof of Lemma 4.8.3.* Fix $n$ and $i, j \in [n]$. We only prove the statement of the lemma if $i \neq j$, the proof of the diagonal case is similar. Denote by

$$X(T) := X_n(T,i,j), \qquad d(T,i) := d(\mathbf{X}_n(T), i), \qquad D(T,i) = \frac{1}{n} d(T,i).$$

Using (291) we get

$$\begin{aligned} &\mathbf{P}\big( X(T+1) = X(T) + 1 \,\big|\, \mathcal{F}_T \big) = \\ &\quad \frac{d(T,i)+\kappa}{2m+n\kappa} \left( \frac{d(T,j)}{2m} \left( 1 - \frac{X(T)}{d(T,j)} \right) \right) + \frac{d(T,j)+\kappa}{2m+n\kappa} \left( \frac{d(T,i)}{2m} \left( 1 - \frac{X(T)}{d(T,i)} \right) \right) \end{aligned} \tag{389}$$

$$\mathbf{P}\big( X(T+1) = X(T) - 1 \,\big|\, \mathcal{F}_T \big) = \frac{X(T)}{2m} \left( 1 - \frac{d(T,i)+\kappa}{2m+n\kappa} \right) + \frac{X(T)}{2m} \left( 1 - \frac{d(T,j)+\kappa}{2m+n\kappa} \right) \tag{390}$$

From this it is straightforward to derive

$$\begin{aligned} &\mathbf{E}\big( e^{\lambda X(T+1)} - e^{\lambda X(T)} \,\big|\, \mathcal{F}_T \big) \leq \\ &e^{\lambda X(T)} \left( (e^{\lambda} - 1) \left( \frac{d(T,i)+\kappa}{2m+n\kappa} \frac{d(T,j)}{2m} + \frac{d(T,j)+\kappa}{2m+n\kappa} \frac{d(T,i)}{2m} \right) + (e^{-\lambda} - 1) \frac{X(T)}{m} \right) \end{aligned} \tag{391}$$

Define the stopping time

$$\tau_y := \min \left\{ T \,:\, \frac{d(T,i)+\kappa}{2m+n\kappa} \frac{d(T,j)}{2m} + \frac{d(T,j)+\kappa}{2m+n\kappa} \frac{d(T,i)}{2m} > \frac{y}{m} \right\}$$

and $X_y(T) := X(T)\mathbb{1}[\tau_y > T]$. Now we prove that for all $T \in \mathbb{N}$

$$\mathbf{E}\big(\, e^{\lambda X_y(T)} \,\big) \leq \max\left\{ C \,,\, \exp(y\lambda e^{\lambda}) \left(1 + \frac{(e^{\lambda}-1)y}{m}\right)\right\}. \tag{392}$$

It is straightforward to check that

$$\mathbf{E}\big(\, e^{\lambda X_y(T+1)} - e^{\lambda X_y(T)} \,\big|\, \mathcal{F}_T \,\big) \leq e^{\lambda X_y(T)} \left( (e^{\lambda}-1)\frac{y}{m} + (e^{-\lambda}-1)\frac{X_y(T)}{m} \right). \tag{393}$$

If we denote $E(T) := \mathbf{E}\big(\, e^{\lambda X_y(T)} \,\big)$, take the expectation of (393) and use Jensen's inequality then we get

$$E(T+1) - E(T) \leq \frac{E(T)}{m} \left( (e^{\lambda}-1)y + (e^{-\lambda}-1)\frac{\log(E(T))}{\lambda} \right). \tag{394}$$

We prove (392) by induction. For $T = 0$ we use (380). If $E(T) > \exp(y\lambda e^{\lambda})$, then by (394) $E(T+1) < E(T)$ and if $E(T) \leq \exp(y\lambda e^{\lambda})$ then

$$E(T+1) \leq E(T) + \exp(y\lambda e^{\lambda})\frac{(e^{\lambda}-1)y}{m} \leq \exp(y\lambda e^{\lambda}) \left(1 + \frac{(e^{\lambda}-1)y}{m}\right).$$

Having established (392) we prove the statement of the lemma by showing that $\mathbf{E}\big(\, X(T)^p \,\big) \leq 1 + \int_1^\infty \mathbf{P}\big(\, X(T)^p \geq x \,\big)\, \mathrm{d}x < +\infty$.

$$\mathbf{P}\big(\, X(T)^p \geq x \,\big) \leq \mathbf{P}\big(\, X_y(T)^p \geq x \,\big) + \mathbf{P}\big(\, X(T) \neq X_y(T) \,\big) \leq \frac{\mathbf{E}\big(\, e^{\lambda X_y(T)} \,\big)}{\exp(\lambda x^{1/p})} + \mathbf{P}\big(\, \tau_y > T \,\big) \overset{(392)}{\leq}$$
$$\frac{C_1 \exp(C_2 y)}{\exp(\lambda x^{1/p})} + \mathbf{P}\big(\, \max_{T \leq 2nmt} D(T,i)D(T,j) > C_3 y \,\big) \overset{(381)}{\leq} C_1 \exp(C_2 y - \lambda x^{1/p}) + C_4 e^{-C_5\sqrt{y}}$$

Now choosing $y = x^{1/2p}$ we indeed get $\int_1^\infty \mathbf{P}\big(\, X(T)^p \geq x \,\big)\, \mathrm{d}x < +\infty$. □

*Proof of Lemma 4.8.4.* Fix $i \in [n]$. We use the notation $D(T) = \frac{1}{n} d(\mathbf{X}_n(T), i)$. We say that $a_n = \mathcal{O}(b_n)$ if there exists a constant $c$ depending only on $\kappa$,$\rho$,$\lambda$,$C$ and $t$ such that $a_n \leq c \cdot b_n$ for all $n \in \mathbb{N}$. It follows from (381) that

$$\forall\, T \leq t \cdot n^3 : \;\; \mathbf{E}\big(\, D(T) \,\big) = \mathcal{O}(1) \qquad \forall\, T, T' \leq t \cdot n^3 : \;\; \mathbf{E}\big(\, D(T)D(T') \,\big) = \mathcal{O}(1). \tag{395}$$

Using (386), (287), (290) and the fact that $\mathcal{V}_{new}(T)$ and $\mathcal{V}_{old}(T)$ are conditionally independent given $\mathcal{F}_T$ we get

$$\mathbf{E}\big(\, D(T+1) - D(T) \,\big|\, \mathcal{F}_T \,\big) = \frac{D(T) + \frac{\kappa}{n}}{2m + n\kappa} - \frac{D(T)}{2m} = \frac{\kappa}{2mn + n^2\kappa} - \frac{n\kappa D(T)}{4m^2 + 2mn\kappa}, \tag{396}$$

$$\mathbf{E}\big( (D(T+1)-D(T))^2 \,\big|\, \mathcal{F}_T \big) = \frac{1}{n^2}\left( \frac{nD(T)+\kappa}{2m+n\kappa} + \frac{nD(T)}{2m} - 2\frac{nD(T)+\kappa}{2m+n\kappa}\frac{nD(T)}{2m} \right). \tag{397}$$

We prove (382) by induction on $T_2 - T_1$.

$$\mathbf{E}\big( (D(T_2+1)-D(T_1))^2 \big) = \mathbf{E}\big( (D(T_2)-D(T_1))^2 \big) +$$
$$2\mathbf{E}\big( \mathbf{E}\big( D(T_2+1)-D(T_2) \,\big|\, \mathcal{F}_{T_2} \big)(D(T_2)-D(T_1)) \big) + \mathbf{E}\big( (D(T_2+1)-D(T_2))^2 \big) \overset{(395)}{=}$$
$$\mathbf{E}\big( (D(T_2)-D(T_1))^2 \big) + \mathcal{O}\left(\frac{1}{n^3}\right)$$

□

*Proof of Lemma 4.8.5.* From (336) it easily follows that the distribution of $Y_s$ converges to $\mathrm{POI}(\mu)$ as $s \to \infty$. Let

$$Y_t^{bin} \sim \mathrm{BIN}(h, e^{-t}), \quad Y_t^{poi} \sim \mathrm{POI}((1-e^{-t})\mu), \quad Y_t^{\infty} \sim \mathrm{POI}(e^{-t}\mu)$$

be mutually independent random variables. By (336) we have $Y_t^{bin} + Y_t^{poi} \sim Y_t$ and $Y_t^{poi} + Y_t^{\infty} \sim \mathrm{POI}(\mu)$.

$$\left| \mathbf{P}\big( Y_t = l \big) - \lim_{s\to\infty} \mathbf{P}\big( Y_s = l \big) \right| = \left| \mathbf{P}\big( Y_t^{bin} + Y_t^{poi} = l \big) - \mathbf{P}\big( Y_t^{poi} + Y_t^{\infty} = l \big) \right| \le$$
$$\mathbf{P}\big( Y_t^{bin} + Y_t^{poi} \ne Y_t^{poi} + Y_t^{\infty} \big) \le \mathbf{P}\big( Y_t^{bin} \ne 0 \big) + \mathbf{P}\big( Y_t^{\infty} \ne 0 \big) =$$
$$1 - (1-e^{-t})^h + (1 - \exp(-e^{-t}\mu)) \le e^{-t}\cdot(h+\mu)$$

□

## 4.9 Proof of Theorem 4.2

In this section we make the sketch proof Theorem 4.2 (see Subsection 4.6.1) rigorous by coupling the evolution of multiple edges between the vertices $1 \le i \le j \le k$ to $\binom{k}{2}$ independent M/M/$\infty$-queues.

Recall the notation of the modified adjacency matrix: if $\mathbf{X}$ is a random element of $\mathcal{A}_k$ let $X^*(i,j) := X(i,j)$ if $i \ne j$ and $X^*(i,i) := \frac{1}{2}X(i,i)$.

We assume that the distribution of $\mathbf{X}_n(T)$ is vertex exchangeable (see the paragraph after (322)). We are going to prove (312) using Lemma 4.5.1 and (326): we only need to show that for all $k \in \mathbb{N}$ and $t \ge 0$ we have

$$\mathbf{X}_n^{[k]}\left( \lfloor t \cdot \frac{\rho(W)\cdot n^2}{2} \rfloor \right) \xrightarrow{\mathrm{d}} \mathbf{X}_{W_t}^{[k]} \quad \text{as} \quad n \to \infty. \tag{398}$$

Note that the evolution of $\left( \mathbf{X}_n^{[k]}(T), (d(\mathbf{X}_n(T), i))_{i=1}^k \right)$ is itself a Markov chain under the edge reconnecting dynamics.

We are going to prove (398) by coupling the $\mathcal{A}_k$-valued discrete-time process $\mathbf{X}_n^{[k]}(T)$ to an $\mathcal{A}_k$-valued continuous-time process $\mathbf{Y}_n^{[k]}(t)$ which we define now. The initial states are the same: $\forall\, i,j \in [k]:\ Y_n(0,i,j) = X_n(0,i,j)$. Given $\mathbf{X}_n(0)$, the evolution of $Y_n(t,i,j)$ is a continuous-time Markov process for each $i,j \in [k]$, the entries $(Y_n(t,i,j))_{i \leq j \leq k}$ evolve independently and $Y_n(t,i,j) \equiv Y_n(t,j,i)$. The process $Y_n^*(t,i,j)$ is an M/M/$\infty$-queue (see (333), (334), (335)) with service rate 1 and arrival rate

$$\mu = \mu_{i,j} := \frac{d(\mathbf{X}_n(0),i)d(\mathbf{X}_n(0),j)}{2m(n)\cdot(1+\mathbb{1}[i=j])}. \tag{399}$$

Now we show that for all $t \geq 0$

$$\mathbf{Y}_n^{[k]}(t) \xrightarrow{\ \mathrm{d}\ } \mathbf{X}_{W_t}^{[k]} \quad \text{as} \quad n \to \infty. \tag{400}$$

From the assumptions $\mathbf{X}_n(0) \xrightarrow{\ \mathrm{d}\ } W$, (380) and Lemma 4.8.1 (ii) it follows that

$$\left(\mathbf{Y}_n^{[k]}(0), \left(\frac{1}{n}d(\mathbf{X}_n(0),i)\right)_{i=1}^{k}\right) \xrightarrow{\ \mathrm{d}\ } \left(\mathbf{X}_W^{[k]}, (D(\mathbf{X}_W,i))_{i=1}^k\right). \tag{401}$$

Now (400) easily follows from this, (336), $\lim_{n\to\infty}\frac{2m(n)}{n^2} = \rho(W)$, Definition 4.5.1 and the definition of $W_t$ (see (313) & (314)).

Denote by $\mathcal{D}_n(T,i) := \frac{1}{n}d(\mathbf{X}_n(T),i)$. We are going to construct a coupling (joint realization on the same probability space) of the discrete time $\mathcal{A}_k$-valued Markov chains $\mathbf{X}_n^{[k]}(T)$ and $\mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)$ for $T = 0,1,\dots$ such that for any $\nu < \frac{5}{2}$ we have

$$\lim_{n\to\infty} \mathbf{P}\big(\forall\, T \leq n^{\nu}: \quad \mathbf{X}_n^{[k]}(T) = \mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)\ \big) = 1. \tag{402}$$

Fix $t \in (0,+\infty)$. If $2 < \nu$ and $n$ is large enough then $2t\cdot m(n) < n^{\nu}$. It is easy to see that (402), (400) and $\lim_{n\to\infty}\frac{2m(n)}{n^2} = \rho(W)$ together imply (398).

For $i,j \in [k]$ we define the matrix $E_{i,j} \in \mathcal{A}_k$ by

$$E_{i,j}(i',j') := \mathbb{1}[i=i', j=j'] + \mathbb{1}[i=j', j=i']$$

Fix $n \in \mathbb{N}$. We introduce the events

$$\begin{aligned}
E_X^{\pm}(T,i,j) &:= \left\{\mathbf{X}_n^{[k]}(T+1) = \mathbf{X}_n^{[k]}(T) \pm E_{i,j}\right\}\\
E_Y^{\pm}(T,i,j) &:= \left\{Y_n^{[k]}\left(\frac{T+1}{m}\right) = Y_n^{[k]}\left(\frac{T}{m}\right) \pm E_{i,j}\right\}\\
E_X(T,\emptyset) &:= \left\{\mathbf{X}_n^{[k]}(T+1) = \mathbf{X}_n^{[k]}(T)\right\}\\
E_Y(T,\emptyset) &:= \left\{Y_n^{[k]}\left(\frac{T+1}{m}\right) = Y_n^{[k]}\left(\frac{T}{m}\right)\right\}
\end{aligned}$$

It is straightforward to derive from (336) that there is an absolute constant $\hat{C}$ such that if we define

$$\mathrm{Err}_Y(T) := \frac{\hat{C}}{m^2}\left(1+\sum_{i,j=1}^{k} Y_n\left(\frac{T}{m}, i, j\right) + \mu_{i,j}\right)^2 \tag{403}$$

then

$$\left|\mathbf{P}\big(\, E_Y^{+}(T,i,j)\,\big|\,\mathcal{F}_T\,\big) - \frac{\mu_{i,j}}{m}\right| \le \mathrm{Err}_Y(T) \tag{404}$$

$$\left|\mathbf{P}\big(\, E_Y^{-}(T,i,j)\,\big|\,\mathcal{F}_T\,\big) - \frac{Y^*\left(\frac{T}{m}, i, j\right)}{m}\right| \le \mathrm{Err}_Y(T) \tag{405}$$

$$\left|\mathbf{P}\big(\, E_Y(T,\emptyset)\,\big|\,\mathcal{F}_T\,\big) - 1 + \frac{\sum_{i\le j\le k} Y^*\left(\frac{T}{m}, i, j\right) + \mu_{i,j}}{m}\right| \le \mathrm{Err}_Y(T) \tag{406}$$

From the definition of the edge reconnecting model it follows (similarly to (389) and (390)) that there is a constant $\tilde{C}$ depending only on $\kappa$ and $\rho$ such that if we define

$$\mathrm{Err}_X(T) := \frac{\tilde{C}}{n^3}\left(1+\sum_{i,j=1}^{k} X_n(T,i,j) + \sum_{i=1}^{k}\mathcal{D}_n(T,i)\right)^2 + \sum_{i,j=1}^{k}\frac{1}{m}\left|\frac{d(\mathbf{X}_n(T),i)d(\mathbf{X}_n(T),j)}{2m\cdot(1+\mathbb{1}[i=j])} - \mu_{i,j}\right| \tag{407}$$

then

$$\left|\mathbf{P}\big(\, E_X^{+}(T,i,j)\,\big|\,\mathcal{F}_T\,\big) - \frac{\mu_{i,j}}{m}\right| \le \mathrm{Err}_X(T) \tag{408}$$

$$\left|\mathbf{P}\big(\, E_X^{-}(T,i,j)\,\big|\,\mathcal{F}_T\,\big) - \frac{X^*(T,i,j)}{m}\right| \le \mathrm{Err}_X(T) \tag{409}$$

$$\left|\mathbf{P}\big(\, E_X(T,\emptyset)\,\big|\,\mathcal{F}_T\,\big) - 1 + \frac{\sum_{i\le j\le k} X^*(T,i,j) + \mu_{i,j}}{m}\right| \le \mathrm{Err}_X(T) \tag{410}$$

For any joint realization (coupling) of the discrete time processes $\mathbf{X}_n^{[k]}(T)$ and $\mathbf{Y}_n^{[k]}\left(\frac{T}{m}\right)$, $T = 0, 1, \dots$ define the event

$$E(T) := \left\{ (E_X(T,\emptyset)\cap E_Y(T,\emptyset)) \cup \bigcup_{\epsilon\in\{+,-\}}\ \bigcup_{i\le j\le k} (E_X^{\epsilon}(T,i,j)\cap E_Y^{\epsilon}(T,i,j)) \right\}.$$

For any coupling the inclusion

$$\left\{\mathbf{X}_n^{[k]}(T) = \mathbf{Y}_n^{[k]}\left(\frac{T}{m}\right)\right\} \cap E(T) \subseteq \left\{\mathbf{X}_n^{[k]}(T+1) = \mathbf{Y}_n^{[k]}\left(\frac{T+1}{m}\right)\right\} \tag{411}$$

holds. Let

$$\mathrm{Err}(T) := 2k^2(\mathrm{Err}_X(T) + \mathrm{Err}_Y(T)).$$

It easily follows from (404) & (408) , (405) & (409) and (406) & (410) that there exists a coupling for which

$$\mathbf{P}\big( E(T) \,\big|\, \mathcal{F}_T \big) \geq \mathbb{1}[\mathbf{X}_n^{[k]}(T) = \mathbf{Y}_n^{[k]}\left(\frac{T}{m}\right)] \cdot (1 - \mathrm{Err}(T))$$

Putting this inequality together with (411), multiplying both sides by

$$\mathbb{1}[\forall\, T' \leq T-1: \;\; \mathbf{X}_n^{[k]}(T') = \mathbf{Y}_n^{[k]}\left(\frac{T'}{m}\right)]$$

and taking the expectation of both sides of the inequality we get

$$\mathbf{P}\big(\forall\, T' \leq T+1: \;\; \mathbf{X}_n^{[k]}(T') = \mathbf{Y}_n^{[k]}\left(\frac{T'}{m}\right) \big) \geq$$
$$\mathbf{P}\big(\forall\, T' \leq T: \;\; \mathbf{X}_n^{[k]}(T') = \mathbf{Y}_n^{[k]}\left(\frac{T'}{m}\right) \big) - \mathbf{E}\big(\,\mathrm{Err}(T)\,\big).$$

Thus in order to prove (402) we only need to show

$$\lim_{n\to\infty} \sum_{T=0}^{n^\nu} \mathbf{E}\big(\,\mathrm{Err}(T)\,\big) = 0 \tag{412}$$

First we show that if $T \leq n^\nu$ then $\mathbf{E}\big(\,\mathrm{Err}_X(T)\,\big) = \mathcal{O}\left(n^{-5/2}\right)$. Since $\nu < \frac{5}{2} < 3$, we have $n^\nu \leq nm$ if $n$ is large enough, thus $\mathbf{E}\big(\,\mathcal{D}_n(T,i)^2\,\big) = \mathcal{O}(1)$ and $\mathbf{E}\big(\,X_n(T,i,j)^2\,\big) = \mathcal{O}(1)$ follow from Lemma 4.8.2 and Lemma 4.8.3.

$$\mathbf{E}\big(\,\frac{1}{m}\left|\frac{d(\mathbf{X}_n(T),i)d(\mathbf{X}_n(T),j)}{2m\cdot(1+\mathbb{1}[i=j])} - \mu_{i,j}\right|\,\big) =$$
$$\mathcal{O}\left(\frac{1}{n^2}\mathbf{E}\big(\,|\mathcal{D}_n(T,i)\mathcal{D}_n(T,j) - \mathcal{D}_n(0,i)\mathcal{D}_n(0,j)|\,\big)\right) =$$
$$\frac{1}{n^2}\mathcal{O}\left(\mathbf{E}\big(\,|\mathcal{D}_n(T,i)-\mathcal{D}_n(0,i)|\,\mathcal{D}_n(T,j)\,\big) + \mathbf{E}\big(\,|\mathcal{D}_n(T,j)-\mathcal{D}_n(0,j)|\,\mathcal{D}_n(0,i)\,\big)\right) =$$
$$\frac{1}{n^2}\mathcal{O}\left(\sqrt{\mathbf{E}\big(\,(\mathcal{D}_n(T,i)-\mathcal{D}_n(0,i))^2\,\big)}\sqrt{\mathbf{E}\big(\,\mathcal{D}_n(T,j)^2\,\big)} + \right.$$
$$\left.\sqrt{\mathbf{E}\big(\,(\mathcal{D}_n(T,j)-\mathcal{D}_n(0,j))^2\,\big)}\sqrt{\mathbf{E}\big(\,\mathcal{D}_n(0,i)^2\,\big)}\right) \overset{(382)}{=}$$
$$\mathcal{O}\left(\frac{1}{n^2}\sqrt{\frac{n^2}{n^3}}\right)\mathcal{O}(1) = \mathcal{O}\left(n^{-5/2}\right)$$

Taking the expectation of (407) we indeed get $\mathbf{E}\big(\,\mathrm{Err}_X(T)\,\big) = \mathcal{O}\left(n^{-3}\right) + \mathcal{O}\left(n^{-5/2}\right)$.

Now we show that if $T \le n^\nu$ then $\mathbf{E}\big(\,\mathrm{Err}_Y(T)\,\big) = \mathcal{O}\left(n^{-4}\right)$: The proof of

$$\mathbf{E}\big(\,Y_n\left(\frac{T}{m}, i, j\right)^2\,\big) = \mathcal{O}(1)$$

is similar to that of Lemma 4.8.3, $\mathbf{E}\big(\,\mu_{i,j}^2\,\big) = \mathcal{O}(1)$ follows from Lemma 4.8.2. Taking the expectation of (403) we get $\mathbf{E}\big(\,\mathrm{Err}_Y(T)\,\big) = \mathcal{O}\left(n^{-4}\right)$.

Thus we have $\mathbf{E}\big(\,\mathrm{Err}(T)\,\big) = \mathcal{O}\left(n^{-5/2}\right)$ from which (412), (402), (398), and the proof of Theorem 4.2 follows.

## 4.10 Proof of Theorem 4.3

In this section we make the sketch proof Theorem 4.3 (see Subsection 4.6.2) rigorous in two stages:

In Subsection 4.10.1 we prove that the joint evolution of the (normed, rescaled) degrees of the vertices $1, 2, \dots, k$ behave like independent C.I.R. processes if $1 \ll n$. Given this result we prove (using the results of Section 4.9) that after $n^2 \ll T$ steps the state of the edge reconnecting model is essentially edge stationary in Subsection 4.10.2.

In Subsection 4.10.3 we formulate Proposition 4.2 which states that the edge reconnecting model indeed exhibits aging (see page 97 for the heuristic description). The proof is in fact a byproduct of the proof of Theorem 4.3 and is left to the reader.

### 4.10.1 Evolution of degrees

**Lemma 4.10.1.** *Let us fix $\kappa \in (0, +\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0, 1, \dots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1, 2, \dots$ satisfying and (380).*

*We assume $\mathbf{X}_n(0) \xrightarrow{d} W$ for some multigraphon $W$.*

*Then for all $t \in [0, +\infty)$ and $k \in \mathbb{N}$ we have*

$$\left(\mathcal{D}_n\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor, i\right)\right)_{i \in [k]} \xrightarrow{d} \left(Z_{t,i}\right)_{i \in [k]} \qquad \text{as } n \to \infty \tag{413}$$

*where $\left(Z_{t,i}\right)_{i \in [k]}$ are i.i.d. with distribution function $F_t(x) = \int_0^x f(t, y)\,\mathrm{d}y$ where $f(t, x)$ is defined by (317).*

In order to prove this lemma, we are going to apply a special case of Corollary 2.2 of [25], which we reformulate to fit our needs and notation:

**Theorem 4.5.** *Let $\beta : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that the stochastic differential equation*

$$\mathrm{d}Z_t = \beta(Z_t)\mathrm{d}t + \gamma(Z_t)\,\mathrm{d}B_t \tag{414}$$

*has a unique weak solution with $Z_0 = z_0$ for all $z_0 \in \mathbb{R}$. Let $F_0(x)$ be a probability distribution function on $\mathbb{R}$.*

*Fix $k \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $(\mathcal{D}_n(T,i))_{i\in[k],T\in\mathbb{N}}$ be a discrete time $\mathbb{R}^k$-valued stochastic process adapted to the filtration $(\mathcal{F}_{n,T})_{T\in\mathbb{N}}$. Let*

$$d\mathcal{D}_n(T,i) := \mathcal{D}_n(T+1,i) - \mathcal{D}_n(T,i).$$

*Suppose*

$$(\mathcal{D}_n(0,i))_{i=1}^k \xrightarrow{d} (Z_{0,i})_{i=1}^k \quad \text{as} \quad n \to \infty \tag{415}$$

*where $(Z_{0,i})_{i=1}^k$ are i.i.d. with distribution function $F_0$. Let $m : \mathbb{N} \to \mathbb{N}$ and suppose that for each $t^* \in [0,+\infty)$ and each $1 \le i,j \le k$ we have*

$$\sup_{t\in[0,t^*]} \left| \sum_{T=0}^{\lfloor 2\cdot m(n)\cdot n\cdot t\rfloor} \mathbf{E}\big(\, d\mathcal{D}_n(T,i) \,\big|\, \mathcal{F}_{n,T}\,\big) - \frac{1}{2m(n)\cdot n} \cdot \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \beta(\mathcal{D}_n(T,i)) \right| \xrightarrow{d} 0 \tag{416}$$

$$\sup_{t\in[0,t^*]} \left| \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \mathbf{Cov}\big(\, d\mathcal{D}_n(T,i)\,,\, d\mathcal{D}_n(T,j),\, \big|\, \mathcal{F}_{n,T}\,\big) - \frac{1}{2m(n)\cdot n} \cdot \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \mathbb{1}[i=j]\cdot \gamma^2(\mathcal{D}_n(t,i)) \right| \xrightarrow{d} 0 \tag{417}$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor} \mathbf{E}\big(\, (d\mathcal{D}_n(T,i))^2 \mathbb{1}[|d\mathcal{D}_n(T,i)| > \varepsilon] \,\big|\, \mathcal{F}_{n,T}\,\big) \xrightarrow{d} 0 \text{ for all } \varepsilon > 0 \tag{418}$$

*as $n \to \infty$.*

*Then the distributions of the $\mathbb{R}^k$-valued continuous-time stochastic processes*

$$(\mathcal{D}_n(\lfloor 2m(n)\cdot n\cdot t\rfloor, i))_{i\in[k],t\ge 0}$$

*converge weakly to the distribution of $(Z_{t,i})_{i\in[k],t\ge 0}$ as $n \to \infty$ in the Skorohod space $\mathbb{D}(\mathbb{R}^k)$, where $(Z_{t,i})_{i\in[k],t\ge 0}$ are i.i.d. solutions of* (414)*, or briefly:*

$$(\mathcal{D}_n(\lfloor 2m(n)\cdot n\cdot t\rfloor, i))_{i\in[k],t\ge 0} \xrightarrow{\mathcal{L}} (Z_{t,i})_{i\in[k],t\ge 0} \tag{419}$$

*Proof of Lemma 4.10.1.*

We are going to use Theorem 4.5 to prove that for all $k$ we have (419) where $(Z_{t,i})_{i\in[k],t\ge 0}$ are i.i.d. solutions of (337) with initial distribution functions $\mathbf{P}\big(\, Z_{0,i} \le x\,\big) = F_0(x)$, where $F_0(x)$ is defined as in Theorem 4.3. From this the claim of Lemma 4.10.1 indeed follows, since from (380) and Lemma 4.8.1 (ii) we get $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho(W)$, thus

$$(\mathcal{D}_n(\lfloor 2m(n)\cdot n\cdot t\rfloor, i))_{i\in[k],t\ge 0} - \big(\mathcal{D}_n\big(\lfloor t\cdot\rho(W)\cdot n^3\rfloor, i\big)\big)_{i\in[k],t\ge 0} \xrightarrow{\mathcal{L}} (\,0\,)_{i\in[k],t\ge 0}\,,$$

from which it follows that for each $t \geq 0$ (413) holds where $(Z_{t,i})_{i\in[k],t\geq 0}$ are i.i.d. solutions of (337), and using (338) we get that $(Z_{t,i})_{i\in[k]}$ are i.i.d. with distribution function $F_t(x) = \int_0^x f(t,y)\,\mathrm{d}y$ where $f(t,x)$ is defined by (317).

We need to check that (415), (416), (417) and (418) holds with $\beta(z) = \kappa - \frac{\kappa}{\rho}z$ and $\gamma(z) = \sqrt{2z}$.

From the assumptions $\mathbf{X}_n(0) \xrightarrow{\mathrm{d}} W$, (380) and Lemma 4.8.1 (ii) it follows that

$$(\mathcal{D}_n(0,i))_{i\in[k]} \xrightarrow{\mathrm{d}} (D(\mathbf{X}_W, i))_{i\in[k]},$$

thus by (328) and the definition of $F_0$ in Theorem 4.3 we get that (415) holds.

$$\sup_{t\in[0,t^*]}\left|\sum_{T=0}^{\lfloor 2\cdot m(n)\cdot n\cdot t\rfloor} \mathbf{E}\big(\, d\mathcal{D}_n(T,i) \,\big|\, \mathcal{F}_{n,T}\,\big) - \frac{1}{2m(n)\cdot n}\cdot \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}\left(\kappa - \frac{\kappa}{\rho}\mathcal{D}_n(T,i)\right)\right| \overset{(287),(290),(386)}{\leq}$$
$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor}\left|\left(\frac{\mathcal{D}_n(T,i)+\frac{\kappa}{n}}{2m(n)+n\kappa} - \frac{\mathcal{D}_n(T,i)}{2m(n)}\right) - \frac{1}{2m(n)\cdot n}\left(\kappa - \frac{\kappa}{\rho}\mathcal{D}_n(T,i)\right)\right| =$$
$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor}\frac{1}{2m(n)\cdot n}\left(\left(\mathcal{O}\left(\frac{1}{n}\right) + \left(\frac{\kappa}{\rho} - \frac{\kappa}{\frac{2m(n)}{n^2}}\right)\right)\mathcal{D}_n(T,i) + \mathcal{O}\left(\frac{n}{m(n)}\right)\right) \tag{420}$$

By Lemma 4.8.2 we have $\mathbf{E}\big(\,\mathcal{D}_n(T,i)\,\big) = \mathcal{O}(1)$, thus $\mathbf{E}\big(\,(420)\,\big) \to 0$ as $n\to\infty$ which implies (416).

We prove (417) by treating the cases $i = j$ and $i \neq j$ separately.

Using (287), (290), (386) and the fact that $\mathcal{V}_{new}(T)$ and $\mathcal{V}_{old}(T)$ are conditionally independent given $\mathcal{F}_{n,T}$ we get

$$\sup_{t\in[0,t^*]}\left|\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}\mathbf{D}^2\left(d\mathcal{D}_n(T,i)\,\big|\,\mathcal{F}_{n,T}\right) - \frac{1}{2m(n)\cdot n}\cdot\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}2\mathcal{D}_n(T,i)\right| \leq$$
$$\sum_{T=0}^{\lfloor 2m(n)\cdot nt^*\rfloor}\left|\frac{1}{n^2}\left(\frac{n\mathcal{D}_n(T,i)}{2m}\left(1-\frac{n\mathcal{D}_n(T,i)}{2m}\right)+\right.\right.$$
$$\left.\left.\frac{n\mathcal{D}_n(T,i)+\kappa}{2m+n\kappa}\left(1-\frac{n\mathcal{D}_n(T,i)+\kappa}{2m+n\kappa}\right)\right) - \frac{1}{2m(n)\cdot n}2\mathcal{D}_n(T,i)\right| =$$
$$\sum_{T=0}^{\lfloor 2m(n)\cdot nt^*\rfloor}\frac{1}{2m(n)\cdot n}\left(\mathcal{O}\left(\frac{n}{m(n)}\right)\mathcal{D}_n(T,i)^2 + \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{n}{m(n)}\right)\mathcal{D}_n(T,i)\right) \tag{421}$$

By Lemma 4.8.2 we have $\mathbf{E}\big(\,\mathcal{D}_n(T,i)\,\big) = \mathcal{O}(1)$ and $\mathbf{E}\big(\,\mathcal{D}_n(T,i)^2\,\big) = \mathcal{O}(1)$, thus $\mathbf{E}\big(\,(421)\,\big) \to 0$ as $n\to\infty$ which implies (417) for $i = j$.

$$\sup_{t\in[0,t^*]}\left|\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}\mathbf{Cov}\big(\,d\mathcal{D}_n(T,i)\,,\,d\mathcal{D}_n(T,j),\,\big|\,\mathcal{F}_{n,T}\,\big)\right|\leq$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor}\Bigg|-\left(\frac{\mathcal{D}_n(T,i)}{2m(n)}\cdot\frac{\mathcal{D}_n(T,j)+\frac{\kappa}{n}}{2m(n)+n\kappa}+\frac{\mathcal{D}_n(T,j)}{2m(n)}\cdot\frac{\mathcal{D}_n(T,i)+\frac{\kappa}{n}}{2m(n)+n\kappa}\right)-$$

$$\left(\frac{\mathcal{D}_n(T,i)+\frac{\kappa}{n}}{2m(n)+n\kappa}-\frac{\mathcal{D}_n(T,i)}{2m(n)}\right)\cdot\left(\frac{\mathcal{D}_n(T,j)+\frac{\kappa}{n}}{2m(n)+n\kappa}-\frac{\mathcal{D}_n(T,j)}{2m(n)}\right)\Bigg|=$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor}\frac{1}{2m(n)\cdot n}\left(\mathcal{O}\left(\frac{n}{m(n)}\right)\left(\mathcal{D}_n(T,i)+\mathcal{D}_n(T,j)\right)^2\right)\quad(422)$$

By Lemma 4.8.2 we have $\mathbf{E}\big(\,\mathcal{D}_n(T,i)^2\,\big)=\mathcal{O}(1)$ and $\mathbf{E}\big(\,\mathcal{D}_n(T,j)^2\,\big)=\mathcal{O}(1)$, thus $\mathbf{E}\big(\,(422)\,\big)\to 0$ as $n\to\infty$ which implies (417) for $i\neq j$.

(418) is trivial since $\mathbf{P}\big(\,|d\mathcal{D}_n(T,i)|\leq\frac{1}{n}\,\big)=1$. □

### 4.10.2 Asymptotic edge-stationarity

Similarly to Section 4.9 we assume that the distribution of $\mathbf{X}_n(T)$ is vertex exchangeable. We are going to prove (315) using Lemma 4.5.1 and (326): we only need to show that for all $k\in\mathbb{N}$ and $t>0$ we have

$$\mathbf{X}_n^{[k]}\left(\lfloor t\cdot\rho(W)\cdot n^3\rfloor\right)\xrightarrow{\ \mathrm{d}\ }\mathbf{X}^{[k]}_{\hat{W}_t}.\quad(423)$$

*Proof of Theorem 4.3.* Let $(Z_{t,i})_{i\in[k]}$ be i.i.d. with distribution function $F_t(x)=\int_0^x f(t,y)\,\mathrm{d}y$ where $f(t,x)$ is defined by (317). Recall the definition of $\mathbf{p}(A,(z_i)_{i=1}^k)$ (see (357) and (358)).

In order to prove (423) we only need to check that for all $A\in\mathcal{A}_k$ we have

$$\lim_{n\to\infty}\mathbf{P}\big(\,\mathbf{X}_n^{[k]}\left(\lfloor t\cdot\rho(W)\cdot n^3\rfloor\right)=A\,\big)=\mathbf{E}\big(\,\mathbf{p}(A,(Z_{t,i})_{i=1}^k)\,\big).\quad(424)$$

We (somewhat arbitrarily) fix $2<\nu<\frac{5}{2}$. Let $T_0^n:=\lfloor t\cdot\rho(W)\cdot n^3\rfloor-\lfloor n^\nu\rfloor$. It easily follows from $\nu<\frac{5}{2}<3$, Lemma 4.8.4 and Lemma 4.10.1 that

$$(\mathcal{D}_n(T_0^n,i))_{i\in[k]}\xrightarrow{\ \mathrm{d}\ }(Z_{t,i})_{i\in[k]}\qquad\text{as } n\to\infty.\quad(425)$$

Now we couple $\mathbf{X}_n^{[k]}(T_0^n+T)$ to $\mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)$ in a similar fashion as in Section 4.9: let $\forall\, i,j\in[k]:\ Y_n(0,i,j)=X_n(T_0^n,i,j)$. Given $\mathbf{X}_n(T_0^n)$, the evolution of $Y_n^*(t,i,j)$ is is an M/M/$\infty$-queue with service rate 1 and arrival rate

$$\mu=\mu_{i,j}:=\frac{d(\mathbf{X}_n(T_0^n),i)d(\mathbf{X}_n(T_0^n),j)}{2m(n)\cdot(1+\mathbb{1}[i=j])}=\frac{\mathcal{D}_n(T_0^n,i)\mathcal{D}_n(T_0^n,j)}{\frac{2m(n)}{n^2}\cdot(1+\mathbb{1}[i=j])}.\quad(426)$$

Now we show that
$$\mathbf{Y}_n^{[k]}\left(\frac{\lfloor n^\nu \rfloor}{m(n)}\right) \xrightarrow{\mathrm{d}} \mathbf{X}_{\hat{W}_t}^{[k]} \quad \text{as} \quad n \to \infty. \tag{427}$$
First note that

$$\lim_{n\to\infty}\lim_{s\to\infty}\mathbf{P}\big(\,\mathbf{Y}_n^{[k]}(s)=A\,\big) \overset{(336)}{=} \lim_{n\to\infty}\mathbf{E}\Big(\,\prod_{i=1}^{k}\prod_{j=i}^{k}\mathbf{p}\left(A^*(i,j),\frac{\mathcal{D}_n(T_0^n,i)\mathcal{D}_n(T_0^n,j)}{\frac{2m(n)}{n^2}\cdot(1+\mathbb{1}[i=j])}\right)\,\Big) \overset{(425)}{=}$$
$$\mathbf{E}\Big(\,\prod_{i=1}^{k}\prod_{j=i}^{k}\mathbf{p}\left(A^*(i,j),\frac{Z_{t,i}\cdot Z_{t,j}}{\rho\cdot(1+\mathbb{1}[i=j])}\right)\,\Big)=\mathbf{E}\big(\,\mathbf{p}(A,(Z_{t,i})_{i=1}^k)\,\big) \tag{428}$$

Let $t_n := \frac{\lfloor n^\nu \rfloor}{m(n)}$. $\lim_{n\to\infty} t_n = +\infty$ follows from $2 < \nu$. By Lemma 4.8.5, Lemma 4.8.2 and Lemma 4.8.3 we have

$$\left|\mathbf{P}\big(\,\mathbf{Y}_n^{[k]}(t_n)=A\,\big)-\lim_{s\to\infty}\mathbf{P}\big(\,\mathbf{Y}_n^{[k]}(s)=A\,\big)\right|\le$$
$$\exp(-t_n)\cdot\sum_{i=1}^{k}\sum_{j=i}^{k}\left(\mathbf{E}\big(\,X_n(T_0^n,i,j)\,\big)+\mathbf{E}\Big(\,\frac{\mathcal{D}_n(T_0^n,i)\mathcal{D}_n(T_0^n,j)}{\frac{2m(n)}{n^2}\cdot(1+\mathbb{1}[i=j])}\,\Big)\right)=\exp(-t_n)\mathcal{O}(1). \tag{429}$$

Thus (427) follows from (428) and (429).

Using the proof of (402) we can construct a coupling such that we have

$$\lim_{n\to\infty}\mathbf{P}\Big(\,\forall\, 0\le T\le n^\nu: \quad \mathbf{X}_n^{[k]}(T_0^n+T)=\mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)\,\Big)=1.$$

Now (423) follows from this, $T_0^n + \lfloor n^\nu \rfloor = \lfloor t\cdot\rho(W)\cdot n^3\rfloor$ and (427). □

### 4.10.3 Aging

Finally we show that the edge reconnecting model indeed exhibits aging (see page 97 for the heuristic discussion of this phenomenon). For the precise formulation we need an additional definition:

Let $k \in \mathbb{N}$, $z_1,\dots,z_k \in \mathbb{R}_+$, $M \in \mathbb{N}$, $s_1 < \cdots < s_M \in \mathbb{R}$, $A_1,\dots,A_M \in \mathcal{A}_k$.

For $1 \le i \le j \le k$ let $Y^*(t,i,j)$ be independent, stationary M/M/∞-queues (see page 111 for the definition) with service rate 1 and arrival rate

$$\mu_{i,j} := \frac{z_i\cdot z_j}{\rho\cdot(1+\mathbb{1}[i=j])}. \tag{430}$$

We assume that these stationary stochastic processes are defined for negative values of $t$ as well. Denote by $\mathbf{Y}^{[k]}(t) = \big(Y(t,i,j)\big)_{i,j\in[k]}$ where $Y(t,i,j) \equiv Y(t,j,i)$, $Y(t,i,j) = Y^*(t,i,j)$ if $i \ne j$ and $Y(t,i,i) := 2Y(t,i,i)$.

From (336) it follows that for all $s \in \mathbb{R}$ we have $Y^*(s,i,j) \sim \mathrm{POI}(\mu_{i,j})$.

From (357) and (358) it follows that $\mathbf{P}\big(\,\mathbf{Y}^{[k]}\,(s) = A\,\big) = \mathbf{p}(A, (z_i)_{i=1}^k))$.

Let

$$\mathbf{p}_{M/M/\infty}\big(\,(A_m, s_m)_{m=1}^M\,, (z_i)_{i=1}^k\,\big) := \mathbf{P}\big(\,\forall\, m \in [M] :\; \mathbf{Y}^{[k]}\,(s_m) = A_m\,\big). \tag{431}$$

Note that $\mathbf{p}_{M/M/\infty}\big(\,(A_m, s_m)_{m=1}^M\,, (z_i)_{i=1}^k\,\big) = \mathbf{p}_{M/M/\infty}\big(\,(A_m, s_m + s)_{m=1}^M\,, (z_i)_{i=1}^k\,\big)$ for any $s \in \mathbb{R}$, because $\big(\mathbf{Y}^{[k]}\,(t)\,\big)_{t\in\mathbb{R}}$ is a stationary process. One could easily write down an explicit formula for (431) using (336).

**Proposition 4.2.** *Let us fix $\kappa \in (0, +\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0, 1, \ldots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1, 2, \ldots$. We assume $\mathbf{X}_n(0) \xrightarrow{d} W$ for some multigraphon $W$ and that (380) holds.*

*Then for any $k \in \mathbb{N}$, $M \in \mathbb{N}$, $s_1 < \cdots < s_M \in \mathbb{R}$, $A_1, \ldots, A_M \in \mathcal{A}_k$, $L \in \mathbb{N}$, $0 < t_1 < \cdots < t_L \in (0, +\infty)$ we have*

$$\lim_{n\to\infty} \mathbf{P}\big(\,\forall\, l \in [L], m \in [M] :\; \mathbf{X}_n^{[k]}\left(\lfloor t_l \cdot \rho(W) \cdot n^3 + s_m \cdot \frac{\rho(W)}{2} n^2 \rfloor\right) = A_m\,\big) =$$
$$\mathbf{E}\big(\,\prod_{l=1}^{L} \mathbf{p}_{M/M/\infty}\big(\,(A_m, s_m)_{m=1}^M\,, (Z_i(t_l))_{i=1}^k\,\big)\,\big),$$

*where $(Z_i(t))_{i\in[k],t\geq 0}$ are i.i.d. solutions of (337) with initial distribution functions $\mathbf{P}\big(\,Z_{0,i} \leq x\,\big) = F_0(x)$, where $F_0(x)$ is defined as in Theorem 4.3.*

This shows that the model indeed exhibits aging: If we fix some $t \in (0, \infty)$, and look at the stochastic process $\left(\mathbf{X}_n^{[k]}\left(t \cdot \rho(W) \cdot n^3 + s \cdot \frac{\rho(W)}{2} n^2\right)\right)_{s\in\mathbb{R}}$ for a some large $n$ then we see a stationary process: $\big(\mathbf{Y}^{[k]}\,(s)\,\big)_{s\in\mathbb{R}}$ with random background parameters $(Z_i(t))_{i=1}^k$. Yet for different values of $t$ the values of these background parameters are different.

We leave the proof of Proposition 4.2 to the reader because it is essentially the same as that of (424) in the proof of Theorem 4.3 in Subsection 4.10.2.

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