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Markov property in non-commutative probability

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Contents

Introduction

Markov chains are the simplest mathematical models for random phenomena evolving in time. Their simple structure makes it possible to say a great deal about their behaviour. At the same time, the class of Markov chains is rich enough to serve in many applications, for example in population growth, mathematical genetics, networks of queues, Monte Carlo simulation and in many others. This makes the Markov chains the first and most important examples of random processes. Indeed, the whole of the mathematical study of random processes can be regarded as a generalization in one way or another of the theory of Markov chains. We shall be concerned exclusively with the case where the process can assume only a finite or countable set of states.

A discrete random variable is a function X with values in a finite set $\mathcal{X}$ and its probability mass function is $p(x) = \Pr\{X = x\}$, $x \in \mathcal{X}$. Each $x \in \mathcal{X}$ is called a state and $\mathcal{X}$ is called the state-space. A stochastic process is an indexed sequence of random variables. In general, there can be an arbitrary dependence among these random variables. The process is characterized by the joint probability mass functions $p(x_1, x_2, \ldots, x_n) = \Pr\{(X_1, X_2, \ldots, X_n) = (x_1, x_2, \ldots, x_n)\}$, where $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ for $n = 1, 2, \ldots$. A simple, but important example of a stochastic process with dependence is one in which each random variable depends on the one preceding it and is conditionally independent of all the other preceding random variables. Such a process is said to be Markovian. For the most part we will attend to the case of three random variables, so we say, that the random variables X, Y and Z form a Markov triplet (denoted by $X \to Y \to Z$) if

$$p(x, y, z) = p(x)p(y|x)p(z|y),$$

where

$$p(y|x) = \frac{\Pr\{X = x, Y = y\}}{\Pr\{X = x\}}$$

is the conditional probability. If Z has the interpretation as "future", Y is the "present" and X is the "past", then having the Markov property means that, given the present state, future states are independent of the past states.

In other words, the description of the present state fully captures all the information that could influence the future evolution of the process.

It is natural to investigate the Markov triplets from information theoretical point of view. In 1948 the electric engineer *C. Shannon* published a remarkable pair of papers laying the foundations for the modern theory of information and communication. Perhaps the key step taken by Shannon was to mathematically define the concept of information. As a measure of uncertainty of a random variable he proposed the following formula

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$$

called Shannon entropy. If the log is to the base 2, the entropy is expressed in bits, while if the base is e, the unit of entropy is sometimes called nat. The Shannon entropy has many properties that are in accord with the intuitive notion of what a measure of information should be, for example it helps us to express the dependence among the random variables. One of its basic properties is subadditivity, i.e.

$$H(X,Y) \leq H(X) + H(Y),$$

where $H(X,Y) = -\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x,y) \log p(x,y)$ is the joint entropy of random variables X and Y, and measures our total uncertainty about the pair (X,Y). The equality holds in the subadditivity if and only if X and Y are independent random variables. The other remarkable property of the Shannon entropy is the strong subadditivity:

$$H(X,Y,Z) + H(Y) \leq H(X,Y) + H(Y,Z),$$

with equality if and only if $X \to Y \to Z$, i.e. X, Y and Z form a Markov triplet. It means that Markov triplets are completely characterized by the strong additivity of their Shannon entropy.

At the turn of the twentieth century a series of crises had arisen in the physics. The problem was that the classical theory were predicting absurdities such as the existence of an "ultraviolet catastrophe" involving infinite energies, or electrons spiraling inexorably into the atomic nucleous. The crisis came to a head in the early 1920's and resulted in the creation of the modern theory of quantum mechanics. *J. von Neumann* worked in Göttingen when *W. Heisenberg* gave the first lectures on the subject. Quantum mechanics motivated the creation of new areas in mathematics, the theory of linear operators on Hilbert spaces was certainly such an area. John von Neumann made an effort towards the mathematical foundations and he initiated the study of what are now called von Neumann algebras. With *F.J. Murray*, they made a first classification of such algebras [47]. While the mathematics of classical probability theory was subsumed into classical measure theory by *A.N. Kolmogorov* [34], the quantum or non-commutative probability theory was induced by the quantum theory and was incorporated into the beginnings of non-commutative measure theory by *J. von Neumann* [48].

In this concept, quantization is a process in which classical observables, i.e. real functions on a phase space, are replaced by self-adjoint operators on a Hilbert space $\mathcal{H}$. Similarly, in quantum or non-commutative probability the role of random variables is played by self-adjoint elements affiliated to some C*-algebra $\mathcal{A}$ with unit element $\mathbf{1}$. Probability measures are replaced by states, i.e positive linear functionals ϕ on $\mathcal{A}$ such that $\phi(\mathbf{1}) = 1$. If $\mathcal{A}$ is a non-commutative algebra then we say that $(\mathcal{A}, \phi)$ is an abstract or algebraic non-commutative probability space. This concept means a generalization: as long as one considers a commutative C*-algebra $\mathcal{A}$, quantum probability reduces to classical probability. One usually represents $\mathcal{A}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$ of bounded operators acting on a complex separable Hilbert space $\mathcal{H}$. If the state ϕ is normal, i.e. positive weakly continous normalized linear functional on the von Neumann algebra $\mathcal{B}(\mathcal{H})$, then it is given by $\phi(A) = \mathrm{Tr}\,(\rho A)$, $A \in \mathcal{B}(\mathcal{H})$, for some unique statistical operator ρ acting on $\mathcal{H}$, i.e. $0 < \rho = \rho^* \in \mathcal{B}(\mathcal{H})$ such that $\mathrm{Tr}\,(\rho) = 1$. If $\mathcal{H}$ is finite dimensional ρ is often called density matrix. From the quantum theoretical point of view the selfadjoint elements of $\mathcal{B}(\mathcal{H})$ are identified with physical observables, while state ϕ represents the state of a physical system.

It is a natural question, that how can we generalize the concept of Markovianity in the non-commutative setting which means the framework of C*-algebras, or matrix algebras in the finite dimensional case.

In von Neumann's unifying scheme for classical and quantum probability an important ingredient was missing: conditioning. In order to study nontrivial statistical dependences, in particular to construct Markov processes, this gap had to be filled. The first step was made in 1962 by the most natural quantum generalization of the notion of conditional expectation, by *H. Umegaki* [75], which is relevant for several problems in operator theory and in quantum probability. By a (Umegaki) conditional expectation $E : \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ we mean a norm-one projection of the C*-algebra $\mathcal{A}$ onto the C*-subalgebra $\mathcal{B}$. The map E is automatically a completely positive identity-preserving $\mathcal{B}$-bimodule map by a theorem of *J. Tomiyama* [73]. E is called compatible with a state ϕ if $\phi \circ E = \phi$. Unfortunately Umegaki's notion is not perfect to express the Markovianity, since the states compatible with norm-one projections tend to be trivial in the extremely non-commutative case. Indeed, a state ϕ on $M_n \otimes M_n$ is compatible with an Umegaki conditional expectation onto $M_n \otimes I$ if and only if it is a product state, which means that our random variables are independent. (Here M_n denotes n by n complex matrices.) To avoid this trivial case *L. Accardi* and *A. Frigerio* proposed the following definition in 1978 [7]. Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of unital C*-algebras. A quasi-conditional or generalized conditional expectation w.r.t the given triplet is a completely positive identity-preserving linear map $E : \mathcal{A} \to \mathcal{B}$ such that

$$E(ca) = cE(a), \qquad a \in \mathcal{A}, c \in \mathcal{C}.$$

The notion of non-commutative or quantum Markov chains was also introduced by Accardi in [2, 3]. Quantum Markov chains are defined on non-commutative C*-algebras, in particular on UHF algebras, and they are determined by an initial state and a sequence of unital completely positive maps, called transition maps. Since in the classical case Markov chains can be defined on abelian C*-algebras and are determined by an initial distribution and a sequence of transition matrices, quantum Markov chains can be regarded as the generalization of the classical ones.

In spite of the abstractness of this definition, several improvements have been made in their applications to physical models. In particular a sub-class of Markov chains, also called finitely correlated states, was shown to coincide with the so-called valence bond states introduced in the late 1980's in the context of antiferromagnetic Heisenberg models. The works of *M. Fannes, B. Nachtergaele* and *R.F. Werner* were appreciable to find the ground states of these modells [23]. As another special class of quantum Markov chains, the notion of quantum Markov states was defined in [7]. A quantum Markov state ϕ is determined by an initial state and a sequence of ϕ-preserving quasi-conditional expectations. If we consider a Markov state with three parts we say that it is a short Markov state or Markov triplet. The question raises, whether similarly to the classical case, there is any characterization of Markov states by the entropy quantities?

If ρ is the density matrix of a normal state ϕ, the von Neumann entropy of the state is defined by

$$S(\phi) \equiv S(\rho) = -\mathrm{Tr}\, \rho \log \rho.$$

Similarly to the classical case the von Neumann entropy plays an important role in the investigations of quantum systems's correlations. The von Neumann entropy is subadditive, i.e.

$$S(\phi_{12}) \leq S(\phi_1) + S(\phi_2),$$

where ϕ_{12} is a normal state of the composite system of $\mathcal{A}_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2$, and the equality holds if and only if ϕ_{12} is product of its marginals, i.e. $\phi_{12} = \phi_1 \otimes \phi_2$, which is the non-commutative analogue of the independent random variables. We also have the remarkable strong subadditivity property which was proved by *E. Lieb* and *M.B. Ruskai* in 1973 [38]. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ be subalgebras of $\mathcal{B}(\mathcal{H})$ representing three quantum systems and set $\mathcal{A}_{123} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$, $\mathcal{A}_{12} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{A}_{23} = \mathcal{A}_2 \otimes \mathcal{A}_3$ as their several compositions. For a state ϕ_{123} of $\mathcal{A}_{123}$ we denote its restrictions to $\mathcal{A}_{12}$, $\mathcal{A}_{23}$ and $\mathcal{A}_2$ with ϕ_{12}, ϕ_{23} and ϕ_2, respectively. The strong subadditivity says, that

$$S(\phi_{123}) + S(\phi_2) \leq S(\phi_{12}) + S(\phi_{23}).$$

On the analogy of the classical Markov property it has been shown that the strong subadditivity of the von Neumann entropy is tightly related to the Markov property invented by *L. Accardi.* A state of a three-fold tensor product system is Markovian if and only if it takes the equality of the strong subadditivity of von Neumann entropy, which is referred to as strong additivity of the von Neumann entropy. In other words, a state of a tensor product system is Markovian if and only if the von Neumann entropy increase is constant. The exact structure of a density ρ_{123} with this property was established in 1994 by *P. Hayden, R. Jozsa, D. Petz* and *A. Winter* [29].

Although a pivotal example of quantum composite systems is tensor product of Hilbert spaces, we can see that the definition of Markov property has been given under a very general setting that is not limited to the most familiar case of tensor-product systems. Which means, it does not require in principle any specific algebraic location among systems imbedded in the total system. A very important example from this point of view is the algebra of the Canonical Anticommutation Relation or briefly CAR algebra, that serves as the description of fermion lattice systems.

The quantum-mechanical states of n identical point particles in the configuration space $\mathbb{R}^\nu$ are given by vectors of the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^{\nu n})$. If $\psi \in \mathcal{H}$ is normalized, then

$$\mathrm{d}p(x_1, \dots, x_n) = |\psi(x_1, \dots, x_n)|^2 \mathrm{d}x_1 \dots \mathrm{d}x_n$$

is the probability density for ψ to describe n particles at the infinitesimal neighborhood of the points $x_1, \dots, x_n$. The normalization of ψ corresponds to the normalization of the total probability to unity. But in microscopic physics identical particles are indistinguishable and this is reflected by the symmetry of the probability density under interchange of the particle coordinates. This interchange defines an unitary representation of the permutation group and the symmetry is assured if the ψ transform under a suitable subrepresentation. There are two cases of paramount importance. The first arises when ψ is symmetric under change of coordinates. Particles whose states transform in this manner are called bosons and are said to satisfy Bose-Einstein statistics. The second case corresponds to anti-symmetry of ψ under interchange of each pair of coordinates. The associated particles are called fermions and said to satisfy Fermi-Dirac statistics. The elementary constituents of matter seem to be fermions, while interactions are mediated by bosons. In the case of fermions the anti-symmetry of the wave function had a deep consequence, namely the Pauli principle: It is impossible to create two fermions in the same state. The main qualitative difference between fermions and bosons is the absence of a Pauli principle for the latter particles. There is no bound on the number of particles which can occupy a given state. Mathematically this is reflected by the unboundedness of the so-called Bose annihilation and creation operators. This unboundedness leads to a large number of technical difficulties which are absent for fermions. These problems can be partially avoided by consideration of bounded functions of the annihilation and creation operators. This idea yields the Weyl operators and their algebra, the algebra of Canonical Commutation Relation (briefly CCR algebra). I investigate both systems from the wievpoint of Markovianity.

Returning to the classical context an important case was solved recently: The characterisation of multivariate normal Markov triplets was given [9]. In classical probability, a Gaussian measure leads to a characteristic function which is the exponential of a quadratic form. Its logarithm is therefore a quadratic polynomial, and all correlations beyond the second order vanish. Following this procedure in some non-commuting systems as the CAR algebra or the CCR algebra it is also possible to define the useful concept of the correlation function (cumulants or truncated function in other words), and we can arrive to the analogues of Gaussian distributions, to the so-called quasi-free states. In these states the n-point functions can be computed from the 2-point functions and in one kind of central limit theorem the limiting state is quasi-free. The quasi-free states are quite tractable, for example the von Neumann entropy has an explicit expression. It is a natural to ask: What can we say about the quasi-free Markov triplets? My goal is the discussion of these questions. The dissertation is organized as follows.

In *Chapter 1* I give an overview of the notion of Markov chains in the classical probability theory. I investigate its connection to the strong additivity property of the Shannon entropy and as an example the characterization of the multivariate Gaussian Markov triplets is given.
In *Chapter 2* I review the basics of the non-commutative probability. After the premilinaries as the notion of C*-algebra, states and maps, I investigate the topic of the non-commutative conditional expectation. I overview the

main properties of the von Neumann entropy and I give a new proof for its strong subadditivity with condition for the equality. I attend to the operator monotone functions too.
In *Chapter 3* I turn to the main topic, the Markov property in the non-commutative setting. I follow the historical line and analyze the connections among the concepts of Markovianity. The characterization of the short Markov states in the three-fold tensor product case is also considered.
In *Chapter 4* I summerize the main properties of CAR algebras and their quasi-free states. I attend to the anti-symmetric Fock space and I give some entropy quantities of quasi-free states.
In *Chapter 5* I prove that an even state on a CAR algebra is Markovian if and only if it saturates the strong subadditivity of the von Neumann entropy with equality, and I slightly generalize this theorem. I also give the complete characterization of quasi-free Markov states.
Chapter 6 contains some crucial properties of the Weyl unitaries, the symmetric Fock space, the CCR algebra and quasi-free states. This is written for the sake of completeness, the results are known but not well accessible in the literature. The main point is the von Neumann entropy formula which is well-known for the CCR quasi-free state. I also give another entropy related quantities.
In *Chapter 7* I investigate the quasi-free Markov triplets. A necessary and sufficient condition described in the block matrix approach is obtained: The block matrix should be block diagonal. There are non-trivial Markovian quasi-free states which are not products in the time localization. The existence of such states is interesting, because it is in contrast with the CAR case. However, the first and the third subalgebras are always independent. The relation to classical Gaussian Markov triplets is also described and I show that the minimizer of relative entropy with respect to a quasi-free Markov state has the Markov property.
The new results are published in the papers [33, 61, 62, 63, 64, 65].

Chapter 1

Markovianity in the classical probability

1.1 Classical Markov chains

This section is about a certain sort of classical random process. The characteristic property of this sort of process is that it retains no memory of where it has been in the past. We shall be concerned exclusively with the case where the process can assume only a finite or countable set of states.

Let $\mathcal{X}$ be a countable set. Each $x \in \mathcal{X}$ is called a **state** and $\mathcal{X}$ is called the **state space**. We say that $p(x)$, $(x \in \mathcal{X})$ is a measure on $\mathcal{X}$ if $0 \leq p(x) < \infty$ for all $x \in \mathcal{X}$. If in addition the total mass $\sum_{x \in \mathcal{X}} p(x) = 1$, we call p a distribution. We work throughout with a probability space $(\Omega, \mathcal{F}, p)$. Recall that a random variable X with values in $\mathcal{X}$ is a function $X : \Omega \to \mathcal{X}$. Let X and Y be random variables with values in the sets $\mathcal{X}$ and $\mathcal{Y}$. The following notations will be used:

$$p(x) = \Pr(X = x) \quad \text{and} \quad p(y) = \Pr(Y = y)$$

for the probability distributions of X and Y, respectively,

$$p(x, y) = \Pr(X = x, Y = y)$$

for their joint distribution and

$$p(x|y) = \Pr(X = x|Y = y) = \frac{p(x, y)}{p(y)}$$

for the conditional distribution.

Definition 1.1.1 *Random variables X, Y and Z are said to form a* ***Markov chain*** *(denoted by $X \to Y \to Z$) if the conditional distribution of Z depends only on Y and is conditionally independent of X, i.e.*

$$p(z|y, x) = p(z|y). \tag{1.1}$$

This is equivalent with the following condition for the joint distribution:

$$p(x, y, z) = p(x)p(y|x)p(z|y) \tag{1.2}$$

or similarly

$$p(x, y, z) = p(y)p(x|y)p(z|y). \tag{1.3}$$

Let suppose that $X \to Y \to Z$. Then the computation

$$p(x, z|y) = \frac{p(x, y, z)}{p(y)} = \frac{p(x, y)p(z|y)}{p(y)} = p(x|y)p(z|y)$$

shows that $X \to Y \to Z$ if and only if X and Z are conditionally independent. If Z has the interpretation as "future", Y is the "present" and X is the "past", then having the Markov property means that, given the present state, future state is independent of the past state. In other words, the description of the present state fully captures all the information that could influence the future evolution of the process. We remark that $X \to Y \to Z$ implies $Z \to Y \to X$. We can extend the notion of Markov chain to a countable set of random variables in a natural way, getting a stochastic process. A **stochastic process** is an indexed sequence of random variables. In general, there can be an arbitrary dependence among the random variables. The process is characterized by the joint probability mass functions

$$p(x_1, x_2, \ldots, x_n) \equiv \Pr\{(X_1, X_2, \ldots, X_n) = (x_1, x_2, \ldots, x_n)\},$$

where $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ for all $n = 1, 2, \ldots$. A stochastic process is said to be **stationary** if the joint distribution of any subset of the sequence of random variables is invariant with respect to the shifts in their index, i.e.

$$\Pr\{(X_1 = x_1, \ldots, X_n = x_n)\} = \Pr\{(X_{1+k} = x_1, \ldots, X_{n+k} = x_n)\},$$

for every shift k for all $x_1, x_2, \ldots, x_n \in \mathcal{X}$. A simple example of a stochastic process with dependence is one in which each random variable depends on the one preceding it and is conditionally independent of all the other preceding random variables, i.e. it forms a Markov chain.

Definition 1.1.2 *A discrete stochastic process $X_1, X_2, \ldots$ is said to be a* ***Markov chain*** *or a* ***Markov process*** *if, for $n = 1, 2, \ldots$*

$$\Pr\{(X_{n+1} = x_{n+1}|X_n = x_n, \ldots, X_1 = x_1)\} = \Pr\{(X_{n+1} = x_{n+1}|X_n = x_n)\},$$

for all $x_1, x_2, \ldots, x_n, x_{n+1} \in \mathcal{X}$.

It is obvious that in this case the joint probability mass function can be written as

$$p(x_1, x_2, \ldots, x_n) = p(x_1)p(x_2|x_1)p(x_3|x_2)\ldots p(x_n|x_{n-1}).$$

The next example shows a very important subclass of Markov chains.

Example 1.1.3 *The Markov chain is said to be* ***time invariant*** *if the conditional probability $p(x_{n+1}|x_n)$ does not depend on n, i.e. for $n = 1, 2, \ldots$*

$$\Pr(X_{n+1} = b|X_n = a) = \Pr(X_2 = b|X_1 = a)$$

for all $a, b \in \mathcal{X}$. If $\{X_i\}$ is a Markov chain, then X_n is often called the ***state*** *at time n. A time invariant Markov chain is characterized by its* ***initial state*** *and a* ***probability transition matrix*** *$P = [P_{ij}]$, $i, j \in \{1, 2, \ldots, m\}$, where*

$$P_{ij} = \Pr(X_{n+1} = j|X_n = i). \tag{1.4}$$

Indeed, the probability mass function of the state $n+1$ is wholly determined by the initial distribution $p(x_1)$ and the transition matrix via

$$p(x_{n+1}) = \sum_{x_n, x_{n-1}, \ldots, x_1} p(x_1)P_{x_1x_2}P_{x_2x_3}\ldots P_{x_nx_{n+1}}. \tag{1.5}$$

□

It is natural to investigate the Markov chains from information theoretical point of view. For this reason we make some preparation in the next section.

1.2 Classical information quantities

For a measure of uncertainty of a random variable *C. Shannon* proposed the following formula. The **Shannon entropy** $H(X)$ of a discrete random variable X is defined by

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x). \tag{1.6}$$

(If the log is to the base 2, the entropy is expressed in **bits**, while if the base of the log is e, the unit of the entropy is sometimes called **nat**.) The Shannon entropy has many properties that agree with the intuitive notion of what a measure of information should be, for example it helps us to express the dependence among the random variables. One of its basic property is the **subadditivity**, i.e.

$$H(X, Y) \leq H(X) + H(Y), \tag{1.7}$$

where

$$H(X, Y) = -\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log p(x, y) \tag{1.8}$$

is the **joint entropy** of random variables X and Y, and measures our total uncertainty about the pair (X, Y). The equality holds in the subadditivity if and only if X and Y are independent random variables. If we introduce the notion **mutual information** followed by Shannon:

$$I(X : Y) = H(X) + H(Y) - H(X, Y), \tag{1.9}$$

then the subadditivity of the entropy is equivalent with $I(X : Y) \geq 0$, and we have equality if and only if X and Y are independent. We define the **conditional mutual information** by the formula

$$I(X : Z|Y) = \sum_{y \in \mathcal{Y}} p(y) I(X|_{Y=y} : Z|_{Y=y}), \tag{1.10}$$

where $I(X|_{Y=y} : Z|_{Y=y})$ is the mutual information between the random variables X and Z conditional on the event "$Z = z$". As (1.10) is a convex combination of mutual informations, i.e. nonnegative quantities,

$$I(X : Z|Y) \geq 0 \tag{1.11}$$

follows immediately. A straightforward computation gives a chain rule

$$I(X : YZ) = I(X : Y) + I(X : Z|Y),$$

which implies the formula

$$I(X : Z|Y) = H(XY) + H(YZ) - H(XYZ) - H(Y). \tag{1.12}$$

This equation and (1.11) leads us to the remarkable property of the Shannon entropy, the **strong subadditivity**:

$$H(X, Y, Z) + H(Y) \leq H(X, Y) + H(Y, Z). \tag{1.13}$$

We can see that the strong subadditivity of the Shannon entropy is an immediate consequence of its subadditivity. It is interesting to find the condition of the equality.

Theorem 1.2.1 *$I(X : Z|Y) = 0$, or equivalently, we have equality in the strong subadditivity of Shannon entropy, i.e.*

$$H(X, Y, Z) + H(Y) = H(X, Y) + H(Y, Z) \tag{1.14}$$

if and only if $X \to Y \to Z$.

Proof. The sufficiency is obvious by (1.3), (1.9) and (1.10). Assume conversely, that $I(X : Z|Y) = 0$ holds. Then by the definition (1.10) it is clear that for all $y \in \mathcal{Y}$, such that $p(y) \neq 0$, $I(X|_{Y=y} : Z|_{Y=y}) = 0$ follows. It means by (1.9) that we have got equality in the subadditivity of the Shannon entropy

$$H(X|_{Y=y}) + H(Z|_{Y=y}) = H(X|_{Y=y}, Z|_{Y=y}),$$

that is $X|_{Y=y}$ and $Z|_{Y=y}$ are independent:

$$p(x, z|y) = p(x|y)p(z|y),$$

so $X \to Y \to Z$ as we stated. □

Further details can be found in [18, 50, 59].

1.3 Gaussian Markov triplets

In this section we investigate the multivariate normal distribution forming a Markov chain. All these results can be found in [9].

Let $\mathbf{X} := (X_1, X_2, \dots, X_n)$ be an n-tuple of real or complex random variables. The (i, j) element of the $n \times n$ **covariance matrix** is given by

$$C_{i,j} := \mathbb{E}(X_i \overline{X_j}) - \mathbb{E}(X_i)\mathbb{E}(\overline{X_j}),$$

where $\mathbb{E}$ denotes the expectation. The covariance matrix is positive semi-definite. The mean $\mathbf{m} := (m_1, m_2, \dots, m_n)$ consists of the expectations $m_i = \mathbb{E}(X_i)$, $(i = 1, 2, \dots, n)$. Let M be a positive definite $n \times n$ matrix and $\mathbf{m}$ be a vector. Then

$$f_{\mathbf{m},M}(\mathbf{x}) := \sqrt{\frac{\operatorname{Det} M}{(2\pi)^n}} \exp\Big(-\tfrac{1}{2}\langle \mathbf{x} - \mathbf{m}, M(\mathbf{x} - \mathbf{m})\rangle\Big) \tag{1.15}$$

is a multivariate Gaussian probability distribution, denoted by $N(\mathbf{m}, M^{-1})$, with expectation $\mathbf{m}$ and with **quadratic matrix** M. If $\mathbf{m} = 0$, then we write simply $f_M(\mathbf{x})$. If M is diagonal, then (1.15) is the product of functions of one-variable which means the independence of the random variables. It is a remarkable fact that the covariance matrix of the distribution (1.15) is M^{-1}. The following lemma is well known, see for example [9, 25].

Lemma 1.3.1 *Let*

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \tag{1.16}$$

be a positive definite $(m+k)$ by $(m+k)$ matrix written in block matrix form. Then the marginal of the Gaussian probability distribution

$$f_M(\mathbf{x}_1, \mathbf{x}_2) := \sqrt{\frac{Det\, M}{(2\pi)^{(m+k)}}} \exp\Big(-\tfrac{1}{2}\langle (\mathbf{x}_1, \mathbf{x}_2), M(\mathbf{x}_1, \mathbf{x}_2)\rangle \Big)$$

on $\mathbb{R}^m$ is the distribution

$$f_1(\mathbf{x}_1) := \sqrt{\frac{Det\, M}{(2\pi)^m Det\, D}} \exp\Big(-\tfrac{1}{2}\langle \mathbf{x}_1, (A - BD^{-1}B^*)\mathbf{x}_1\rangle \Big). \tag{1.17}$$

The matrix $(M|D) := A - BD^{-1}B^*$ appears often in the matrix analysis, and called the **Schur complement** of D in M [66].

Now turn to the conditional distributions. Given the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$, the conditional density is given by

$$f(\mathbf{x}_2|\mathbf{x}_1) := \frac{f_{(\mathbf{x}_1,\mathbf{x}_2)}(\mathbf{x}_1, \mathbf{x}_2)}{f_{\mathbf{x}_1}(\mathbf{x}_1)}, \tag{1.18}$$

which is a function of $\mathbf{x}_2$, since $\mathbf{x}_1$ is fixed. If $(\mathbf{X}_1, \mathbf{X}_2)$ is Gaussian with a quadratic matrix (1.16), then the conditional distribution (1.18) is the Gaussian $N(-D^{-1}B^*\mathbf{x}_1, D^{-1})$.

Let $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ be random variables with joint probability distribution $f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. The distribution of the appropriate marginals are $f(\mathbf{x}_1, \mathbf{x}_2)$, $f(\mathbf{x}_2, \mathbf{x}_3)$ and $f(\mathbf{x}_2)$. In accordance with the foregoing $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ is called a **Markov triplet** if

$$f(\mathbf{x}_3|\mathbf{x}_1, \mathbf{x}_2) = f(\mathbf{x}_3|\mathbf{x}_2). \tag{1.19}$$

We use the notation $\mathbf{X}_1 \to \mathbf{X}_2 \to \mathbf{X}_3$ for the Markov triplets as before. Let $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ be a Gaussian random variable with the quadratic matrix

$$M = \begin{bmatrix} A_1 & A_2 & B_1 \\ A_2^* & A_3 & B_2 \\ B_1^* & B_2^* & D \end{bmatrix} \tag{1.20}$$

and with expectation $\mathbf{m} = 0$. The next theorem gives the characterization of Gaussian Markov triplets by the form of its block covariance matrix and by the form of its quadratic matrix [9].

Theorem 1.3.1 *For the Gaussian triplet $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ with quadratic matrix (1.20) and with expectation 0, the following conditions are equivalent.*

1. $\mathbf{X}_1 \to \mathbf{X}_2 \to \mathbf{X}_3$.

2. $B_1 = 0$.

3. *The conditional distribution* $f(\mathbf{x}_3|\mathbf{x}_1, \mathbf{x}_2)$ *does not depend on* $\mathbf{x}_1$.

4. *The covariance matrix of* $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ *is of the form*

$$\begin{bmatrix} S_{11} & S_{12} & S_{12}S_{22}^{-1}S_{23} \\ S_{12}^* & S_{22} & S_{23} \\ S_{23}^*S_{22}^{-1}S_{12}^* & S_{23}^* & S_{33} \end{bmatrix}. \tag{1.21}$$

The Markov property should be characterized by entropy quantities, as we expected. For an n-tuple of real random variables $\mathbf{X} := (X_1, X_2, \dots, X_n)$ with density $f(x_1, x_2, \dots, x_n)$ the **Boltzmann-Gibbs entropy** (also called **differential entropy**) is defined as

$$h(\mathbf{X}) = -\int f(\mathbf{x}) \log f(\mathbf{x}) \mathrm{d}\mathbf{x},$$

whenever this has a meaning. In particular, if $\mathbf{X}$ has a multivariate Gaussian distribution (1.15), then

$$h(\mathbf{X}) = \frac{1}{2} \left(n \log(2\pi e) - \log \operatorname{Det} M \right).$$

The **relative entropy** of the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$ with probability densities f_1 and f_2, respectively is defined as

$$D(\mathbf{X}_1||\mathbf{X}_2) = \int f_1(\mathbf{x})(\log f_1(\mathbf{x}) - \log f_2(\mathbf{x}))\mathrm{d}\mathbf{x}.$$

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be n-tuples of real variable and assume that the distribution of $\mathbf{X}_2$ is $N(0, C)$. Then

$$D(\mathbf{X}_1||\mathbf{X}_2) = -h(\mathbf{X}_1) + \frac{1}{2} \left(\operatorname{Tr} C_{\mathbf{X}_2} M - \log \operatorname{Det} M - n \log 2\pi \right), \tag{1.22}$$

where $M = C^{-1}$ and $C_{\mathbf{X}_2}$ is the covariance matrix of $\mathbf{X}_2$. The next theorem shows an important property of positive block matrices [9].

Theorem 1.3.2 *Let*

$$S := \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^* & S_{22} & S_{23} \\ S_{13}^* & S_{23}^* & S_{33} \end{bmatrix}$$

be a positive definite block matrix. Then

$$Det\, S\, Det\, S_{22} \le Det \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix} Det \begin{bmatrix} S_{22} & S_{23} \\ S_{23}^* & S_{33} \end{bmatrix} \tag{1.23}$$

and the condition for equality is $S_{13} = S_{12} S_{22}^{-1} S_{23}$.

As a consequence we get a characterization of Markov triplets via the strong additivity of Boltzmann-Gibbs entropy [9].

Theorem 1.3.3 *Let* $\mathbf{X}_i$ *be a random variable with values in* $\mathbb{R}^{n_i}$, $i = 1, 2, 3$. *Assuming that the differential entropy and the covariance of* $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ *are well defined the strong subadditivity of entropy holds:*

$$h(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) + h(\mathbf{X}_2) \le h(\mathbf{X}_1, \mathbf{X}_2) + h(\mathbf{X}_2, \mathbf{X}_3).$$

The equality holds if and only if $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ *form a Markov triplet.*

In the next chapter we turn to the non-commutative case.

Chapter 2

Basics of non-commutative probability

2.1 C*-algebras

In this chapter we summarize briefly some basic notions and technical tools from the theory of non-commutative probability, needed later in our investigations. In order to treat classical and non-commutative or quantum systems in the same formalism, it is useful to use a C*-algebraic language. Excellent and more detailed discussions can be found in several places [8, 16, 21, 45, 50, 52, 53, 68].

Let $\mathcal{A}$ be an algebra with a norm $||.||$ on it. If $(\mathcal{A}, ||.||)$ is a Banach space as a vector space and the submulticative property of the norm

$$||xy|| \leq ||x||||y||$$

holds for all $x, y \in \mathcal{A}$, then it called a **Banach algebra**. If the algebra is equipped with an **involution**, i.e. with a map $* : \mathcal{A} \to \mathcal{A}$ with the following properties:

1. $(x^*)^* = x$
2. $(x + y)^* = x^* + y^*$
3. $(\lambda x)^* = \overline{\lambda} x^*$
4. $(xy)^* = y^* x^*$

for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, moreover $||x^*|| = ||x||$ holds for all $x \in \mathcal{A}$ then it called a **Banach*-algebra**. If the involution and the norm are also related by the so-called C*-property

$$||x^* x|| = ||x||^2, \qquad x \in \mathcal{A}$$

then the algebra is a **C*-algebra**. A C*-algebra is called **unital**, if there exists an element $\mathbf{1} \in \mathcal{A}$ such that $\mathbf{1}x = x\mathbf{1} = x$ holds for all $x \in \mathcal{A}$. In the following, under a C*-algebra we always mean a unital one.

Example 2.1.1 *Let $B(\mathcal{H})$ denote the set of all bounded operators on a Hilbert space $\mathcal{H}$. Then $B(\mathcal{H})$ is a non-commutative C*-algebra with the standard operations, with the operator norm and with the adjoint as an involution. Every subalgebra of $B(\mathcal{H})$, which is closed under the involution and the operator norm is again a C*-algebra, and called a C*-subalgebra of $B(\mathcal{H})$.*

The importance of the example above is given by the following theorem of *I.M. Gelfand*, *M. Naimark* and *I.E. Segal*.

Theorem 2.1.2 *Every C*-algebra is isomorphic to a C*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

A C*-algebra is called finite dimensional if it is finite dimensional as a vector space. The following theorem gives the characterization of such algebras.

Theorem 2.1.3 *Every finite dimensional C*-algebra is isomorphic to $\oplus_{i=1}^{n}\mathcal{B}(\mathcal{H}_i)$ for some finite dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$.*

Given a concrete C*-algebra $\mathcal{A}$, i.e. a C*-algebra of bounded operators on a given Hilbert space $\mathcal{H}$, it is natural to consider notions of convergence on $\mathcal{A}$ weaker than those induced by the operator norm.

Definition 2.1.4 *A **von Neumann algebra** $\mathcal{N}$ is a C*-subalgebra of $B(\mathcal{H})$ which is also closed in the strong operator topology.*

The latter requirement means that if $\{x_n\}$ is a sequence of operators from $\mathcal{N}$, such that for all $\xi \in \mathcal{H}$ one has $x_n\xi \to x\xi$ for some $x \in B(\mathcal{H})$, then $x \in \mathcal{N}$. So, in particular, $B(\mathcal{H})$ is also a von Neumann algebra. The adjoint operation is not continous in the strong operator topology. This fact motivates to introduce a weaker topology. A sequence of bounded operators $\{x_n\}$ converges to $x \in B(\mathcal{H})$ in the weak operator sense if, $x_n\xi \to x\xi$ weakly for all $\xi \in \mathcal{H}$, i.e. $\langle \eta, x_n\xi\rangle \to \langle \eta, x\xi\rangle$ for all $\xi, \eta \in \mathcal{H}$. The adjoint operation is continous in the weak operator topology. The operator norm topology is strictly stronger than the strong operator topology which is stronger than the weak operator topology whenever $\mathcal{H}$ is infinite dimensional. When $\mathcal{H}$ is finite dimensional, these topologies coincide and there is no distinction between C*-algebras and von Neumann algebras.

If S is any subset of $B(\mathcal{H})$, then its **commutant** S' is the set of bounded operators which commute with every element in S, i.e.

$$S' \equiv \{y \in B(\mathcal{H}) : xy = yx, \forall x \in S\}.$$

It is a unital subalgebra of $B(\mathcal{H})$, closed in the weak operator topology, hence in the strong operator topology. The operation of taking the commutant can be iterated, $S'' \equiv (S')'$, and it is clear that $S \subset S''$. The so called **von Neumann's double commutant theorem** characterizes the bicommutant of a *-subalgebra of $B(\mathcal{H})$:

Theorem 2.1.5 *For any *-subalgebra $\mathcal{A} \subset B(\mathcal{H})$, $\mathcal{A}''$ coincides with the strong and weak operator closures of $\mathcal{A}$.*

The double commutant theorem implies that a subset $S \subset B(\mathcal{H})$ which is closed under the involution is a von Neumann algebra if and only if $S = S''$. It follows that S' is a von Neumann algebra itself. A von Neumann algebra $\mathcal{N}$ is called a **factor** if its center is trivial, i.e. if the only elements in $\mathcal{N}$ which commute with every other element in $\mathcal{N}$ are the constant multiplies of the identity: $\mathcal{N} \cap \mathcal{N}' = \mathbb{C}\mathbf{1}$. $B(\mathcal{H})$ is a factor, since $B(\mathcal{H})' = \mathbb{C}\mathbf{1}$, and the only abelian factor is $\mathbb{C}\mathbf{1}$. Note that a von Neumann algebra $\mathcal{N}$ is abelian if and only if $\mathcal{N} \subset \mathcal{N}'$ and $\mathcal{N}$ is said to be **maximally abelian** if $\mathcal{N} = \mathcal{N}'$.

2.2 States and representations

Since every C*-algebra $\mathcal{A}$ is a Banach space, its topological dual $\mathcal{A}^*$, consisting of continous linear maps $\mathcal{A} \to \mathbb{C}$, is also a Banach space. States on a C*-algebra are general expectation functionals.

Definition 2.2.1 *A **state** on a unital C*-algebra $\mathcal{A}$ is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$, which is positive, i.e. $\phi(x^*x) \geq 0$, for all $x \in \mathcal{A}$, and normalized: $\phi(\mathbf{1}) = 1$.*

The set of the states on $\mathcal{A}$ is denoted by $\mathcal{S}(\mathcal{A})$. A state ϕ defines an inner product on $\mathcal{A}$ through the formula

$$\langle x, y \rangle := \phi(x^*y),$$

which implies the **Schwarz inequality** for states:

$$|\phi(x^*y)|^2 \leq \phi(x^*x)\phi(y^*y). \tag{2.1}$$

This inequality implies that a state is continous and in the same time it also enables us to compute its norm. Indeed, with $y = x$

$$|\phi(x)| \leq \phi(x^*x)^{1/2} \leq ||x^*x||^{1/2}\phi(\mathbf{1}) = ||x||$$

shows that ϕ is bounded, and hence continous with $||\phi|| \leq 1$. Moreover, $||\phi|| = 1$ as $\phi(\mathbf{1}) = 1$. A remarkable fact, that a continous functional on $\mathcal{A}$ is positive if and only if $||\phi|| = \phi(\mathbf{1})$. It is easy to check, that $\mathcal{S}(\mathcal{A})$ is a convex subset of the dual space $\mathcal{A}^*$ of $\mathcal{A}$. The extremal points of this convex set are called **pure states**. In other words, a state is pure if and only if it cannot be decomposed into a nontrivial convex combination of two states. The next example motivates to introduce a very important class of states.

Example 2.2.2 *Consider the matrix algebra $\mathcal{M}_n \equiv \mathcal{M}(\mathbb{C})$ of $n \times n$ complex matrices. As a vector space $\mathcal{M}_n$ is isomorphic to $\mathbb{C}^{n^2}$. By taking the matrix units as a canonical basis for $\mathcal{M}_n$, we have an inner product on $\mathcal{M}_n$ defined by*

$$\langle A, B \rangle = \sum_{i,j=1}^{n} \overline{A_{ij}} B_{ij} = Tr\, A^* B,$$

for all $A, B \in \mathcal{M}_n$. Thus if ϕ is a linear functional on $\mathcal{M}_n$, then there exists by the Riesz-Frèchet theorem a unique $D \in \mathcal{M}_n$ such that for all $A \in \mathcal{M}_n$

$$\phi(A) = \langle D, A \rangle = Tr\, D^* A.$$

Moreover if ϕ is a state, i.e. positive and normalized, then it follows that $D \geq 0$ and $Tr\, D = 1$. □

The state considered above is an example for normal state. We say that a state ϕ on a von Neumann algebra $\mathcal{N}$ is **normal** if for any countable family $\{P_n\}$ of mutually orthogonal projections in $\mathcal{N}$

$$\phi\left(\sum_n P_n\right) = \sum_n \phi(P_n)$$

holds. Every normal state ϕ on $B(\mathcal{H})$ is given by

$$\phi(x) = \mathrm{Tr}\,(\rho x), \qquad x \in B(\mathcal{H}), \tag{2.2}$$

for some unique **statistical operator** ρ acting on $\mathcal{H}$, i.e. $0 \leq \rho \in B(\mathcal{H})$ such that $\mathrm{Tr}\,\rho = 1$. If $\mathcal{H}$ is finite dimensional ρ is often called **density matrix**. In the commutative case the set of normal states can be identified with the set of probability measures on a set, and the functional as the expectation value in the classical sense.

The most simple examples of normal states are the **vector states**. If $\Psi \in \mathcal{H}$ is a unit vector, then the corresponding density matrix is $P_\Psi \in B(\mathcal{H})$ the orthogonal projection onto the one dimensional subspace of $\mathcal{H}$ spanned by Ψ, and the state is given by:

$$\phi(x) = \langle \Psi, x\Psi \rangle = \mathrm{Tr}\,(P_\Psi x), \quad x \in B(\mathcal{H}).$$

A state ϕ on a C*-algebra $\mathcal{A}$ is **faithful** if $0 \leq x \in \mathcal{A}$ and $\phi(x) = 0$ entail $x = 0$ and the state τ is **tracial**, if $\tau(xy) = \tau(yx)$, for all $x, y \in \mathcal{A}$. If $\dim\mathcal{H} = n$, then τ is a normal, faithful state with the density matrix $\frac{1}{n}\mathbf{1}$. If $\mathcal{H}$ is infinite dimensional, then there exists no faithful tracial state on $B(\mathcal{H})$.

Definition 2.2.3 *A* ***representation*** *of a C*-algebra* $\mathcal{A}$ *is a C*-algebra morphism* $\pi : \mathcal{A} \to B(\mathcal{H})$ *with some Hilbert space* $\mathcal{H}$*. The representation is called* ***faithful****, if* $\mathrm{Ker}\pi = \{0\}$*, and* ***irreducible*** *if there is no nontrivial invariant closed subspace of* $\mathcal{H}$ *under the action of* π*.*

We remark, that the irreducibility is equivalent to $\pi(\mathcal{A})' = \mathbb{C}\mathbf{1}$. Given a state ϕ on $\mathcal{A}$, one can construct a Hilbert space $\mathcal{H}_\phi$, a distinguished unit vector $\Omega_\phi \in \mathcal{H}_\phi$ and a C*-homomorphism $\pi_\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi)$, so that $\pi_\phi(\mathcal{A})$ is a C*-algebra acting on the Hilbert space $\mathcal{H}_\phi$, the set of vectors $\pi_\phi(\mathcal{A})\Omega_\phi$ is dense in $\mathcal{H}_\phi$ (i.e. Ω is **cyclic**) and

$$\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle, \qquad \forall x \in \mathcal{A}.$$

The triple $(\mathcal{H}_\phi, \Omega_\phi, \pi_\phi)$ is uniquely determined up to unitary equivalence by these properties, π_ϕ is called the **GNS representation** of $\mathcal{A}$ determined by ϕ, in honour of *I.M. Gelfand*, *M. Naimark* and *I.E. Segal.* The GNS representation of a C*-algebra $\mathcal{A}$ is faithful, and any state of $\mathcal{A}$ can be given as a vector state in this representation. Now we are ready to give the definition of the non-commutative probability space.

Definition 2.2.4 *An abstract or* ***algebraic probability space*** *consists of a unital C*-algebra* $\mathcal{A}$ *and a state* ϕ *on* $\mathcal{A}$*. If* $\mathcal{A}$ *is non-commutative, then the pair* $(\mathcal{A}, \phi)$ *is called a* ***non-commutative probability space****. An algebraic* ***random variable*** *is an embedding* $j : \mathcal{B} \to \mathcal{A}$ *of an algebra* $\mathcal{B}$ *into* $\mathcal{A}$*. The state* $\phi \circ j$ *of* $\mathcal{B}$ *is called the* ***distribution*** *of the random variable* j*.*

The next example shows that from C*-algebraic point of view the classical probability theory is simply the commutative case.

Example 2.2.5 *Let* X *be a compact Hausdorff topological space, and* $\mathcal{C}(X)$ *the set of all continous complex-valued functions on* X*. Then* $\mathcal{C}(X)$ *forms an algebra with respect to the pointwise operations. If we take the norm to the supremum norm, i.e.*

$$||f|| := \sup\{|f(x)| : x \in X\}$$

and the involution is the pointwise conjugation, then $\mathcal{C}(X)$ *becomes a* ***commutative C*-algebra****. Moreover a theorem of I.M. Gelfand and M. Naimark* *says that every commutative C*-algebra is isomorphic to $\mathcal{C}(X)$ for some compact Hausdorff space X. States on commutative C*-algebras can be concretely described as follows. Suppose μ is a finite signed measure on X. Then*

$$\phi(f) = \int_X f(x)\mathrm{d}\mu(x), \quad f \in \mathcal{C}(X)$$

defines a continous linear functional on $\mathcal{C}(X)$. Moreover, ***Riesz-Kakutani theorem*** *tells us that any continous linear functional ϕ on $\mathcal{C}(X)$ is of this form, for some measure μ, and ϕ is a state if and only if μ is a positive normalized measure. Thus states on a commutative C*-algebra should be regarded as analogous to probability measures on a compact Hausdorff topological space. Regarding this case, we can form the Hilbert space $L^2(X,\mu)$, as the completion of $\mathcal{C}(X)$ with respect to the inner product*

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}\mathrm{d}\mu(x) = \phi(f\overline{g}) \quad f, g \in \mathcal{C}(X).$$

There is then a canonical representation π of $\mathcal{C}(X)$ on $L^2(X,\mu)$ given by pointwise multiplication:

$$\pi(f)g = fg \qquad f, g \in \mathcal{C}(X),$$

where g and fg are regarded as elements of $L^2(X,\mu)$. Moreover, if $\Omega = 1$, the identity of $\mathcal{C}(X)$ is also regarded as a vector in $L^2(X,\mu)$, then

$$\phi(f) = \int_X (f(x)1)1\mathrm{d}\mu(x) = \langle \pi(f)\Omega, \Omega\rangle,$$

$$L^2(X,\mu) = \overline{\pi(\mathcal{C}(X))\Omega}.$$

□

2.3 Positive and completely positive maps

An element x in the C*-algebra $\mathcal{A}$ is called **positive**, $x \geq 0$, in notation, if it can be written in the form $x = y^*y$ for some $y \in \mathcal{A}$. The set of positive elements, denoted by $\mathcal{A}_+$ forms a convex cone, with $\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}$. The positivity induces a partial ordering on the set of self-adjoint elements: we say that $x \geq y$ if and only if $x - y \geq 0$.

A map $\alpha : \mathcal{A} \to \mathcal{B}$ between C*-algebras is positive, if it preserves the positivity, i.e. $\alpha(\mathcal{A}_+) \subset \mathcal{B}_+$. Convex combinations and compositions of positive maps are positive again, but the positivity is not preserved under forming tensor product. Indeed, the transposition map $\alpha(E_{ij}) = E_{ji}$, where E_{ij} are the matrix units is positive, but one can easily check that the map $\mathbf{1}_2 \otimes \alpha$ is not positive even though both factors are positive. This motivates to define a subclass of positive maps:

Definition 2.3.1 *A map $\alpha : \mathcal{A} \to \mathcal{B}$ is called d-**positive**, if*

$$\mathbf{1}_d \otimes \alpha : \mathcal{M}_d \otimes \mathcal{A} \to \mathcal{M}_d \otimes \mathcal{B}$$

*is positive, where $\mathcal{M}_d$ is the algebra of $d \times d$ complex matrices with the identity $\mathbf{1}_d$. If α is d-positive for all $d \geq 1$, then we called it **completely positive**.*

Sums, positive multiples and compositions of d-positive maps are d-positive again. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional C*-algebras, and $\alpha^* : \mathcal{B} \to \mathcal{A}$ is the adjoint map of $\alpha : \mathcal{A} \to \mathcal{B}$ with respect to the Hilbert-Schmidt scalar product, i.e.

$$\operatorname{Tr} \alpha(A)B = \operatorname{Tr} A\alpha^*(B), \quad A \in \mathcal{A}, B \in \mathcal{B},$$

then it easy to see that α^* is d-positive if and only if α is d-positive and α is trace preserving if and only if α^* is unit preserving.

The complete positivity is a typical non-commutative notion. Indeed, if $\mathcal{A}$ or $\mathcal{B}$ is a commutative C*-algebra then any positive $\alpha : \mathcal{A} \to \mathcal{B}$ map is completely positive. Completely positive maps are very extensively studied, for details see for example [15, 21, 39, 53]. Here we only mention the important **Kraus representation** theorem.

Theorem 2.3.2 *Let $\mathcal{H}$ and $\mathcal{K}$ finite dimensional Hilbert spaces, and $\mathcal{A} \subset B(\mathcal{H})$ and $\mathcal{B} \subset B(\mathcal{K})$ two C*-subalgebras. The map $\alpha : \mathcal{A} \to \mathcal{B}$ is completely positive if and only if there exist operators $V_k : \mathcal{H} \to \mathcal{K}$, $k = 1, \ldots, n$ such that*

$$\alpha(A) = \sum_k V_k A V_k^*, \qquad A \in \mathcal{A} \tag{2.3}$$

holds.

The operators V_k are not unique, they are called **Kraus operators**. If $\sum_k V_k V_k^* = \mathbf{1}_\mathcal{K}$, then α is **unital**, i.e. preserves the unit, while if $\sum_k V_k^* V_k = \mathbf{1}_\mathcal{H}$, then α preserves the trace. The trace preserving completely positive maps are often called **stochastic maps**. Moreover, if α is unital to, we say that is **bistochastic**.

2.4 Conditional expectations

In the classical probability the notion of conditional expectation was clearified by *A.N. Kolmogorov* and *J.L. Doob* [25]. Consider an arbitrary commutative probability space. Let $\mathcal{A}$ be the underlying σ-algebra of sets and $\mathcal{B}$ an arbitrary σ-subalgebra in $\mathcal{A}$. Let Y be a random variable with expectation. A random variable U is called a conditional expectation of Y relative to $\mathcal{B}$ if it is $\mathcal{B}$-measurable and $\mathbb{E}(Y\mathbf{1}_B) = \mathbb{E}(U\mathbf{1}_B)$ for every set $B \in \mathcal{B}$, where $\mathbb{E}$ denotes the expectation, $\mathbf{1}_B$ is the characteristic function of the set B and we write $U = E(Y|\mathcal{B})$. It follows from the **Radon-Nikodym theorem** that a conditional expectation always exists. Moreover, whenever Z is a $\mathcal{B}$-measurable and the expectations exist, then

$$\mathbb{E}(YZ|\mathcal{B}) = Z\mathbb{E}(Y|\mathcal{B}). \tag{2.4}$$

If $\mathcal{B}_0 \subset \mathcal{B}$ a σ-subalgebra and $U_0 = \mathbb{E}(Y|\mathcal{B}_0)$ the restriction of U, then the following important properties of the conditional expectations hold:

$$\mathbb{E}(Y\mathbf{1}_B) = \mathbb{E}(U\mathbf{1}_B) = \mathbb{E}(U_0\mathbf{1}_B), \quad B \in \mathcal{B}_0, \tag{2.5}$$

$$\mathbb{E}(Y|B_0) = \mathbb{E}(\mathbb{E}(Y|B)|B_0) = \mathbb{E}(\mathbb{E}(Y|B_0)|B), \quad B \in \mathcal{B}_0. \tag{2.6}$$

Now let us turn to the non-commutative case. A detailed description of this topic can be found in [59]. Followed the concept of the classical case which is expressed in (2.4), (2.5) and (2.6), *H. Umegaki* proposed the following definition [75].

Definition 2.4.1 *Let $\mathcal{A}$ be a von Neumann algebra, and $\mathcal{B} \subset \mathcal{A}$ be its subalgebra. A* ***conditional expectation*** *$E : \mathcal{A} \to \mathcal{B}$ is a unital linear positive mapping such that*

$$E(ab) = E(a)b \quad \textit{for all} \quad a \in \mathcal{A}, b \in \mathcal{B}. \tag{2.7}$$

Choosing $a = \mathbf{1}$, we obtain that E acts identically on $\mathcal{B}$. From the positivity $E(a^*) = E(a)^*$ follows, which implies the modular property $E(ba) = bE(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Moreover,

$$\sum_{i,j} b_i^* E(a_i^* a_j) b_j = E((\sum_i a_i b_i)^* (\sum_j a_j b_j)) \geq 0$$

shows that E is completely positive due to the positivity and the modular property. Heuristically, $E(a)$ is a kind of best approximation of a from $\mathcal{B}$. Indeed, let τ be a faithful, tracial functional on $\mathcal{A}$. Then $\mathcal{A}$ becomes a Hilbert space when it is endowed with the **Hilbert-Schmidt inner product**

$$\langle a_1, a_2 \rangle := \tau(a_1^* a_2),$$

and the conditional expectation $E : \mathcal{A} \to \mathcal{B}$ will be an orthogonal projection onto $\mathcal{B}$ with respect to the above defined inner product.

Let ϕ be a normal state on $\mathcal{A}$. We say that the conditional expectation $E : \mathcal{A} \to \mathcal{B}$ preserves the state ϕ or E is **compatible** with ϕ if

$$\phi \circ E = \phi. \tag{2.8}$$

The following example shows that in contrast to the classical case a state preserving conditional expectation does not always exist.

Example 2.4.2 *Let $E : \mathcal{M}_n \otimes \mathcal{M}_n \to \mathcal{M}_n \otimes \mathbf{1}_n$ be a conditional expectation compatible with the state ϕ on $\mathcal{M}_n \otimes \mathcal{M}_n$. Then*

$$E(x \otimes y) = (x \otimes \mathbf{1}_n)E(\mathbf{1}_n \otimes y) = (E(\mathbf{1}_n \otimes y))(x \otimes \mathbf{1}_n)$$

shows that $E(\mathbf{1}_n \otimes y) \in (\mathcal{M}_n \otimes \mathbf{1}_n)' = \mathbf{1}_n \otimes \mathcal{M}_n$ and since $E(\mathbf{1}_n \otimes y) \in \mathcal{M}_n \otimes \mathbf{1}_n$ we have $E(\mathbf{1}_n \otimes y) = c(\mathbf{1}_n \otimes \mathbf{1}_n)$ for some constant c, but by the compatible property $\phi(E(\mathbf{1}_n \otimes y)) = \phi(\mathbf{1}_n \otimes y) = \phi_2(y)$ and $c = \phi_2(y)$ follows, where ϕ_2 is the restriction of ϕ to the factor $\mathbf{1}_n \otimes \mathcal{M}_n$. Then any such conditional expectation has the form

$$E(x \otimes y) = (x \otimes \mathbf{1}_n)\phi_2(y) \tag{2.9}$$

and ϕ must be a product state: $\phi = \phi_1 \circ \phi_2$. □

Before clarifying the conditions for the existence of the conditional expectation we need to introduce the concept of the modular operator. Let $\mathcal{N} \in B(\mathcal{H})$ be a von Neumann algebra with an $\Omega \in \mathcal{H}$ **cyclic** and **separating** vector for $\mathcal{N}$, i.e. the set $\{x\Omega : N \in \mathcal{N}\}$ is dense in $\mathcal{H}$ and for any $x \in \mathcal{N}$, $x\Omega = 0$ implies $x = 0$. By the double commutant theorem (Theorem 2.1.5) one can easily verify that Ω is cyclic for $\mathcal{N}$ if and only if Ω is separating for $\mathcal{N}'$. One can introduce two natural antilinear operators S_0 and F_0 on $\mathcal{H}$ by the relations

$$S_0(x\Omega) := x^*\Omega, \quad \forall x \in \mathcal{N}, \tag{2.10}$$

$$F_0(x'\Omega) := (x')^*\Omega, \quad \forall x' \in \mathcal{N}'. \tag{2.11}$$

S_0 and F_0 are well defined on the dense domains $\mathrm{dom}S_0 = \mathcal{N}\Omega$ and $\mathrm{dom}F_0 = \mathcal{N}'\Omega$, respectively. It can be shown that the operators S_0 and F_0 are closable and that $S = F_0^*$ and $F = S_0^*$, where S and F denotes the closures of S_0 and F_0, respectively. The closed, antilinear operator S is called the **Tomita operator** for the pair $(\mathcal{N}, \Omega)$. Let Δ be the unique positive, selfadjoint operator and J the unique antilinear operator occuring in the polar decomposition of S, i.e.

$$S = J\Delta^{1/2}.$$

We call Δ the **modular operator** and J the **modular conjugation** associated to the pair $(\mathcal{N}, \Omega)$. Given the modular operator Δ, we can construct the strongly continous unitary group, called **modular group** by

$$\Delta^{it} = \exp(it(\ln \Delta)), \quad t \in \mathbb{R}, \tag{2.12}$$

via functional calculus. Denote

$$\sigma_t(x) := \Delta^{it} x \Delta^{-it}, \quad \forall x \in \mathcal{N}, t \in \mathbb{R}. \tag{2.13}$$

The main result of the so-called modular theory is the **Tomita-Takesaki theorem**.

Theorem 2.4.3 *With the notations above we have*

$$J\mathcal{N}J = \mathcal{N}' \qquad \text{and} \tag{2.14}$$

$$\sigma_t(\mathcal{N}) = \mathcal{N}, \quad t \in \mathbb{R}. \tag{2.15}$$

σ_t is a one parameter automorphism group on $\mathcal{N}$, the so-called **modular automorphism group**. The modular automorphism group is the one of the most useful elements in the further analysis of von Neumann algebras. For details many excellent books should be proposed [16, 17, 27, 73]. Our next example is crucial from the point of view of our further investigations.

Example 2.4.4 *If $\mathcal{N} = B(\mathcal{H})$, where $\mathcal{H}$ is separable, then as we have seen before, each normal state ϕ is given by a density operator ρ in the form*

$$\phi(x) = Tr(\rho x),$$

moreover ϕ is faithful if and only if ρ is invertible. In this case one may calculate that the modular automorphism group is given by

$$\sigma_t(x) = \rho^{it} x \rho^{-it}. \tag{2.16}$$

□

The following theorem of *M. Takesaki* clarifies the existence of the conditional expectation compatible with a given state by means of the modular automorphism group of the state [52].

Theorem 2.4.5 *Let $\mathcal{M}$ be a von Neumann subalgebra of the von Neumann algebra $\mathcal{N}$ and let ϕ be a faithful normal state of $\mathcal{N}$. Then the ϕ-preserving conditional expectation of $\mathcal{N}$ onto $\mathcal{M}$ exists if and only if $\mathcal{M}$ is stable under the modular group of ϕ, i.e.*

$$\sigma_t(\mathcal{M}) \subset \mathcal{M}, \quad t \in \mathbb{R}. \tag{2.17}$$

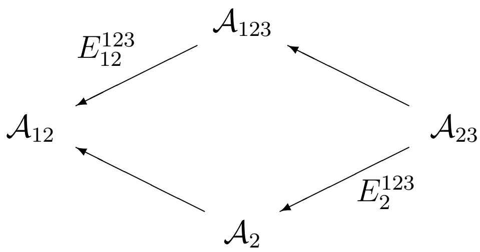
If a conditional expectation exists, then it is unique.

An immediate consequence of the theorem above, that the conditional expectation compatible with a faithful tracial state τ always exists. A remarkable property of the conditional expectation is the so called **commuting square** property. We briefly collect the facts applying to us, for details we suggest [21, 26].

Theorem 2.4.6 *Let $\mathcal{A}_{123}$ be a matrix algebra with subalgebras $\mathcal{A}_{12}$, $\mathcal{A}_{23}$ and $\mathcal{A}_2$, with $\mathcal{A}_2 \subset \mathcal{A}_{12}, \mathcal{A}_{23}$. Assume that a conditional expectation $E_{12}^{123} : \mathcal{A}_{123} \to \mathcal{A}_{12}$, $E_2^{23} : \mathcal{A}_{23} \to \mathcal{A}_2$ and $E_2^{12} : \mathcal{A}_{12} \to \mathcal{A}_2$ exist. Then the following conditions are equivalent:*

1. $E_{12}^{123}|\mathcal{A}_{23} = E_2^{23}$
2. $E_{23}^{123}|\mathcal{A}_{12} = E_2^{12}$
3. $E_{12}^{123}E_{23}^{123} = E_{23}^{123}E_{12}^{123}$ and $\mathcal{A}_{12} \cap \mathcal{A}_{23} = \mathcal{A}_2$
4. $E_{12}^{123}E_{23}^{123} = E_2^{123}$
5. $E_{23}^{123}E_{12}^{123} = E_2^{123}$.

Note that if any of these properties hold, then the following diagram is commutative,



where the arrows $\nwarrow$ consist of embeddings and the arrows $\swarrow$ consist of conditional expectations.

2.5 Coarse-grainings

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Recall that **2-positivity** of the map $\alpha : \mathcal{A} \to \mathcal{B}$ means that

$$\begin{bmatrix} \alpha(A) & \alpha(B) \\ \alpha(C) & \alpha(D) \end{bmatrix} \geq 0, \quad \text{whenever} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0,$$

where $A, B, C, D \in \mathcal{A}$. It is well known that a 2-positive unit preserving mapping α satisfies the **Schwarz inequality**

$$\alpha(x)^*\alpha(x) \leq \alpha(x^*x), \quad \forall x \in \mathcal{A}. \tag{2.18}$$

Indeed, since

$$\begin{bmatrix} x^*x & x \\ x^* & \mathbf{1} \end{bmatrix} \geq 0, \quad \forall x \in \mathcal{A},$$

for a unit preserving, 2-positive α, we have

$$\begin{bmatrix} \alpha(x^*x) & \alpha(x) \\ \alpha(x^*) & \mathbf{1} \end{bmatrix} \geq 0,$$

which implies (2.18). A 2-positive unital mapping between C*-algebras is called a **coarse-graining**.

Example 2.5.1 *Let $\mathcal{A} = B(\mathcal{H})$ and let $\mathcal{B}$ be the infinite tensor product $\mathcal{A} \otimes \mathcal{A} \otimes \dots$. If γ denotes the right shift on $\mathcal{B}$, then we can define a sequence α_n of coarse grainings $\mathcal{A} \to \mathcal{B}$ as follows*

$$\alpha_n(A) := \frac{1}{n}(A + \gamma(A) + \cdots + \gamma^{n-1}(A)).$$

α_n can be regarded to the non-commutative analogue of the ***sample mean****.*

One can check easily that the set

$$\mathcal{A}_\alpha := \{x \in \mathcal{A} : \alpha(x)^*\alpha(x) = \alpha(x^*x) \quad \text{and} \quad \alpha(x)\alpha(x)^* = \alpha(xx^*)\} \tag{2.19}$$

is a C*-algebra, and the restriction of α to $\mathcal{A}_\alpha$ is a C*-algebra morphism. Moreover, a stronger multiplicativity property

$$\alpha(x)\alpha(y) = \alpha(xy), \quad x \in \mathcal{A}_\alpha, y \in \mathcal{A} \tag{2.20}$$

holds. This suggests to call $\mathcal{A}_\alpha$ to be the **multiplicative domain** of α. Consider the fixed point set $\mathcal{F}_\alpha$ of the 2-positive unital map α

$$\mathcal{F}_\alpha := \{x \in \mathcal{A} : \alpha(x) = x\}. \tag{2.21}$$

In general $\mathcal{F}_\alpha$ is a linear subspace of $\mathcal{A}$, that is also closed under the *-operation. Moreover, the following is true.

Theorem 2.5.2 *If there exists a faithful invariant state ϕ of α, i.e. $\phi \circ \alpha = \phi$, then $\mathcal{F}_\alpha$ is an algebra.*

Proof. Assume, that $x \in \mathcal{F}_\alpha$. Then $\alpha(x^*x) - \alpha(x)^*\alpha(x) \geq 0$ by the Schwarz inequality, and by using the invariance of ϕ, we get

$$\phi(\alpha(x^*x) - \alpha(x)^*\alpha(x)) = \phi(\alpha(x^*x) - x^*x) = 0.$$

Then the faithfulness of ϕ implies $\alpha(x^*x) - \alpha(x)^*\alpha(x) = 0$. Taking the linear combinations of fixed points, and repeating the above argument, we obtain the statement. □

We say that $\mathcal{F}_\alpha$ is the **fixed point algebra** of α. We also get the relation

$$\mathcal{F}_\alpha \subset \mathcal{A}_\alpha. \tag{2.22}$$

Example 2.5.3 *Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a coarse graining, and ϕ a faithful α-invariant state. If $\mathcal{A}$ is endowed with the inner product*

$$\langle x|y\rangle_\phi := \phi(x^*y), \quad x, y \in \mathcal{A},$$

then the Schwarz inequality implies that α is a contraction in the norm induced by $\langle .|.\rangle_\phi$. Due to the ***von Neumann ergodic theorem*** *the following limit exists:*

$$s_n(x) := \frac{1}{n}(x + \alpha(x) + \cdots + \alpha^{n-1}(x)) \to E(x), \tag{2.23}$$

where E is the projection onto $\mathcal{F}_\alpha$ the fixed point algebra of α, self-adjoint in the scalar-product $\langle .|.\rangle_\phi$. Since s_n's are coarse-grainings, so is their limit E. In fact, E is a conditional expectation onto $\mathcal{F}_\alpha$. This result, namely: the limit of s_n is a conditional expectation $E : \mathcal{A} \to \mathcal{F}_\alpha$ onto the fixed point algebra, if there is a faithful state left invariant by α, is the special case of the so-called ***Kovács-Szűcs theorem****.*

As an immediate consequence of Theorem 2.4.5 we get the following.

Corollary 2.5.4 *The modular group of a faithful α-invariant state leaves the fixed point algebra of α invariant.*

Note that the conditional expectation given by the limit (2.23) is independent of the invariant state ϕ, we have only used the existence of such ϕ.

Let us make an important remark regarding the physical applications. A quantum mechanical system is described by a C*-algebra, observables correspond to self-adjoint elements and the physical states of the system are modelled by the states of the algebra. The evolution of the system $\mathcal{A}$ can be described in the **Heisenberg picture** in which an observable $x \in \mathcal{A}$ moves into $\alpha(x)$, where α is a coarse-graining, i.e. 2-positive, unital map, which transforms an observable to observable. The **Schrödinger picture** is dual, it gives the transformation of states. Let $\alpha : \mathcal{A} \to \mathcal{B}$ be a unital, 2-positive linear map between the C*-algebras $\mathcal{A}$ and $\mathcal{B}$. Its adjoint map $\Phi := \alpha^*$ is also 2-positive and trace-preserving map and maps a density matrix to density matrix, and it will be called coarse-graining as well.

2.6 The von Neumann entropy

In this section we give a brief overview about the quantities related to the information used in the non-commutative probability theory. From the point of view of information theory, as well as physics, it is very interesting to know what kind of inequalities hold among these quantities and when these inequalities are saturated. We will also investigate these questions. Our main references are [50, 52, 59]. We have seen that in the classical probability the Shannon entropy measures the uncertainty associated with a classical probability. In the non-commutative case, in a similar fashion, the probability distributions are replaced with states.

Consider a non-commutative probability space $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a finite C*-algebra, and ϕ is a normal, faihful state on $\mathcal{A}$ given by a density matrix ρ. The **von Neumann entropy** is defined by the formula

$$S(\phi) \equiv S(\rho) := -\mathrm{Tr}\,\rho \log \rho. \tag{2.24}$$

Usually, the logarithms are taken to base two. From the definition it is clear that

$$S(\rho) \geq 0,$$

and the equality holds if and only if ρ is a one-rank projection, i.e. the state is pure. As the von Neumann entropy is the trace of a continous function of the density matrix, hence it is a continous function on the states. To deduce some properties of the von Neumann entropy, it is useful to introduce the concept of relative entropy. Originally the relative entropy was introduced by *H. Umegaki* in the setting of von Neumann algebras [75]. Later it was widely used in the mathematical physics by *G. Lindblad* [39] and extended to arbitrary von Neumann algebras by *H. Araki* [10]. Assume that ρ and σ are density matrices of the states ϕ and ω, respectively on a Hilbert space $\mathcal{H}$, then their **relative entropy** is defined by

$$S(\phi||\omega) \equiv S(\rho||\sigma) = \begin{cases} \mathrm{Tr}\,\rho(\log\rho - \log\sigma) & \text{if } \mathrm{supp}\,\rho \subset \mathrm{supp}\,\sigma, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.25}$$

Theorem 2.6.1 $S(\rho||\sigma) \geq 0$ *with equality if and only if* $\rho = \sigma$.

Proof. If ρ and σ are statistical operators one can check [52], that

$$\mathrm{Tr}\,\rho(\log\rho - \log\sigma) \geq \frac{1}{2}\mathrm{Tr}\,(\rho - \sigma)^2, \tag{2.26}$$

with an immediate consequence for the non-negativeness of the relative entropy. □

The theorem above motivates to consider the relative entropy as some kind of a distance measure on the set of states, even though it is not symmetric in its arguments and does not satisfy the triangle inequality. However, relative entropy can be interpreted as a measure of statistical distinguishability in a simple decision scheme [59]. In a d-dimensional Hilbert space apply the non-negativity for the tracial state τ with the density matrix $\mathbf{1}_d/d$.

$$0 \leq S(\rho||\mathbf{1}_d/d) = -S(\rho) + \log d$$

shows that the relative entropy can be regarded to the extension of the von Neumann entropy.

If we consider a composite system 'AB' (for example a bipartite system), it corresponds to tensor product in mathematical terms. Let $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$ be the algebras of bounded operators acting on $\mathcal{H}_A$ and $\mathcal{H}_B$. The Hilbert space of the composite system 'AB' is $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and a state of the composite system is given by a density matrix $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB}) = \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$. The marginal states ρ_A and ρ_B on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively are determined by the relations

$$\begin{aligned} \mathrm{Tr}\,(\rho_A a) &= \mathrm{Tr}\,(\rho_{AB} a) \quad \text{for all} \quad a \in \mathcal{B}(\mathcal{H}_A), \\ \mathrm{Tr}\,(\rho_B b) &= \mathrm{Tr}\,(\rho_{AB} b) \quad \text{for all} \quad b \in \mathcal{B}(\mathcal{H}_B). \end{aligned}$$

The subsystems of a bipartite system are **independent** if and only if the state of the composite system is the product of marginals, i.e. $\rho_{AB} = \rho_A \otimes \rho_B$. A possible way to quantify correlation between the subsystems is to measure it by the relative entropic distance of the state from the product of its marginals, by the so called **mutual information**:

$$I(A:B) := S(\rho_{AB}||\rho_A \otimes \rho_B). \tag{2.27}$$

Theorem 2.6.2 *The von Neumann entropy is **subadditive**, i.e.*

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B), \tag{2.28}$$

with equality if and only if the subsystems are independent.

Proof. The non-negativity of relative entropy implies that mutual information is positive, unless the two marginals are independent. Since

$$S(\rho_{AB}||\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

the non-negativity implies the subadditivity of the von Neumann entropy. □
As relative entropy is interpreted as statistical distinguishability of states, it is interesting to ask what happens if the states are subjected to a stochastic operation? The answer is given by the important **Uhlmann's monotonicity theorem** [74].

Theorem 2.6.3 *Let $\alpha : \mathcal{A} \to \mathcal{B}$ be a unital, 2-positive linear map between the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then its adjoint map $T := \alpha^*$ is also 2-positive and trace-preserving map and maps a density matrix to density matrix. Then, for states ϕ and ω on $\mathcal{A}$ with densities ρ and σ, respectively we have*

$$S(T\rho||T\sigma) \leq S(\rho||\sigma), \tag{2.29}$$

or equivalently

$$S(\omega \circ \alpha||\phi \circ \alpha) \leq S(\omega||\phi). \tag{2.30}$$

We can get a remarkable property of the von Neumann entropy as a consequence of Uhlmann's theorem.

Theorem 2.6.4 *The von Neumann entropy is* ***strongly subadditive****, i.e.*

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}). \tag{2.31}$$

Proof. As we have seen before, a conditional expectation which preserves the tracial state τ always exists. Consider a three-partite state ρ_{ABC} on $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and the trace preserving conditional expectation $E_{AB}^{ABC} : \mathcal{B}(\mathcal{H}_{ABC}) \to \mathcal{B}(\mathcal{H}_{AB})$, where $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. By the commuting square property of the conditional expectation we have for the marginals

$$E_{AB}^{ABC}(\rho_{ABC}) = \rho_{AB} \tag{2.32}$$

and

$$E_{AB}^{ABC}(\rho_{BC}) = \rho_B. \tag{2.33}$$

As E_{AB}^{ABC} is completely positive and trace preserving, we have

$$S(\rho_{AB}||\rho_B) \leq S(\rho_{ABC}||\rho_{BC}),$$

which is equivalent with the strong subadditivity. □

It is interesting to find the states which saturate the strong subadditivity of von Neumann entropy with equaliy. Uhlmann's theorem gives an immediate characterization of such states [46, 58, 60].

Theorem 2.6.5 *Let T be a coarse graining, i.e. a trace preserving, 2-positive linear map. For states ρ and σ,*

$$S(\rho||\sigma) = S(T\rho||T\sigma)$$

holds if and only if there exists a coarse graining $\hat{T}$ such that

$$\hat{T}T\rho = \rho, \quad \textit{and} \quad \hat{T}T(\sigma) = \sigma.$$

The strong subadditivity of von Neumann entropy has got important consequences in the mathematical physics. *D.W. Robinson* and *D. Ruelle* first noted the importance of classical strong subadditivity for statistical physics [69]. The non-commutative, in other words *quantum* version was conjectured in 1968 by *O.E. Lanford* and Robinson [36]. Obtaining a proof of the result was rather difficult however. It is important remark that in contrast to the classical case, the strong subadditivity of the von Neumann entropy does not follows from the subadditivity. Finally, in 1973 the theorem was proved in two papers: [37] contains *E.H. Lieb* famous theorem which is a generalization of the crucial *Wigner-Yanase-Dyson* conjecture made in 1963, while the surprising connection to strong subadditivity was developed by Lieb and *M.B Ruskai* [38]. The proof based on the Uhlmann's monotonicity theorem comes from [74]. A simple proof based on the relative modular operators can be found in [54].

Because of its importance we give an alternative proof for the strong subadditivity of von Neumann entropy with a condition for equality based on *E.H. Lieb*'s extension of Golden-Thompson inequality. The original **Golden-Thompson inequality** says, that if A and B are selfadjoint matrices, then the inequality

$$\mathrm{Tr}\, e^A e^B \geq \mathrm{Tr}\, e^{A+B}$$

holds. It is really an inequality except for the trivial case $[A, B] = 0$ and one may deduce it from the monotonicity of the relative entropy [52]. The so called **Golden-Thompson-Lieb inequality** is the following [52].

Theorem 2.6.6 *Let $\mathcal{A}$ be a matrix algebra, and $A, B, C \in \mathcal{A}$ selfadjoint operators. Then the following inequality holds:*

$$Tr e^{A+B+C} \leq \int_0^\infty Tr (t + e^{-A})^{-1} e^B (t + e^{-A})^{-1} e^C \,\mathrm{d}t. \tag{2.34}$$

One can check, that the $B = 0$ case leads back to the original inequality. Now we are ready to prove the following theorem [65].

Theorem 2.6.7 *With the notation as before, the strong subadditivity of von Neumann entropy, i.e. (2.31) holds. We have equality if and only if*

$$\log \rho_{ABC} + \log \rho_B = \log \rho_{AB} + \log \rho_{BC}. \tag{2.35}$$

Proof. Since the operator

$$\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}) \tag{2.36}$$

is positive it can be written as $\lambda\sigma$ for a density matrix σ with a suitable normalizing constant $\lambda > 0$. Then we have

$$\begin{aligned} S(\rho_{AB}) &+ S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B) \\ &= \operatorname{Tr} \rho_{ABC}(\log \rho_{ABC} - (\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) \\ &= S(\rho_{ABC}||\lambda\sigma) = S(\rho_{ABC}||\sigma) - \log \lambda. \end{aligned}$$

We can observe that $\lambda \leq 1$ implies the non-negativity of the left-hand-side, i.e. the strong subadditivity. Due to Theorem 2.6.6, with the choices $A = -\log \rho_B$, $B = \log \rho_{BC}$ and $C = \log \rho_{AB}$ we have

$$\begin{aligned} &\operatorname{Tr} \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC}) \\ &\qquad \leq \int_0^\infty \operatorname{Tr} \left\{ \rho_{AB}(t\mathbf{1} + \rho_B)^{-1} \rho_{BC}(t\mathbf{1} + \rho_B)^{-1} \right\} \mathrm{d}t. \end{aligned}$$

Now applying the existence of the trace preserving conditional expectation E_{AB}^{ABC}, with (2.33) and by the Theorem 2.4.6 on the commuting square property, we have

$$\operatorname{Tr} \left\{ \rho_{AB}(t\mathbf{1} + \rho_B)^{-1} \rho_{BC}(t\mathbf{1} + \rho_B)^{-1} \right\} = \operatorname{Tr} \left\{ \rho_B(t\mathbf{1} + \rho_B)^{-1} \rho_B(t\mathbf{1} + \rho_B)^{-1} \right\}.$$

Now we are able to integrate out by the formula

$$\int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} \, \mathrm{d}t = \lambda \tag{2.37}$$

where $\lambda > 0$ and we get:

$$\int_0^\infty \operatorname{Tr} \left\{ \rho_{AB}(t\mathbf{1} + \rho_B)^{-1} \rho_{BC}(t\mathbf{1} + \rho_B)^{-1} \right\} \mathrm{d}t = \operatorname{Tr} \rho_B = 1.$$

This means that $\operatorname{Tr}(\lambda\sigma) \leq 1$ which implies that $\lambda \leq 1$ and the strong subadditivity follows. Moreover we have got equality if and only if $\lambda = 1$, i.e. $\exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})$ is a density matrix and

$$S(\rho_{ABC}|| \exp(\log \rho_{AB} - \log \rho_B + \log \rho_{BC})) = 0$$

implies

$$\log \rho_{ABC} = \log \rho_{AB} - \log \rho_B + \log \rho_{BC},$$

as we stated. This is a necessary and sufficient condition for the equality. □

Actually, this condition is strongly related to statistical sufficiency and we have further equivalent conditions [58, 60, 32].

Theorem 2.6.8 *We have equality in (2.31) if and only if the following equivalent conditions hold.*

(i) $\rho_{ABC}^{it}\rho_{BC}^{-it} = \rho_{AB}^{it}\rho_B^{-it}$ *for all* $t \in \mathbb{R}$.

(ii) $\rho_{ABC}^{1/2}\rho_{BC}^{-1/2} = \rho_{AB}^{1/2}\rho_B^{-1/2}$.

(iii) $\log \rho_{ABC} = \log \rho_{AB} - \log \rho_B + \log \rho_{BC}$.

(iv) There are positive matrices $X, Y \in \mathcal{B}(\mathcal{H}_{AB})$ *and* $0 \le Z \in \mathcal{B}(\mathcal{H}_{ABC})$, *such that* $\rho_{ABC} = XZ$, $\rho_{AB} = YZ$ *and the commutation relation* $ZX = XZ$ *and* $ZY = YZ$ *hold.*

Remark that some of the equivalent conditions are valid also in infinite dimensional Hilbert space, for example, the equivalence of (i) and (iv) is obtained in [32].

2.7 Operator monotone functions

In this section we briefly study an important class of functions called operator monotone functions. Recall that for selfadjoint matrices A and B, we use the notation $A \le B$ to mean $B - A$ is positive. The relation $\le$ is a partial order on selfadjoint matrices. Let f be a real function defined on an interval I. If a diagonal matrix $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_i \in I$ for all $i = 1, \dots, n$, we define $f(D) = \operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n))$. If A is a selfadjoint matrix whose eigenvalues λ_j are in I, we choose a unitary U such that $A = UDU^*$, where D is diagonal, and then define $f(A) = Uf(D)U^*$. In this way we can define $f(A)$ for all selfadjoint matrices of any order whose eigenvalues are in I. The operator monotone functions are real functions whose extensions to selfadjoint matrices preserve order. They are closely related to operator convex functions, so we shall study both of these together. All results with proofs can be found in [15, 31].

Definition 2.7.1 *A real function* f *is said to be* ***matrix monotone of order*** **n** *if on* n *by* n *selfadjoint matrices* $A \le B$ *implies* $f(A) \le f(B)$. *If* f *is matrix monotone of order* n *for all* n *we say* f *is* ***matrix monotone*** *or* ***operator monotone****. A real function* f *is said to be* ***matrix convex of order*** **n** *if for all* $n \times n$ *selfadjoint matrices* A *and* B *and for all* $\lambda \in [0, 1]$,

$$f((1 - \lambda)A + \lambda B) \le (1 - \lambda)f(A) + \lambda f(B).$$

If f *is matrix convex of all orders, we say that* f *is* ***matrix convex*** *or* ***operator convex****. A function* f *is called* ***operator concave*** *if the function* $-f$ *is operator convex.*

It is clear that the set of operator monotone functions and the set of operator convex functions are both closed under positive linear combinations and also under pointwise limits. From the fact that any convex function $f : \mathbb{R} \to \mathbb{R}$ is continous it follows that operator convexity is equivalent to **operator midpoint convexity**, $f(\frac{A+B}{2}) \leq \frac{f(A)+f(B)}{2}$. The next two theorems are among the several results that describe the connections between operator convexity and operator monotonicity.

Theorem 2.7.2 *Let f be a continous function mapping the positive half-line $[0, +\infty)$ into itself. Then f is operator monotone if and only if it is operator concave.*

Theorem 2.7.3 *Let f be a continous real function on $[0, \alpha)$. Then f is operator convex and $f(0) \leq 0$ if and only if $\frac{f(x)}{x}$ is operator monotone on $(0, \alpha)$.*

We find some important cases in the next example.

Example 2.7.4 *Some computation shows that $\alpha + \beta t$, ($\beta \geq 0$), t^r on $[0, \infty)$ for $0 \leq r \leq 1$, $-\frac{1}{t}$ on $(0, \infty)$ and $\log t$ on $(0, \infty)$ are operator monotone functions, but t^r is not operator monotone, whenever $r > 1$.*

The functions t^r for $1 \leq r \leq 2$, $\frac{1}{t}$ and $t \log t$ are operator convex, but for example t^3 is not, as a simple computation shows.

It is a remarkable fact, that operator monotone functions are smooth to all orders, in fact they are analytic. This result is used to establish the following integral representation for operator monotone functions, the so-called **Löwner theorem** [15].

Theorem 2.7.5 *If f is an operator monotone function on $\mathbb{R}^+$, then*

$$f(t) = f(0) + \beta t + \int_0^\infty \frac{\lambda t}{\lambda + t} \mathrm{d}\mu(\lambda), \tag{2.38}$$

where μ is a measure such that the integral

$$\int_0^\infty \frac{\lambda}{\lambda + 1} \mathrm{d}\mu(\lambda)$$

is finite.

The Löwner theorem represents a real breakthrough that allows us to connect the theory of operator monotonicity to complex analysis. In particular, we see that we can define an analytic continuation of f to the entire complex plane except for $z \in (-\infty, 0)$:

$$f(z) = f(0) + \beta z + \int_0^\infty \frac{\lambda z}{\lambda + z} \mathrm{d}\mu(\lambda).$$

Defining the upper half plane $H_+ \equiv \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, one can observe that f maps H_+ into itself. These observations can be used to prove the forward implication in the following theorem.

Theorem 2.7.6 *A function f is operator monotone on (a, b) if and only if f has an analytic continuation to the upper half plane H_+ that maps H_+ into itself.*

The following lemma is very useful for our further investigations.

Lemma 2.7.1 *The function*

$$\kappa(t) = -t \log t + (t + 1) \log(t + 1) \tag{2.39}$$

is operator monotone on $(0, \infty)$.

Proof. The integral representation

$$\kappa(x) = \int_1^\infty t^{-2} \log(tx + 1) \mathrm{d}t$$

and the fact, that $\log x$ is an operator monotone function implies the statement. □

In next chapter we overview the concept of Markovianity in the non-commutative probability.

Chapter 3

Markovianity in the non-commutative probability

3.1 Markov chains, states and triplets

In classical probability theory, Markov chains are defined on abelian C*-algebras and there is a standard way to associate a Markov chain to a given initial distribution and a given sequence of transition matrices as in the Example 1.1.3 was shown. The notion of non-commutative or quantum Markov chains was introduced by *L. Accardi* extending this technique [2, 3, 7].

Recall that a pair $(\mathcal{A}, \phi)$ consisting of an C*-algebra and its state ϕ is called an algebraic probability space. An algebraic random variable is an embedding $j : \mathcal{B} \to \mathcal{A}$ of an algebra $\mathcal{B}$ into $\mathcal{A}$. The state $\phi \circ j$ of $\mathcal{B}$ is called the distribution of the random variable j. By an **algebraic stochastic process** we mean a family of algebraic random variables $j_t : \mathcal{B} \to \mathcal{A}$, indexed by a set T. The simplest example of an algebraic stochastic process is an infinite tensor product over the natural numbers. To each $i \in \mathbb{N}$ a copy $\mathcal{A}_i$ of the finite dimensional C*-algebra $\mathcal{B}$ is associated and $\mathcal{A}$ is the infinite C*-tensor product $\otimes_{i\in\mathbb{N}}\mathcal{A}_i$. In this case the algebraic random variable $\mathcal{A}_i$ is the ith factor of the tensor product. For example, let $\mathcal{A}_i = \mathcal{M}_d(\mathbb{C})$ the $d \times d$ complex matrix algebra, for $i \in \mathbb{N}$ with the identity $\mathbf{1}$ and $\mathcal{A}$ be the infinite C*-tensor product $\otimes_{i=0}^{\infty}\mathcal{A}_i$. The algebra $\mathcal{A}$ is often called a **spin chain** algebra. We denote $\mathcal{A}_\Lambda = \otimes_{i\in\Lambda}\mathcal{A}_i$ for arbitrary subset $\Lambda \subset \mathbb{N}$. If ϕ is a state on $\mathcal{A}$, we write $\phi_{[0,n]}$ for the restriction $\phi|\mathcal{A}_{[0,n]}$, and particularly ϕ_0 for $n = 0$. The **right shift** automorphism of the algebra $\mathcal{A}$ will be denoted by γ. A state ϕ is called **stationary** if it is γ-invariant, i.e. $\phi \circ \gamma = \phi$.

Let $\mathcal{E}_i : \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be a completely positive unital mapping, for all $i \in \mathbb{N}$, called **transition expectation**. Then there exists a unique completely positive identity-preserving map $E_{0]} : \mathcal{A} \equiv \otimes_{i\in\mathbb{N}}\mathcal{A}_i \to \mathcal{A}_0$ such that for each positive integer n, and for any $a_0, a_1, \dots, a_n \in \mathcal{M}_d(\mathbb{C})$, one has

$$E_{0]}(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots) = \mathcal{E}_0\left(a_0 \otimes \mathcal{E}_1(a_1 \otimes \cdots \otimes \mathcal{E}_n(a_n \otimes \mathbf{1}))\right). \tag{3.1}$$

We can use the map $E_{0]}$ to lift any state ϕ_0 on $\mathcal{A}_0$ to a state ϕ on $\mathcal{A} \equiv \otimes_{i\in\mathbb{N}}\mathcal{A}_i$ defined by

$$\phi = \phi_0 \circ E_{0]}. \tag{3.2}$$

Now we are ready to generalize the notion of Markov chains in the non-commuting setting, followed Accardi:

Definition 3.1.1 *The state ϕ on $\mathcal{A} \equiv \otimes_{i\in\mathbb{N}}\mathcal{A}_i$ associated to $(\phi_0, \{\mathcal{E}_i\})$, is called a* ***Markov chain*** *if (3.2) holds, i.e.*

$$\phi(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes \mathbf{1} \otimes \mathbf{1} \otimes \dots) = \phi_0\left\{\mathcal{E}_0\left(a_0 \otimes \mathcal{E}_1(a_1 \otimes \cdots \otimes \mathcal{E}_n(a_n \otimes \mathbf{1}))\right)\right\}, \tag{3.3}$$

for each positive integer n, where ϕ_0 is the ***initial distribution*** *and $\{\mathcal{E}_i\}$ is a sequence of transition expectations. If for each $n \in \mathbb{N}$, $\mathcal{E}_n = \mathcal{E}_0 \equiv \mathcal{E}$ holds, then we speak of an* ***homogenous*** *Markov chain.*

A light modification of quantum (non-commutative) Markov chains was introduced by *M. Fannes, B. Nachtergaele* and *R.F. Werner* in a quite different context [23].

Definition 3.1.2 *A state ϕ on $\mathcal{A}$ is called a* ***C*-finitely correlated state*** *generated by $(\mathcal{B}, E, \omega)$ if $\mathcal{B}$ is a finite C*-algebra called an* ***auxiliary algebra****, $E : \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{B} \to \mathcal{B}$ is a unital completely positive map called a* ***transfer operator*** *and ω is a state on $\mathcal{B}$ such that*

$$\omega(a) = \omega(E(\mathbf{1} \otimes a)) \tag{3.4}$$

for all $a \in \mathcal{B}$ and

$$\phi(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \omega(E(a_1 \otimes E(a_2 \otimes \dots E(a_n \otimes \mathbf{1}_{\mathcal{B}}) \dots))) \tag{3.5}$$

for all $a_1, a_2, \dots, a_n \in \mathcal{M}_d(\mathbb{C})$.

From (3.4) follows that the C*-finitely correlated states are translation-invariant automatically, moreover the class of C*-finitely correlated states is shown to be a *-weakly dense convex subset of the set of translation-invariant states, which is important for the possibility of using these states as trial states in variational computations [23]. It is clear that a C*-finitely correlated state is a homogenous quantum Markov chain with the choice $\mathcal{B} = \mathcal{M}_d(\mathbb{C})$ and $\omega = \phi_0$. The topic of finitely correlated states is an extendedly investigated field in mathematical physics [23, 24, 42].

We arrived to the quantum Markov chains based on the generalization of the concept of transition matrix, but it is possible to make the non-commutative generalization via the concept of conditional expectation too. As we have seen in the Example 2.4.2, a state-preserving conditional expectation does not exist in general: states compatible with norm one projections tend to be trivial in the extremely non-commutative case, i.e. factorial case, that is algebras with trivial center. This fact led *L. Accardi* and *C. Cecchini* to modify the definition in the following way [5].

Definition 3.1.3 *Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of C^*-subalgebras. A* ***quasi-conditional expectation*** *w.r.t the given triplet, is a completely positive, identity-preserving linear map $E : \mathcal{A} \to \mathcal{B}$ such that*

$$E(ca) = cE(a), \qquad a \in \mathcal{A}, c \in \mathcal{C}. \tag{3.6}$$

One can check that $E(ac) = E(a)c$ also holds for all $a \in \mathcal{A}$ and $c \in \mathcal{C}$. Equivalently E can be characterized as a completely positive identity-preserving map $\mathcal{A} \to \mathcal{B}$ whose fixed point algebra contains $\mathcal{C}$. Condition (3.6) implies that

$$E(\mathcal{A} \setminus \mathcal{C}) \subset \mathcal{B} \setminus \mathcal{C} \tag{3.7}$$

and is natural to refer (3.7) as **quantum Markov property**. The terminology being justified by the fact that if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian algebras and E is a conditional expectation in the usual sense, then (3.7) is an equivalent form of the classical Markov property. *L. Accardi* and *A. Frigerio* give the definition of the Markov state on $\mathcal{A} \equiv \otimes_i \mathcal{A}_i$ in the following way [7].

Definition 3.1.4 *A state ϕ on $\mathcal{A}$ is called a* ***Markov state*** *with respect to the localization $\{\mathcal{A}_{[0,n]}\}$ if for each $n \in \mathbb{N}$ there exists a quasi-conditional expectation E_n with respect to the triple $\mathcal{A}_{[0,n-1]} \subset \mathcal{A}_{[0,n]} \subset \mathcal{A}_{[0,n+1]}$ such that*

$$\phi(a) = \phi_{[0,n]}(E_n(a)), \qquad a \in \mathcal{A}_{[0,n+1]}. \tag{3.8}$$

Instead of (3.8) we can write $\phi_{[0,n+1]} = \phi_{[0,n]} \circ E_n$ too, and we shall say that the quasi-conditional expectation E_n is **compatible** with the state ϕ. Moreover,

$$\phi_{[0,n+1]} = \phi_0 \circ E_1 \circ E_2 \circ \cdots \circ E_n \tag{3.9}$$

holds, which is the analogoue of (1.5). It is clear that the quantum Markov property (3.7) has got the following form:

$$E_n(\mathcal{A}_{[n,n+1]}) \subset \mathcal{A}_n. \tag{3.10}$$

In the light of the Example 2.5.3, we can take a conditional expectation instead of the quasi-conditional expectation E_n, as the following theorem shows [52, 57, 59].

Theorem 3.1.5 *For all $a \in \mathcal{A}_{[0,n+1]}$, there exists*

$$F_n = \lim_{k\to\infty} \frac{1}{k+1} \sum_{i=0}^{k} E_n^i(a),$$

and F_n is a conditional expectation from $\mathcal{A}_{[0,n+1]}$ into $\mathcal{A}_{[0,n]}$, with

$$\mathcal{A}_{[0,n-1]} \subset \mathrm{Ran}(F_n) = \mathcal{F}(E_n),$$

where $\mathcal{F}(E_n)$ is the fixed point algebra of E_n. Moreover F_n satisfies

$$\phi_{[0,n+1]} = \phi_{[0,n]} \circ F_n.$$

To clarify the connection between Markov states and Markov chains, observe that $\mathcal{A}_{[0,n-1]} \subset \mathrm{Ran}(E_n)$ and E_n acts on $\mathcal{A}_{[0,n-1]}$ identically, so there exists a conditional expectation $\mathcal{E}_n : \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ such that

$$E_n = \mathbf{1}_{\mathcal{A}_{[0,n-1]}} \otimes \mathcal{E}_n$$

holds for all $n \in \mathbb{N}$. According to (3.9), one can give the following characterization.

Theorem 3.1.6 *The following statements are equivalent.*

1. *ϕ is a Markov state on $\mathcal{A}$.*

2. *There exists a sequence of conditional expectation $E_n : \mathcal{A}_{[0,n+1]} \to \mathcal{A}_{[0,n]}$, with $\mathcal{A}_{[0,n-1]} \subset \mathrm{Ran}(E_n)$ such that*

$$\phi_{[0,n+1]} = \phi_{[0,n]} \circ E_n.$$

3. *There exists a sequence of conditional expectation $\mathcal{E}_n : \mathcal{M}_d \otimes \mathcal{M}_d \to \mathcal{M}_d$ such that*

$$\phi(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \phi_0(\mathcal{E}_1(a_1 \otimes \mathcal{E}_2(a_2 \otimes \ldots \mathcal{E}_{n-1}(a_{n-1} \otimes a_n) \ldots))) \tag{3.11}$$

   for all $a_1, a_2, \ldots, a_n \in \mathcal{M}_d(\mathbb{C})$.

We have seen in the Theorem 1.2.1, that in the classical case the Markov chains of three random variables are completely characterized by the strong additivity of Shannon entropy. This induces to restrict ourself to three-partite Markov systems, which can be regarded to the building bricks of Markov chains or states. Our general definition is the following.

Definition 3.1.7 *Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of C*-subalgebras. A state ϕ on $\mathcal{A}$ is called a **short Markov state** or **Markov triplet**, if there exists a quasi-conditional expectation E w.r.t the given triplet, i.e. a completely positive, identity-preserving linear map $E : \mathcal{A} \to \mathcal{B}$ such that*

1. $E(ca) = cE(a), \qquad a \in \mathcal{A}, c \in \mathcal{C},$
2. $E(\mathcal{A} \setminus \mathcal{C}) \subset \mathcal{B} \setminus \mathcal{C}$

compatible with ϕ, that is

$$\phi = \phi_{\mathcal{B}} \circ E. \tag{3.12}$$

Our goal is the analysis of such systems.

3.2 Characterization of Markov triplets in the three-fold tensor product case

Forming direct sums and tensor products of Hilbert spaces provides two general ways of building larger spaces from given ones. If we like to describe a composite system of different particles it is usual to use the tensor product. In the tensor product case the characterization of the Markov triplets is well known [59].

Theorem 3.2.1 *Let ϕ_{123} be a state on the tensor product of matrix algebras $\mathcal{A}_{123} \equiv \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$. The reduced states will be denoted by ϕ_{12}, ϕ_{23} and ϕ_2. The following conditions are equivalent.*

(i) ϕ_{123} *is a Markov triplet on* $\mathcal{A}_{123}$.

(ii) *The equality*

$$S(\phi_{123}) + S(\phi_2) = S(\phi_{12}) + S(\phi_{23})$$

holds in the strong subadditivity of the von Neumann entropy.

(iii) *If τ denotes the tracial state of $\mathcal{A}_1$, then*

$$S(\phi_{123} \| \tau \otimes \phi_{23}) = S(\phi_{12} \| \tau \otimes \phi_2).$$

(iv) There exists a subalgebra $\mathcal{B}$ and a conditional expectation from $\mathcal{A}_{123}$ onto $\mathcal{B}$ such that

$$\mathcal{A}_1 \subset \mathcal{B} \subset \mathcal{A}_{12}$$

and E leaves the state ϕ_{123} invariant.

(v) There is a state transformation

$$\mathcal{E} : \mathcal{A}_2 \to \mathcal{A}_{23}$$

such that $(\mathbf{1}_1 \otimes \mathcal{E})(\phi_{12}) = \phi_{123}$.

Moreover, it is possible to give the exact form of the density matrix of Markov triplets: every Markov state is the convex combination of orthogonal product type states [29].

Theorem 3.2.2 *Let ρ_{ABC} be a density matrix on the finite dimensional tensor product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Then ρ_{ABC} is a density matrix of a Markov triplet if and only if there exists an orthogonal decomposition of $\mathcal{H}_B$*

$$\mathcal{H}_B = \bigoplus_k \mathcal{H}_k^L \otimes \mathcal{H}_k^R$$

and for every k there are density matrices $\rho_{AL}^k \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_k^L)$ and $\rho_{RC}^k \in \mathcal{B}(\mathcal{H}_k^R \otimes \mathcal{H}_C)$ such that ρ_{ABC} is a convex combination

$$\rho_{ABC} = \sum_k p_k \rho_{AL}^k \otimes \rho_{RC}^k.$$

The theorem also holds in infinite dimensional Hilbert space if all the von Neumann entropies $S(\rho_{AB})$, $S(\rho_B)$ and $S(\rho_{BC})$ are finite [32]. It is a remarkable fact that the restriction ρ_{AC} is a product state of its marginals, i.e. the 'past' and the 'future' are indepedent in the tensor product case.

In the next chapter we turn to a system which localization does not follow the tensor product localization, hence the conditions of the Theorems 3.2.1 and 3.2.2 are not realized.

Chapter 4

CAR algebras and quasi-free states

4.1 The CAR algebra

The quantum-mechanical states of n identical point particles in the configuration space $\mathbb{R}^\nu$ are given by vectors of the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^{\nu n})$. If $\psi \in \mathcal{H}$ is normalized, then

$$\mathrm{d}p(x_1, \dots, x_n) = |\psi(x_1, \dots, x_n)|^2 \mathrm{d}x_1 \dots \mathrm{d}x_n$$

is the probability density for ψ to describe n particles at the infinitesimal neighborhood of the points $x_1, \dots, x_n$. The normalization of ψ corresponds to the normalization of the total probability to unity. But in microscopic physics identical particles are indistinguishable and this is reflected by the symmetry of the probability density under interchange of the particle coordinates. This interchange defines an unitary representation of the permutation group and the symmetry is assured if the ψ transform under a suitable subrepresentation. There are two cases of paramount importance. The first arises when ψ is symmetric under change of coordinates. Particles whose states transform in this manner are called **bosons** and are said to satisfy **Bose-Einstein statistics**. The second case corresponds to anti-symmetry of ψ under interchange of each pair of coordinates. The associated particles are called **fermions** and said to satisfy **Fermi-Dirac statistics**. The elementary constituents of matter seem to be fermions, while interactions are mediated by bosons. Our goal to investigate both systems from the wievpoint of Markovianity. We start with the fermions.

The models of interacting fermions should suffice to understand the properties of matter in standard conditions, but some quasi-particles in solid state physics are also well-described as fermionic systems. Systems of fermions, even if they are non-interacting, still exhibit some strong correlations due to their statistics. In this section we introduce the abstract CAR algebra (the algebra of the canonical anti-commutation relation), which is used to the description of fermion systems and we will pass to a particular, but very important and suggestive representation in the next one. The works [8, 11, 16] contain all what we need with other details.

Definition 4.1.1 *For $I \subset \mathbb{Z}$, the unital C^*-algebra $\mathcal{A}(I)$ generated by the elements satisfying the* ***canonical anti-commutation relations****, i.e.*

$$\begin{aligned} a_i a_j + a_j a_i &= 0, && (4.1)\\ a_i a_j^* + a_j^* a_i &= \delta_{i,j}\mathbf{1} && (4.2) \end{aligned}$$

for $i, j \in I$ is called a ***CAR algebra*** *over I. The operators a^* and a are often called* ***creation*** *and* ***annihilation operators****, respectively.*

It is easy to see that $\mathcal{A}(I)$ is the linear span of the identity and monomials of the form

$$A_{i(1)}A_{i(2)}\dots A_{i(k)}, \tag{4.3}$$

where $i(1) < i(2) < \cdots < i(k)$ and each factor $A_{i(j)}$ is one of the four operators $a_{i(j)}, a^*_{i(j)}, a_{i(j)}a^*_{i(j)}$ and $a^*_{i(j)}a_{i(j)}$. The CAR algebra $\mathcal{A}$ is defined by

$$\mathcal{A} \equiv \overline{\bigvee_{l\in L} \mathcal{A}\left(\{l\}\right)}^{C^*}.$$

It is known that for $I = \{1, 2, \dots, n\}$, $\mathcal{A}(I)$ is isomorphic to a matrix algebra $M_{2^n}(\mathbb{C}) \simeq \overbrace{M_2(\mathbb{C})}^{1} \otimes \cdots \otimes \overbrace{M_2(\mathbb{C})}^{n}$. An explicit isomorphism is given by the so-called **Jordan-Wigner isomorphism**. Namely, the relations

$$\begin{aligned} e_{11}^{(i)} &:= a_i a_i^* & e_{12}^{(i)} &:= V_{i-1}a_i\\ e_{21}^{(i)} &:= V_{i-1}a_i^* & e_{22}^{(i)} &:= a_i^* a_i \end{aligned}$$

$$V_i := \prod_{j=1}^{i}(I - 2a_j^* a_j)$$

determine a family of mutually commuting 2×2 matrix units for $i \in I$. Since

$$a_i = \prod_{j=1}^{i-1}\left(e_{11}^{(j)} - e_{22}^{(j)}\right)e_{12}^{(i)},$$

the above matrix units generate $\mathcal{A}(I)$ and give an isomorphism between $\mathcal{A}(I)$ and the n-fold tensor product $M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$:

$$e^{(1)}_{i_1 j_2} e^{(2)}_{i_2 j_2} \ldots e^{(n)}_{i_n j_n} \longleftrightarrow e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_n j_n}. \tag{4.4}$$

(Here e_{ij} stand for the standard matrix units in $M_2(\mathbb{C})$.) For any subset $J \subset \mathbb{Z}$, there exists a unique automorphism Θ_J of $\mathcal{A}(\mathbb{Z})$, called **parity automorphism** such that

$$\begin{aligned} &\Theta_J(a_i) = -a_i \text{ and } \Theta_J(a_i^*) = -a_i^*, \quad \text{whenever } i \in J \\ &\Theta_J(a_i) = a_i \text{ and } \Theta_J(a_i^*) = a_i^*, \quad \text{whenever } i \notin J. \end{aligned}$$

In particular, we write Θ instead of $\Theta_{\mathbb{Z}}$. Θ_J is inner i.e. there exists a self-adjoint unitary v_J in $\mathcal{A}(J)$ given by

$$v_J \equiv \prod_{i \in J} v_i, \quad v_i \equiv a_i^* a_i - a_i a_i^*, \tag{4.5}$$

such that $\Theta_J(A) = (\mathrm{Ad}v_J)A \equiv v_J A v_J^*$ for any $A \in \mathcal{A}(J)$. Then the **odd** and **even parts** of $\mathcal{A}_I$ are defined as

$$\mathcal{A}(I)^+ := \{A \in \mathcal{A}(I) : \Theta_I(A) = A\}, \quad \mathcal{A}(I)^- := \{A \in \mathcal{A}(I) : \Theta_I(A) = -A\}. \tag{4.6}$$

Remark that $\mathcal{A}(I)^+$ is a subalgebra but $\mathcal{A}(I)^-$ is not. The **graded commutation relation** for CAR algebras is known: if $A \in \mathcal{A}(K)$ and $B \in \mathcal{A}(L)$ where $K \cap L = \emptyset$, then

$$AB = \epsilon(A, B)BA \tag{4.7}$$

where

$$\epsilon(A, B) = \begin{cases} -1 & \text{if A and B are odd} \\ +1 & \text{otherwise.} \end{cases} \tag{4.8}$$

The parity automorphism is the special case of the action of the **gauge group** $\{\alpha_\vartheta : 0 \le \vartheta < 2\pi\}$ with

$$\alpha_\vartheta(a_i) = e^{-i\vartheta} a_i.$$

An element $A \in \mathcal{A}$ is **gauge-invariant** if $\alpha_\vartheta(A) = A$ for all $0 \le \vartheta < 2\pi$. A state ϕ on the CAR algebra $\mathcal{A}$ is called **even state** if it is Θ-invariant:

$$\phi(\Theta(A)) = \phi(A) \tag{4.9}$$

for all $A \in \mathcal{A}$. Note that $\phi(A) = 0$ for all $A \in \mathcal{A}^-$ is equivalent to the condition that ϕ is an even state of $\mathcal{A}$. Let I and J be two disjoint subsets of $\mathbb{Z}$. We say that ϕ is a **product state** with respect to $\mathcal{A}(I)$ and $\mathcal{A}(J)$, if

$$\phi(AB) = \phi(A)\phi(B) \tag{4.10}$$

holds for all $A \in \mathcal{A}(I)$ and $B \in \mathcal{A}(J)$. If a state ϕ of the joint system $\mathcal{A}(I \cup J)$ (which is the same as the C*-algebra generated by $\mathcal{A}(I)$ and $\mathcal{A}(J)$) coincides with ϕ_I on $\mathcal{A}(I)$ and ω_J on $\mathcal{A}(J)$, i.e.

$$\begin{aligned} \phi(A) &= \phi_I(A), \quad \forall A \in \mathcal{A}(I), & (4.11) \\ \phi(B) &= \omega_J(B), \quad \forall B \in \mathcal{A}(J), & (4.12) \end{aligned}$$

then ϕ is called the **joint extension** of ϕ_I and ω_J. As a special case, if

$$\phi(AB) = \phi_I(A)\omega_J(B) \equiv \phi_I \wedge \omega_J(AB), \tag{4.13}$$

then $\phi = \phi_I \wedge \omega_J$ is called a **product state extension** of ϕ_I and ω_J. A product state extension does not exist unconditionally. Indeed, suppose that both A and B are odd elements. If the product state extension of ϕ_I and ω_J exist, then

$$\begin{aligned} \phi_I(A)\omega_J(B) &= \phi_I \wedge \omega_J(AB) = \overline{\phi_I \wedge \omega_J((AB)^*)} \\ &= \overline{\phi_I \wedge \omega_J(B^*A^*)} = -\overline{\phi_I \wedge \omega_J(A^*B^*)} \\ &= -\overline{\phi_I(A^*)\omega_J(B^*)} = -\phi_I(A)\omega_J(B), \end{aligned}$$

where we have used (4.7). This shows that at least one of two states must be even, i.e. must vanish on odd elements. This result was generalized in [12] in the following form:

Theorem 4.1.2 *Let $I_1, I_2, \ldots$ be an arbitrary (finite or infinite) number of mutually disjoint subsets of $\mathbb{Z}$ and ϕ_i be a given state of $\mathcal{A}(I_i)$ for each i.*

1. *A product state extension of ϕ_i, $i = 1, 2, \ldots$ exists if and only if all states ϕ_i except at most one are even. It is unique if it exists. It is even if and only if all ϕ_i are even.*

2. *Suppose that all ϕ_i are pure. If there exists a joint extension of ϕ_i, $i = 1, 2, \ldots,$ then all states except at most one have to be even. If this is the case, the joint extension is uniquely given by the product state extension and is a pure state.*

A state τ is called **tracial state** if $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$. We remark that the existence of a tracial state follows from the isomorphism (4.4) immediately. We will use the following lemma [11].

Lemma 4.1.1 *A tracial state τ is an even product state.*

The **right shift automorphism** γ on $\mathcal{A} \equiv \mathcal{A}(\mathbb{Z})$ is defined by $\gamma(a_i) = a_{i+1}$ and $\gamma(a_i^*) = a_{i+1}^*$ for all $a_i, a_i^* \in \mathcal{A}$, $i \in \mathbb{Z}$. A state ϕ on $\mathcal{A}$ is **translation-invariant** if $\phi \circ \gamma = \phi$ holds. It is important to know that any translation-invariant state is automatically even [16]. Recall, that for a subset $\mathcal{B}$ of a C*-algebra $\mathcal{A}$ the commutant of $\mathcal{B}$ is defined by

$$\mathcal{B}' = \{A \in \mathcal{A} : AB = BA \quad \text{for all} \quad B \in \mathcal{B}\}. \tag{4.14}$$

It is a unital subalgebra of $\mathcal{A}$ and plays an important role in our investigations. The commutants in the CAR algebra are given by the following theorem [11].

Theorem 4.1.3 *For a finite $I \subset \mathbb{Z}$,*

1. $\mathcal{A}(I)' \cap \mathcal{A} = \mathcal{A}\left(I^c\right)^+ + v_I \mathcal{A}\left(I^c\right)^-$
2. $\left(\mathcal{A}(I)^+\right)' \cap \mathcal{A} = \mathcal{A}\left(I^c\right) + v_I \mathcal{A}\left(I^c\right),$

where I^c denotes the complementer set of I.

As we have seen, the fundamental object of the Markov states is the conditional expectation. Now we investigate the existence of the conditional expectation in the CAR algebra case. Recall that by a (Umegaki) conditional expectation $E : \mathcal{A} \to \mathcal{B} \subset \mathcal{A}$ we mean a norm-one projection of the C^*-algebra $\mathcal{A}$ onto the C^*-subalgebra $\mathcal{B}$. One can check that the map E is automatically a completely positive identity-preserving $\mathcal{B}$-bimodule map. For CAR algebras the existence of a conditional expectation which preserves the tracial state τ follows from generalities about conditional expectations or the isomorphism (4.4). Inspite of these, it is useful to give a construction for it followed the original proof [11].

Lemma 4.1.2 *Let $J \subset I$. Then $\mathcal{A}(J) \subset \mathcal{A}(I)$ and there exists a unique conditional expectation $E_J^I : \mathcal{A}(I) \to \mathcal{A}(J)$ which preserves the trace, i.e. $\tau \circ E_J^I = \tau$.*

Proof. The C*-algebra generated by the commuting subalgebras $\mathcal{A}(I)$ and $\mathcal{A}(I \setminus J)^+$ is isomorphic to their tensor product. We have a conditional expectation

$$F_1 : \mathcal{A}(I) \to \mathcal{A}(J) \otimes \mathcal{A}(I \setminus J)^+, \quad F_1(A) = \frac{1}{2}\big(A + \Theta_{I \setminus J}(A)\big) \tag{4.15}$$

and another

$$F_2 : \mathcal{A}(J) \otimes \mathcal{A}(I \setminus J)^+ \to \mathcal{A}(J), \quad F_2(A \otimes B) = \tau(B)\, A. \tag{4.16}$$

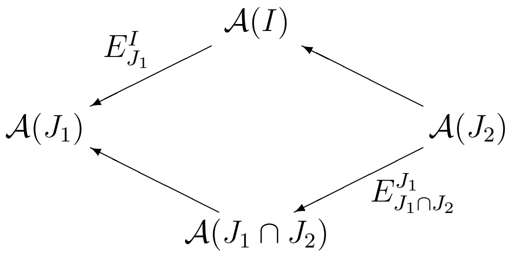
The composition $F_2 \circ F_1$ is E_J^I. □

Example 4.1.4 *Assume that $I = [1,4]$, $J = [1,2]$ and consider the action of the above conditional expectations on terms like (4.3). F_1 keeps $a_1 a_2^* a_2 a_3^* a_4$ fixed and F_2 sends it to $a_1 a_2^* a_2 \tau(a_3^*)\tau(a_4) = 0$. Moreover, E_J^I sends $a_1 a_2^* a_2 a_3 a_3^* a_4^* a_4$ to $a_1 a_2^* a_2 \tau(a_3 a_3^*)\tau(a_4^* a_4)$.*

One can observe, that for arbitrary subsets $J_1, J_2 \subset I$,

$$E_{J_1}^I | \mathcal{A}(J_2) = E_{J_1 \cap J_2}^{J_2} \tag{4.17}$$

holds. This means that we have a commuting square, as in general:



where the arrows $\nwarrow$ consist of embeddings and the arrows $\swarrow$ consist of conditional expectations.

4.2 The anti-symmetric Fock space

In the previous section we summerized the elements of the abstract CAR algebra. Now we pass to a particular, but very important representation of this algebra: to the Fock representation. Assume that the states of each particle form a complex Hilbert space $\mathcal{H}$ and let $\mathcal{H}^{n\otimes} := \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$ denote the n-fold tensor product of $\mathcal{H}$ with itself. We introduce the **Fock space** $\mathcal{F}(\mathcal{H})$ over the Hilbert space $\mathcal{H}$ by

$$\mathcal{F}(\mathcal{H}) = \oplus_{n \geq 0} \mathcal{H}^{n\otimes}, \tag{4.18}$$

where $\mathcal{H}^{0\otimes} \equiv \mathbb{C}$. In general, the permutation group S_n of n objects acts naturally on the n-fold tensor product $\mathcal{H}^{n\otimes}$:

$$U_\pi(f_1 \otimes f_2 \otimes \ldots \otimes f_n) = f_{\pi(1)} \otimes f_{\pi(2)} \otimes \ldots \otimes f_{\pi(n)},$$

where $\pi \in S_n$ and U_π is a unitary representation of S_n on $\mathcal{H}^{n\otimes}$. A vector $\psi \in \mathcal{H}^{n\otimes}$ is **totally anti-symmetric** if

$$U_\pi \psi = \epsilon(\pi)\psi$$

for all $\pi \in S_n$, where $\epsilon(\pi) = \pm 1$ according to whether π is even or odd permutation. The anti-symmetric vectors form the closed subspace $\mathcal{H}^{n\wedge}$ of

$\mathcal{H}^{n\otimes}$. We introduce the notation $f_1 \wedge f_2 \wedge \ldots \wedge f_n$ for the anti-symmetrized tensor product of n vectors and P_a for the projection onto their subspace $\mathcal{H}^{n\wedge}$:

$$f_1 \wedge f_2 \wedge \ldots \wedge f_n \equiv P_a(f_1 \otimes f_2 \otimes \ldots \otimes f_n) = \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} \epsilon(\pi) f_{\pi(1)} \otimes f_{\pi(2)} \otimes \cdots \otimes f_{\pi(n)} \,,$$

where $f_1, f_2, \ldots, f_n \in \mathcal{H}$. We have $\mathcal{H}^{n\wedge} \equiv P_a \mathcal{H}^{n\otimes}$ with this notation. One can compute easily the inner product of two totally anti-symmetric vectors:

$$< f_1 \wedge f_2 \wedge \ldots \wedge f_n | g_1 \wedge g_2 \wedge \ldots \wedge g_n >= \text{Det}\left([< f_i|g_j >]_{i,j}\right). \qquad (4.19)$$

We can construct an orthonormal base of $\mathcal{H}^{n\wedge}$ by anti-symmetrizing a product basis of $\mathcal{H}^{n\otimes}$. Assume that $\{e_1, e_2, \ldots, e_m\}$ is a basis in $\mathcal{H}$. Then the vectors

$$\{e_{i(1)} \wedge e_{i(2)} \wedge \cdots \wedge e_{i(n)} | i_1 < i_2 < \ldots < i_n\}$$

form an orthonormal base in $\mathcal{H}^{n\wedge}$. The dimension of $\mathcal{H}^{n\wedge}$ is therefore given by

$$\dim\left(\mathcal{H}^{n\wedge}\right) = \binom{\dim\mathcal{H}}{n}.$$

In particular, the anti-symmetric tensor powers of $\mathcal{H}$ becomes trivial for $n > \dim\mathcal{H}$. If $\mathcal{H}$ is separable, then $\mathcal{H}^{n\wedge}$ is also separable. The **anti-symmetric** or **fermionic Fock space** $\mathcal{F}_a$ is then defined by

$$\mathcal{F}_a(\mathcal{H}) := P_a \mathcal{F}(\mathcal{H}) = \oplus_{n=\geq 0} \mathcal{H}^{n\wedge}$$

where $\mathcal{H}^{0\wedge} \equiv \mathbb{C}$. The element $1 \in \mathcal{H}^{0\wedge} \equiv \mathbb{C}$ plays an important role. One denotes it by Φ and one calls it the **vacuum vector**, which describes the state without particles. In this spirit the summand $\mathcal{H}^{n\wedge}$ is called the n-particle subspace. By (4.19) the dimension of $\mathcal{F}_a(\mathcal{H})$ with a finite-dimensional one-particle state $\mathcal{H}$ is

$$\dim\left(\mathcal{F}_a(\mathcal{H})\right) = \sum_{n=0}^{\dim\mathcal{H}} \binom{\dim\mathcal{H}}{n} = 2^{\dim\mathcal{H}}. \qquad (4.20)$$

This suggests that if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then

$$\mathcal{F}_a(\mathcal{H}) = \mathcal{F}_a(\mathcal{H}_1) \otimes \mathcal{F}_a(\mathcal{H}_2). \qquad (4.21)$$

Indeed, if $\{f_1, f_2, \ldots\}$ and $\{g_1, g_2, \ldots\}$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then a basis of $\mathcal{H}^{n\wedge}$ is given by

$$f_{i(1)} \wedge f_{i(2)} \wedge \cdots \wedge f_{i(l)} \wedge g_{j(1)} \wedge g_{j(2)} \wedge \cdots \wedge g_{j(n-l)}$$

can be identified for any ordered set $(i(1),\dots,i(l))$ and $(j(1),\dots,j(n-l))$ with

$$f_{i(1)} \wedge f_{i(2)} \wedge \cdots \wedge f_{i(l)} \otimes g_{j(1)} \wedge g_{j(2)} \wedge \cdots \wedge g_{j(n-l)}$$

which is a basis vector in $\mathcal{H}_1^{l\wedge} \otimes \mathcal{H}_2^{(n-l)\wedge}$. We remark that the isomorphism (4.21) is crucial in statistical physics, for example this implies that the entropy of a free fermion gas in equilibrium is an extensive quantity: it is proportional to the volume occupied by the system. The isomorphism would fail without the anti-symmetrization procedure and therefore lead to the **Gibbs paradoxon**, namely, that the entropy density is always infinite or zero. We will see that a similar isomorphism holds in the case of the symmetric or bosonic Fock spaces.

The peculiar structure of Fock space allows the amplification of operators on $\mathcal{H}$ to the whole space $\mathcal{F}_a(\mathcal{H})$ by a method commonly referred to as second quantization. Consider an operator $U \in B(\mathcal{H})$ acting on the one-particle space. Denote

$$U_n(f_1 \wedge f_2 \wedge \cdots \wedge f_n) = Uf_1 \wedge Uf_2 \wedge \cdots \wedge Uf_n, \tag{4.22}$$

and

$$\Gamma(U) = \bigoplus_{n\geq 0} U_n. \tag{4.23}$$

This operator $\Gamma(U)$ is called the **second quantization** of U on $\mathcal{F}_a(\mathcal{H})$. It is easy to see that

$$\Gamma(U_1U_2) = \Gamma(U_1)\Gamma(U_2) \qquad \text{and} \qquad \Gamma(U^*) = \Gamma(U)^*,$$

moreover

$$\Gamma(U)^s = \Gamma(U^s), \quad \forall s \in \mathbb{R} \tag{4.24}$$

are holds. In particular if U is a unitary, then $\Gamma(U)$ is unitary as well. Moreover, if $U(t)$ is a strongly continuous one-parameter group of unitary operators, then so is $\Gamma(U(t))$. In other words, if $U(t) = e^{itA}$ for some self-adjoint operator A, then $\Gamma(U(t)) = e^{it\Delta(A)}$ for some self-adjoint operator $\Delta(A)$ called **differential second quantization of** A. From this reason sometimes it is denoted by $\mathrm{d}\Gamma(A)$ in the literature. One easily checks that

$$\Delta(A) \equiv \mathrm{d}\Gamma(A) = \bigoplus_{n\geq 0} A_n, \tag{4.25}$$

where

$$A_n(f_1 \wedge f_2 \wedge \cdots \wedge f_n) = \sum_{i=1}^{n} f_1 \wedge \cdots \wedge f_{i-1} \wedge Af_i \wedge f_{i+1} \wedge \cdots \wedge f_n. \tag{4.26}$$

A simple computation shows, that if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $U = U_1 \oplus U_2$ such that U_1 and U_2 acts on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then $\Gamma(U_1 \oplus U_2) = \Gamma(U_1) \otimes \Gamma(U_2)$ holds.

Lemma 4.2.1 *Consider an operator $U \in B(\mathcal{H})$, where $\mathcal{H}$ is a finite dimensional Hilbert space. Then*

$$Tr\,\Gamma(U) = Det(\mathbf{1} + U). \tag{4.27}$$

Proof. Suppose that $\dim\mathcal{H} = N$ and let $\{\lambda_i\}$ be the eigenvalues of the operator U with multiplicities. Denote Λ_n the subsets of $\{1, \ldots, \mathrm{Rank}U\}$ with n elements. Then

$$\mathrm{Tr}\,\Gamma(U) = \sum\nolimits_{n=0}^{N} \prod_{i \in \Lambda_n} \lambda_i = \prod_{i=1}^{N} 1 + \lambda_i = \mathrm{Det}\,(\mathbf{1} + U). \tag{4.28}$$

□

Consider the dense linear subspace $\mathcal{F}_a^n(\mathcal{H})$ spanned by vectors

$$0 \oplus \cdots \oplus 0 \oplus (f_1 \wedge \cdots \wedge f_n) \oplus 0 \oplus \ldots.$$

For $f \in \mathcal{H}$ the **creation operator** $a^*(f)$ is defined on $\mathcal{F}_a^n(\mathcal{H})$ as

$$\begin{aligned} a^*(f):\quad & 0 \oplus \cdots \oplus 0 \oplus (f_1 \wedge \cdots \wedge f_n) \oplus 0 \oplus \ldots \mapsto \\ & 0 \oplus \cdots \oplus 0 \oplus (f \wedge f_1 \wedge \cdots \wedge f_n) \oplus 0 \oplus \ldots. \end{aligned}$$

$a^*(f)$ is linear in the variable f and it maps the n-particle subspace into the $(n+1)$-particle subspace. Some computation shows that its adjoint is given by

$$\begin{aligned} & a(f): 0 \oplus \cdots \oplus 0 \oplus (f_1 \wedge \cdots \wedge f_n) \oplus 0 \oplus \ldots \mapsto \\ & \sum_{j=1}^{n} (-1)^{j+1} \langle f | f_j \rangle 0 \oplus \cdots \oplus 0 \oplus (f_1 \wedge \cdots \wedge f_{j-1} \wedge f_{j+1} \wedge \cdots \wedge f_n) \oplus 0 \oplus \ldots \end{aligned}$$

and called the **annihilation operator**. One can check that they satisfy the anticommutation relations

$$a(f)a(g) + a(g)a(f) = 0, \tag{4.29}$$

$$a(f)a(g)^* + a(g)^*a(f) = \langle f | g \rangle \mathbf{1} \tag{4.30}$$

on $\mathcal{F}_a^n(\mathcal{H})$ and $a(f)$ and $a^*(f)$ continously extends to bounded linear operators on $\mathcal{F}_a(\mathcal{H})$, which will denoted by the same letters and this extensions also satisfy (4.29). Hence, we arrive to a representation of the CAR algebra on the anti-symmetric Fock space. Indeed, consider the Hilbert space $l^2(\mathbb{Z})$ with the canonical orthonormal basis $\{\delta_k : k \in \mathbb{Z}\}$. For $I \subset \mathbb{Z}$, the CAR algebra $\mathcal{A}(I)$ is isomorph with the C*-algebra generated by $\{a(\delta_k) : k \in I\}$. Any annihilation operator $a(f)$ kills the vacuum, i.e.

$$a(f)\Phi = 0$$

for all $f \in \mathcal{H}$ and if we apply successively n creation operators on Φ, then we end up with an n-particle vector

$$a^*(f_1)a^*(f_2)\ldots a^*(f_n)\Phi = f_1 \wedge f_2 \wedge \cdots \wedge f_n. \tag{4.31}$$

(This fact motivates the title 'creation'.) Note that if $f_i = f_j$ for some pair i, j with $1 \leq i < j \leq n$, then

$$a^*(f_1)a^*(f_2)\ldots a^*(f_n)\Phi = f_1 \wedge f_2 \wedge \cdots \wedge f_n = 0,$$

by anti-symmetry. This is the famous **Pauli principle**, which tells that it is impossible to create two fermions in the same state. It is a remarkable property that all bounded linear operators on the antisymmetric Fock space can be approximated by non-commutative polynomials in the $a(f)$ and $a^*(f)$ in the weak operator topology. For details we refer [8, 16].

4.3 Quasi-free states

In classical probability, a Gaussian measure leads to a characteristic function which is the exponential of a quadratic form. Its logarithm is therefore a quadratic polynomial, and all correlations beyond the second order vanish. In the CAR case for even states it is also possible to define the useful concept of the correlation function (cumulants or truncated function in other words), see [16] for details. *E. Baslev* and *A. Verbeure* defined a quasi-free state of the CAR algebra to be an even state for which the correlation functions vanish for $n \geq 3$ [13]. For gauge-invariant states this is equivalent to the earlier definition of *D. Shale* and *W. Stinespring* [70]. Before giving the definition, we introduce some notations. Let $\mathcal{H}$ be a separable Hilbert space. The CAR algebra $\mathcal{A}(\mathcal{H})$ is the universal C*-algebra generated by the unit element $\mathbf{1}$ and by $\{a(f)|f \in \mathcal{H}\}$ such that the map $f \mapsto a(f)$ is complex antilinear and the usual canonical anticommutation relations (4.29) and (4.30) are hold. With this notations one can define the following state.

Definition 4.3.1 *Let Q be a bounded linear operator on $\mathcal{H}$, such that $0 \leq Q \leq \mathbf{1}$. A linear functional ω_Q on the CAR algebra $\mathcal{A}(\mathcal{H})$ defined by*

$$\omega_Q(\mathbf{1}) = 1, \tag{4.32}$$

$$\omega_Q(a^*(f_1)\dots a^*(f_m)a(g_n)\dots a(g_1)) := \delta_{mn} Det\left([\langle g_i|Qf_j\rangle]_{ij}\right) \tag{4.33}$$

*is called a **quasi-free state** with the **symbol** Q.*

We remark that the choice of Q to be a positive contraction is necessary and sufficient to having a state. A state ω_Q is gauge-invariant automatically by the construction. (4.33) shows that for a gauge-invariant quasi-free state, the n-point functions, that is the correlations are wholly determined by these 2-point functions. That was the origin of the earlier definition of Baslev and Verbeure [13].

Example 4.3.2 *Suppose an interaction-free fermionic system described by a Hamiltonian H with discrete spectrum $\{h_i\}$, such that $e^{-\beta\hat{H}}$ is a trace-class operator for $\beta > 0$, where $\hat{H} \equiv \mathrm{d}\Gamma(H)$ is the second-quantized Hamiltonian. The Gibbs state ρ_β of the system in the Fock representation has density operator $\frac{e^{-\beta\hat{H}}}{Tr e^{-\beta\hat{H}}}$, and its value on monomials can be expressed as*

$$\rho_\beta(a^*(f_1)\dots a^*(f_m)a(g_n)\dots a(g_1)) := \delta_{mn} \det\left[\langle g_i, Qf_j\rangle\right]_{i,j=1}^n, \tag{4.34}$$

where Q has the same eigenvectors as H with corresponding eigenvalues $\frac{e^{-\beta h_i}}{Tr e^{-\beta h_i}}$. So the quasi-free states can be considered as the generalization of canonical Gibbs states for non-interacting systems.

We give the statistical operator of a fermionic quasi-free state in Fock representation.

Lemma 4.3.1 *The density of a quasi-free state ω_Q in the Fock representation is given by*

$$\rho_Q = \frac{\Gamma(Q(\mathbf{1}-Q)^{-1}))}{Tr\Gamma(Q(\mathbf{1}-Q)^{-1}))}. \tag{4.35}$$

Proof. Since ω_Q is normal, there exists a positive trace class operator S which for

$$\rho_Q = \frac{\Gamma(S)}{\mathrm{Tr}\,\Gamma(S)}. \tag{4.36}$$

One can check easily that

$$\Gamma(U)a^*(f) = a^*(Uf)\Gamma(U) \tag{4.37}$$

holds for any $U \in B(\mathcal{H})$. From (4.33) follows that the 2-point functions are

$$\omega_Q(a^*(f)a(g)) = \langle g|Qf\rangle. \tag{4.38}$$

Then we have

$$\begin{aligned}
\omega_A(a^*(f)a(g)) &= \frac{1}{\mathrm{Tr}\,\Gamma(S)}\mathrm{Tr}\;(\Gamma(S)a^*(f)a(g)) \\
&= \frac{1}{\mathrm{Tr}\,\Gamma(S)}\mathrm{Tr}\;(a(g)a^*(Sf)\Gamma(S)) \\
&= \langle g|Sf\rangle - \omega_Q\,(a^*(Sf)a(g)) \\
&= \langle g|Sf\rangle - \langle g|QSf\rangle = \langle g|Qf\rangle,
\end{aligned}$$

for all $f, g \in \mathcal{H}$, where we have use the trace property and (4.37) in the second line and the anticommutation relations for the creation and the annihilation operators in the third one. This means that $Q = S - QS$ which gives

$$S = \frac{Q}{\mathbf{1} - Q}$$

as we stated. □

We collect some facts about the quasi-free states. All these results with proofs can be found in [8, 16, 21, 72]. Obviously the quasi-free states are even. A quasi-free state is pure if and only if its symbol is a projection.

Consider the Hilbert space $l^2(\mathbb{Z})$ with the canonical orthonormal basis $\{\delta_k : k \in \mathbb{Z}\}$, then for any $I \subset \mathbb{Z}$, the C*-algebra $\mathcal{A}(I)$ generated by $\{a_k \equiv a(\delta_k) : k \in I\}$ satisfies the anticommutation relations (4.29). Let P_n be the projection from $l^2(\mathbb{Z})$ onto the finite-dimensional subspace spanned by $\{\delta_1, \dots \delta_n\}$. The restriction of ω_Q to the subalgebra $\mathcal{A}(P_n l^2(\mathbb{Z}))$ is again quasi-free with symbol $P_n Q P_n$. The quasi-free state ω_Q is translation-invariant if and only if its symbol Q is a Töplitz-operator, i.e. there exists a sequence $\{q_k : k \in \mathbb{Z}\}$ such that $Q_{lk} = q(k-l)$. By the Fourier transform,

$$\hat{q}(\vartheta) = \sum_{k \in \mathbb{Z}} q_k e^{ik\vartheta} \tag{4.39}$$

and its inverse

$$q_k = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{d}\vartheta \hat{q}(\vartheta) e^{-ik\vartheta}, \tag{4.40}$$

with the circle $\mathbb{T}$ parametrized by $[0, 2\pi)$, the symbol of a translation-invariant quasi-free state is unitarily equivalent with the multiplication operator by $\hat{q}$ on $L^2(\mathbb{T}, \mathrm{d}\vartheta)$. Moreover, $0 \le \hat{q} \le 1$ holds almost everywhere. For a pure (i.e. Q is a projection) translation-invariant quasi-free state ω_Q this means that the Fourier transform of the symbol is a characteristic function, i.e. there exists a measurable set $K \subset \mathbb{T}$ such that $\hat{q}(\vartheta) = \chi_K(\vartheta)$.

Now we turn to the characterization of quasi-free product states. This result can be found in [63]. At first let recall the basic notions. For a pair of disjoint subsets I_1 and I_2 of $\mathbb{Z}$, let ϕ_1 and ϕ_2 be given states of the CAR algebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$, respectively. If for a state ϕ of the joint system $\mathcal{A}(I_1 \cup I_2)$ (which coincides with the C^*-algebra generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$)

$$\phi(A_1 A_2) = \phi_1(A_1)\phi_2(A_2) \tag{4.41}$$

holds for all $A_1 \in \mathcal{A}(I_1)$ and all $A_2 \in \mathcal{A}(I_2)$, then ϕ is called a **product state extension** of ϕ_1 and ϕ_2 (or with respect to the subalgebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$). We will use the notation $\phi = \phi_1 \circ \phi_2$. As we have seen in the Theorem 4.1.2, for an arbitrary (finite or infinite) number of given subsystems, a product state extension is shown to exist if and only if all states of subsystems except at most one are even. Since any restriction of a quasi-free state is again quasi-free, it is clear that if a quasi-free state is product state it must be a product of quasi-free states. Let $Q \in M_n(\mathbb{C})$ be a positive contraction, the symbol of the gauge-invariant CAR quasi-free state ω_Q. Denote $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ two disjoint ordered subsets of the index set $K = (1, \dots, n) \subset \mathbb{Z}$, such that $I \cup J = K$, i.e. $k + l = n$. Let P_I be the projection from $\mathbb{C}^n$ onto the finite-dimensional subspace spanned by the subset of the canonical orthonormal basis $\{\delta_{i_1}, \dots, \delta_{i_k}\}$. Obviously $P_J = \mathbf{1}_n - P_I$ is a projection onto the orthogonal subspace spanned by $\{\delta_{j_1}, \dots, \delta_{j_l}\}$. Consider the CAR algebras $\mathcal{A}\left(P_I l^2(\mathbb{Z})\right)$ and $\mathcal{A}\left(P_J l^2(\mathbb{Z})\right)$ as the subsystems of a bipartite system. The symbols of the restrictions of the quasi-free state ω_Q to the subalgebras are given by $Q_I = P_I Q P_I$ and $Q_J = P_J Q P_J$. Then for the quasi-free product states the following is hold.

Theorem 4.3.3 *With the conditions above ω_Q is a product state with respect to the subsystems $\mathcal{A}(I)$ and $\mathcal{A}(J)$, i.e. $\omega_Q = \omega_{Q_I} \circ \omega_{Q_J}$ if and only if for its symbol Q the condition*

$$Q_{i_r, j_s} = 0 \tag{4.42}$$

holds for all $r = 1, \dots, k$ and $s = 1, \dots, l$.

Proof. For the necessity let suppose that ω_Q is a product state, i.e.

$$\omega_Q(A_I A_J) = \omega_{Q_I}(A_I)\omega_{Q_J}(A_J)$$

holds for all $A_I \in \mathcal{A}(I)$ and $A_J \in \mathcal{A}(J)$. As the quasi-free states are even states, it must be disappear on the products $A_I A_J$ either A_I or A_J is odd. Particularly

$$\omega_Q(a^*_{i_r} a_{j_s}) = Q_{i_r, j_s} = 0, \tag{4.43}$$

as we stated. We show that it is also a sufficient condition. Indeed, this condition make possible to arrange Q to a block-diagonal form by interchangements of columns and rows. Suppose that by interchanging the columns N times we can obtain a matrix in which the indices of the first k columns are $i_1, \dots, i_k$ respectively. Since Q is symmetric, with N appropriate interchangements of the rows we arrive to the following block-diagonal matrix:

$$\begin{bmatrix} Q_I & 0 \\ 0 & Q_J \end{bmatrix}. \tag{4.44}$$

As we get this matrix by $2N$ interchangements of rows and columns, the determinant did not changed. We get

$$\det Q = \det Q_I \det Q_J, \tag{4.45}$$

which is equivalent with our statement. $\square$

In the next section we compute some entropy quantities of CAR quasi-free states.

4.3.1 Entropy related quantities of CAR quasi-free states

A great advantage of the quasi-free states ω_Q that the most of entropy related quantities have an explicit expression by their symbol Q. For details we refer [8, 19].

Lemma 4.3.2 *Let ω_Q be a gauge-invariant quasi-free state determined by the symbol $Q = \sum_i q_i |e_i\rangle\langle e_i|$. Then the density matrix of ω_Q on the antisymmetric Fock space $\mathcal{F}_a(\mathbb{C}^n)$ factorizes into*

$$\rho_Q = \otimes_{j=1}^n \begin{pmatrix} q_i & 0 \\ 0 & 1 - q_i \end{pmatrix}$$

with respect to the decomposition

$$\mathcal{F}_a(\mathbb{C}^n) = \otimes_{j=1}^n \mathcal{F}_a(\mathbb{C}|e_j\rangle) \equiv \otimes^n \mathbb{C}^2 \equiv \mathbb{C}^{2^n}.$$

Its spectrum is given by

$$\mathrm{Sp}(\rho_Q) = \{r_\Lambda | \Lambda \subset \{1, 2, \dots, n\}\}$$

with

$$r_\Lambda = \prod_{j \in \Lambda} q_j \prod_{l \in \{1,2,\dots,n\}/\Lambda} (1 - q_l).$$

Proof. All of the expression is based on the following observations: we can always find a 1-dimensional decomposition of the finite dimensional Hilbert space $\mathcal{H} = \oplus_i \mathcal{H}_i$ such that it is amenable with the eigen-decomposition of Q, that is

$$Q = \sum_i q_i |e_i\rangle\langle e_i|$$

and $\mathcal{H}_i = \mathbb{C}|e_i\rangle$. It is straightforward that $\omega_Q = \wedge_i \omega_{Q_i}$, with $Q_i = q_i|e_i\rangle\langle e_i|$. For the 1-dimensional Hilbert space $\mathcal{H}_i$, the anti-symmetric Fock space $\mathcal{F}_a(\mathcal{H}_i)$ is 2-dimensional and the associated representation can be expressed on $\mathbb{C}^2$ as

$$Q_i = \begin{pmatrix} q_i & 0 \\ 0 & 1 - q_i \end{pmatrix}.$$

According to the uniqueness of the product extension of the even states by the Theorem (4.1.2), the statements are immediate. □

From the above lemma the expression of the von Neumann entropy is a simple corollary.

Theorem 4.3.4 *The von Neumann entropy of a CAR quasi-free state ω_Q is given by*

$$S(\omega_Q) = -\,Tr Q \log Q - Tr(\mathbf{1} - Q)\log(\mathbf{1} - Q). \tag{4.46}$$

Proof. It is immediate from the spectrum of the density matrix ρ_Q. □

We remark that, the p-Rényi entropy of a quasi-free state ω_Q was computed recently [19], as

$$H_p(\omega_Q) = \frac{1}{1-p}\mathrm{Tr}\,\log\left((\mathbf{1} - Q)^p + Q^p\right). \tag{4.47}$$

The von Neumann entropy (4.46) can be also deduced from the p-Rényi entropy by taking the limit $p \searrow 0$. Now we compute the relative entropy of two quasi-free states.

Theorem 4.3.1 *Let Q_1 and Q_2 be two symbols such that* $\ker Q_2 \subset \ker Q_1$ *and* $\ker(\mathbf{1} - Q_2) \subset \ker(\mathbf{1} - Q_1)$, *then the relative entropy of ω_{Q_1} with respect to ω_{Q_2} is given by*

$$\begin{aligned} &S(\omega_{Q_1}||\omega_{Q_2}) \\ &\quad = Tr\left\{Q_1(\log Q_1 - \log Q_2) + (\mathbf{1} - Q_1)(\log(\mathbf{1} - Q_1) - \log(\mathbf{1} - Q_2))\right\}. \end{aligned} \tag{4.48}$$

Proof. We use the following formula for the relative entropy

$$S(\rho||\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma) = -S(\rho) - \frac{\mathrm{d}}{\mathrm{d}p}\,\mathrm{Tr}\,\rho\sigma^p\big|_{p=0}\,. \tag{4.49}$$

Thinking to the form of the statistical operator of a quasi-free state (4.35), we use the notations $D_1 = Q_1(\mathbf{1} - Q_1)^{-1}$ and $D_2 = Q_2(\mathbf{1} - Q_2)^{-1}$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{Tr}\,\rho_{Q_1}\rho_{Q_2}^p &= \frac{\mathrm{d}}{\mathrm{d}p}\mathrm{Tr}\,\frac{\Gamma(D_1)}{\mathrm{Tr}\,\Gamma(D_1)}\left[\frac{\Gamma(D_2)}{\mathrm{Tr}\,\Gamma(D_2)}\right]^p \\
&= \frac{1}{\mathrm{Tr}\,\Gamma(D_1)}\frac{\mathrm{d}}{\mathrm{d}p}\left\{\frac{1}{(\mathrm{Tr}\,\Gamma(D_2))^p}\mathrm{Tr}\,\Gamma(D_1{D_2}^p)\right\} \\
&= \frac{1}{\mathrm{Tr}\,\Gamma(D_1)}\frac{\mathrm{d}}{\mathrm{d}p}\left\{(\mathrm{Det}\,(\mathbf{1}+D_2))^{-p}\mathrm{Det}\,(\mathbf{1}+D_1{D_2}^p)\right\},
\end{aligned}$$

where we have used (4.24) and the Lemma (4.2.1) for the trace of the second quantized operator. After using Jacobi's formula for differentiating a determinant [15]

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{Det}\,A = \mathrm{Tr}\,\left(A^{-1}\frac{\mathrm{d}A}{\mathrm{d}p}\right)\mathrm{Det}\,A, \tag{4.50}$$

and after taking $p = 0$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{Tr}\,\rho_{Q_1}\rho_{Q_2}^p\bigg|_{p=0} &= \frac{1}{\mathrm{Det}\,(\mathbf{1}+D_1)}\{-\log\mathrm{Det}\,(\mathbf{1}+D_2)\mathrm{Det}\,(\mathbf{1}+D_1) + \\
&\quad \mathrm{Det}\,(\mathbf{1}+D_1)\mathrm{Tr}\,(\mathbf{1}+D_1)^{-1}D_1\log D_2\} \\
&= -\mathrm{Tr}\,\log(I+D_2) + \mathrm{Tr}\,(I+D_1)^{-1}D_1\log D_2 \\
&= \mathrm{Tr}\,Q_1\log Q_2 + \mathrm{Tr}\,(\mathbf{1}-Q_1)\log(\mathbf{1}-Q_2),
\end{aligned}$$

where we used the remarkable fact, that $\log\mathrm{Det}\,A = \mathrm{Tr}\,\log A$. Substituting this result into (4.49) and using the Theorem 4.3.4 for the von Neumann entropy of quasi-free states, the statement is obtained. We remark that the conditions for the kernels of symbols are necessary to the finiteness of the relative entropy. □

Chapter 5

Markov states on CAR algebras

5.1 The strong additivity of von Neumann entropy and the Markov property

It has been shown that the Markovianity is tightly related to the strong subadditivity of von Neumann entropy. Namely, Theorem 3.2.1 shows, that a state of a three-composed tensor-product system forms a Markov triplet if and only if it takes the equality for the strong subadditivity inequality of entropy. Moreover Theorem 3.2.2 gives the complete characterization of Markov triplets in this case. It is also possible to extend this result to an infinite tensor product system: a translation invariant quantum Markov state of the quantum spin algebra has a constant entropy increment at each step by the strong additivity, see *Proposition 11.5* in [52]. Our goal to investigate this situation in the CAR case.

The root of the problem is the difference between the three-fold tensor product system and the CAR algebra from the point of view of the commutation of the subsystems. Indeed, however when the set I is countable, the CAR algebra is isomorphic to the C^*-infinite tensor product $\overline{\otimes_I M_2(\mathbb{C})}^{C^*}$ as we saw, but the isomorphism does not preserve the natural localization. The elements of the disjoint subsystems do not commute in contrast to the tensor product case. In spite of these difficulties the strong subadditivity of von Neumann entropy also holds for CAR algebras as was proved by Araki and Moriya [11] and as our proof of Theorem 2.6.7 shows: let I and J be two arbitrary subsets of $\mathbb{Z}$ and denote $\mathcal{A}(I \cup J)$, $\mathcal{A}(I)$, $\mathcal{A}(J)$ and $\mathcal{A}(I \cap J)$ the CAR algebras corresponding to the sets $I \cup J$, I, J and $I \cap J$, respectively with the states $\rho_{I\cup J}$, ρ_I, ρ_J and $\rho_{I\cap J}$, as usual. Then

$$S(\rho_I) + S(\rho_J) \geq S(\rho_{I\cap J}) + S(\rho_{I\cup J}) \tag{5.1}$$

holds. We prove that the equality case is equivalent with the Markov property also for CAR algebras if we restrict ourself for even states [43, 65, 64].

Theorem 5.1.1 *Let I and J be two arbitrary subsets of $\mathbb{Z}$. Let $\phi_{I\cup J}$ be an even state on the CAR algebra $\mathcal{A}(I\cup J)$ with the density matrix $\rho_{I\cup J}$. Then $\phi_{I\cup J}$ is a Markov triplet corresponding to the localization $\{\mathcal{A}(I\setminus J), \mathcal{A}(I), \mathcal{A}(I\cup J)\}$, i.e. there exists a quasi-conditional expectation γ w.r.t the triplet $\mathcal{A}(I\setminus J)\subset\mathcal{A}(I)\subset\mathcal{A}(I\cup J)$ satisfying*

$$\phi_I\circ\gamma=\phi_{I\cup J}, \tag{5.2}$$

$$E\left(\mathcal{A}(J)\right)\subset\mathcal{A}(I\cap J), \tag{5.3}$$

if and only if it saturates the strong subadditivity inequality of entropy with equality, ie.

$$S(\rho_I)+S(\rho_J)=S(\rho_{I\cap J})+S(\rho_{I\cup J}), \tag{5.4}$$

where ρ_J, ρ_I and $\rho_{I\cap J}$ are the density matrices of the appropriate restrictions of $\phi_{I\cup J}$.

Proof. At first remember, that a quasi-conditional expectation w.r.t the given triplet, is a completely positive, identity-preserving linear map $\gamma:\mathcal{A}(I\cup J)\to\mathcal{A}(I)$ such that

$$\gamma(xy)=x\gamma(y),\qquad x\in\mathcal{A}(I\setminus J), y\in\mathcal{A}(I\cup J). \tag{5.5}$$

Let suppose that we (5.4) holds, then by expressing with the relative entropy we have

$$S(\rho_{I\cup J}||\rho_J)=S(\rho_I||\rho_{I\cap J}). \tag{5.6}$$

Let's define the map

$$\gamma(X)={\rho_{I\cap J}}^{-1/2}E_I^{I\cup J}({\rho_J}^{1/2}X{\rho_J}^{1/2}){\rho_{I\cap J}}^{-1/2},\quad X\in\mathcal{A}(I\cup J), \tag{5.7}$$

where $E_I^{I\cup J}$ is the trace preserving conditional expectation onto $\mathcal{A}(I)$, constructed in Lemma 4.1.2. It is clear, that $\gamma:\mathcal{A}(I\cup J)\to\mathcal{A}(I)$. We show that γ is a quasi-conditional expectation with respect to the desired triplet, which preserves the even state $\phi_{I\cup J}$, that is $\phi_{I\cup J}$ is a Markov state. It is obvious that γ is completely positive and preserves the identity. We show that $\mathcal{A}(I\setminus J)\subset\mathrm{Fix}(\gamma)$, the fixpoint algebra of γ. For any $X\in\mathcal{A}(I\setminus J)\subset\mathcal{A}(I)$ we have

$$\begin{aligned}\gamma(X) &= {\rho_{I\cap J}}^{-1/2}E_I^{I\cup J}({\rho_J}^{1/2}X{\rho_J}^{1/2}){\rho_{I\cap J}}^{-1/2}\\ &= X{\rho_{I\cap J}}^{-1/2}E_I^{I\cup J}(\rho_J){\rho_{I\cap J}}^{-1/2}=X,\end{aligned}$$

where we have used that $\rho_{I\cap J}$ commutes with any element of $\mathcal{A}(I \setminus J)$ as a consequence of Theorem 4.1.3, since $\phi_{I\cup J}$ and its all restrictions are even. We also used, that $E_I^{I\cup J}(\rho_J) = \rho_{I\cap J}$ follows from the commuting square property. So, we get that γ leaves the elements of the algebra $\mathcal{A}(I\setminus J)$ fixed. We remark that

$$\mathcal{A}(I \setminus J)^+ \subset \mathrm{Fix}(\gamma) \tag{5.8}$$

holds, even if $\phi_{I\cup J}$ is not an even state. If $X \in \mathcal{A}(I \setminus J)$ and $Y \in \mathcal{A}(I \cup J)$, then

$$\begin{aligned} \gamma(XY) &= {\rho_{I\cap J}}^{-1/2} E_I^{I\cup J}({\rho_J}^{1/2} XY {\rho_J}^{1/2}) {\rho_{I\cap J}}^{-1/2} & (5.9)\\ &= X {\rho_{I\cap J}}^{-1/2} E_I^{I\cup J}({\rho_J}^{1/2} Y {\rho_J}^{1/2}) {\rho_{I\cap J}}^{-1/2} = X\gamma(Y), & (5.10) \end{aligned}$$

which shows the modular property. We remark again, that $\gamma(XY) = X\gamma(Y)$ holds for all $X \in \mathcal{A}(I \setminus J)^+$, $Y \in \mathcal{A}(I \cup J)$, even if $\phi_{I\cup J}$ is not even. For any $X \in \mathcal{A}(J)$

$$\begin{aligned} \gamma(X) &= E_I^{I\cup J}({\rho_{I\cap J}}^{-1/2} {\rho_J}^{1/2} X {\rho_J}^{1/2} {\rho_{I\cap J}}^{-1/2}) \\ &= E_I^{I\cup J}\left(E_J^{I\cup J}({\rho_{I\cap J}}^{-1/2} {\rho_J}^{1/2} X {\rho_J}^{1/2} {\rho_{I\cap J}}^{-1/2})\right) \\ &= E_{I\cap J}^{I\cup J}({\rho_{I\cap J}}^{-1/2} {\rho_J}^{1/2} X {\rho_J}^{1/2} {\rho_{I\cap J}}^{-1/2}), \end{aligned}$$

by the commuting square property, that is $\gamma(\mathcal{A}(J)) \subset \mathcal{A}(I \cap J)$ holds. These properties show that γ is a quasi-conditional expectation with respect to the triple

$$\mathcal{A}(I \setminus J) \subset \mathcal{A}(I) \subset \mathcal{A}(I \cup J).$$

Our assumption (5.6), according to the Theorem 2.6.8, is equivalent with

$${\rho_{I\cup J}}^{it} {\rho_J}^{-it} = {\rho_I}^{it} {\rho_{I\cap J}}^{-it}, \quad t \in \mathbb{R},$$

or by the analytic continuation for $t = -i$ we have

$$\rho_{I\cup J} {\rho_J}^{-1} = \rho_I {\rho_{I\cap J}}^{-1}. \tag{5.11}$$

With the help of this relation we get for any $X \in \mathcal{A}(I \cup J)$

$$\begin{aligned} \phi_I(\gamma(X)) &= \tau\left(\rho_I E_I^{I\cup J}({\rho_{I\cap J}}^{-1/2} {\rho_J}^{1/2} X {\rho_J}^{1/2} {\rho_{I\cap J}}^{-1/2})\right) \\ &= \tau\left(E_I^{I\cup J}({\rho_I}^{1/2} {\rho_{I\cap J}}^{-1/2} {\rho_J}^{1/2} X {\rho_J}^{1/2} {\rho_{I\cap J}}^{-1/2} {\rho_I}^{1/2})\right) \\ &= \tau\left({\rho_I}^{1/2} {\rho_{I\cap J}}^{-1/2} {\rho_J}^{1/2} X {\rho_J}^{1/2} {\rho_{I\cap J}}^{-1/2} {\rho_I}^{1/2}\right) \\ &= \tau\left({\rho_{I\cup J}}^{1/2} X {\rho_{I\cup J}}^{1/2}\right) = \phi_{I\cup J}(X), \end{aligned}$$

which means that $\phi_I \circ \gamma = \phi_{I\cup J}$, so $\phi_{I\cup J}$ is a Markov state, as we stated. For the converse statement let consider a Markov state $\phi_{I\cup J}$ ie.

$$\phi_I \circ F = \phi_{I\cup J} \tag{5.12}$$

and

$$F\left(\mathcal{A}(J)\right) \subset \mathcal{A}(I \cap J) \tag{5.13}$$

where F is a quasi-conditional expectation with respect to the triple $\mathcal{A}(I \setminus J) \subset \mathcal{A}(I) \subset \mathcal{A}(I \cup J)$. Let denote F^* the dual map of F with respect to the Hilbert-Schmidt scalar product $\langle X, Y\rangle = \tau(X^*Y)$. In this case for any $X \in \mathcal{A}(I \cup J)$ we have

$$\begin{aligned}
\phi_{I\cup J}(X) &= \tau\left(\rho_{I\cup J}X\right) = \langle \rho_{I\cup J}, X\rangle \\
&= \phi_I\left(F(X)\right) = \tau\left(\rho_I F(X)\right) \\
&= \langle \rho_I, F(X)\rangle = \langle F^*(\rho_I), X\rangle
\end{aligned}$$

which yields

$$F^*(\rho_I) = \rho_{I\cup J}. \tag{5.14}$$

Now suppose that $X \in \mathcal{A}(J)$. Then

$$\begin{aligned}
\phi_{I\cup J}(X) &= \phi_J(X) = \tau\left(\rho_J X\right) \\
&= \langle \rho_J, X\rangle = \phi_I\left(F(X)\right) \\
&= \phi_{I\cap J}\left(F(X)\right) = \tau\left(\rho_{I\cap J}F(X)\right) \\
&= \langle \rho_{I\cap J}, F(X)\rangle = \langle F^*(\rho_{I\cap J}), X\rangle
\end{aligned}$$

holds, where we used that (5.12) and (5.13). The computation above shows that

$$F^*(\rho_{I\cap J}) = \rho_J \tag{5.15}$$

also holds. As F^* is a dual of a quasi-coditional expectation, it is completely positive and trace preserving, i.e. it is a coarse graining, so with the equations (5.14) and (5.15) F^* fulfill the the necessary and sufficient conditions of the Theorem 2.6.5. We proved that we have equality in the strong subadditivity of von Neumann entropy for all Markovian state without any restriction for its evenness. □

From the proof it turns out attending to (5.8) and the remark after (5.9), that we can leave the condition of the evenness of our state, if we require a stronger condition to our localization:

Corollary 5.1.2 *Let I and J be two arbitrary subsets of $\mathbb{Z}$. Let $\phi_{I\cup J}$ be any state on the CAR algebra $\mathcal{A}(I\cup J)$ with the density matrix $\rho_{I\cup J}$. Then $\phi_{I\cup J}$ is forms a Markov triplet corresponding to the localization $\{\mathcal{A}(I\setminus J)^+, \mathcal{A}(I), \mathcal{A}(I\cup J)\}$ if and only if it saturates the strong subadditivity inequality of entropy , ie.*

$$S(\rho_I) + S(\rho_J) = S(\rho_{I\cap J}) + S(\rho_{I\cup J}),$$

where ρ_J, ρ_I and $\rho_{I\cap J}$ are the density matrices of the appropriate restrictions of $\phi_{I\cup J}$. □

It is natural to consider the Markov triplets, as the builduing blocks of Markov states.

Corollary 5.1.3 *An even state ϕ on $\mathcal{A}$ is quantum Markov state, ie. for each $n \in \mathbb{N}$, there exists a quasi-conditional expectation E_n w.r.t the triplet $\mathcal{A}(n-1]) \subset \mathcal{A}(n]) \subset \mathcal{A}(n+1])$ satisfying*

$$\phi_{n]} \circ E_n = \phi_{n+1]},$$

$$E_n(\mathcal{A}([n, n+1])) \subset \mathcal{A}(\{n\}).$$

if and only if

$$S\left(\phi_{n+1]}\right) + S\left(\phi_{\{n\}}\right) = S\left(\phi_{[n,n+1]}\right) + S\left(\phi_{n]}\right)$$

for all n.

Proof. For a fixed n, with the choice $I = n]$, $J = [n, n+1]$ the quantum Markov state become a short quantum Markov state. By using the theorem above for all n we get the statement. □

Further characterization of CAR Markov states and chains can be found in [6]. In the next section we investigate the structure of Markovian quasi-free states.

5.2 Characterisation of quasi-free Markov states on CAR algebras

Let $Q \in M_n(\mathbb{C})$ be a positive contraction, the symbol of the gauge-invariant quasi-free state ω_Q on the CAR algebra $\mathcal{A}(Pl^2(\mathbb{Z}))$, where P is the projection from $l^2(\mathbb{Z})$ onto the finite-dimensional subspace spanned by the subset of the canonical orthonormal basis $\{\delta_1, \ldots \delta_n\}$. Similarly, let R_1, R_2 and R_3 be the projections from $l^2(\mathbb{Z})$ onto the finite dimensional subspaces spanned by $\{\delta_1, \ldots \delta_k\}$, $\{\delta_{k+1}, \ldots \delta_{k+l}\}$ and $\{\delta_{k+l+1}, \ldots \delta_{k+l+m}\}$, respectively, where $k+l+m = n$. Obviously R_1, R_2 and R_3 are mutually orthogonal projections. The restrictions of ω_Q to the subalgebras $\mathcal{A}_{12} \equiv \mathcal{A}((R_1 \vee R_2)l^2(\mathbb{Z}))$, $\mathcal{A}_{23} \equiv \mathcal{A}((R_2 \vee R_3)l^2(\mathbb{Z}))$ and $\mathcal{A}_2 \equiv \mathcal{A}(R_2 l^2(\mathbb{Z}))$ are again quasi-free with symbols $Q_{12} = (R_1 \vee R_2)Q(R_1 \vee R_2)$, $Q_{23} = (R_2 \vee R_3)Q(R_2 \vee R_3)$ and $Q_2 = R_2 Q R_2$, respectively. It is useful to write Q in the following block-matrix form

$$Q = \begin{bmatrix} A & X & Z \\ X^* & B & Y \\ Z^* & Y^* & C \end{bmatrix}, \qquad (5.16)$$

where $A = A^* \in M_k(\mathbb{C})$, $B = B^* \in M_l(\mathbb{C})$, $C = C^* \in M_m(\mathbb{C})$, $X \in M_{k,l}(\mathbb{C})$, $Y \in M_{l,m}(\mathbb{C})$ and $Z \in M_{k,m}(\mathbb{C})$. Here $M_{k,l}(\mathbb{C})$ denotes the k by l complex matrices, and $M_k(\mathbb{C}) \equiv M_{k,k}(\mathbb{C})$ for simplicity. Our goal is the characterization of quasi-free states which form a Markov triplet.

Theorem 5.2.1 *A quasi-free state on a CAR algebra form a Markov triplet if and only if it is the product of its reduced states.*

Proof. As a quasi-free state is even state, by Theorem 5.1.1 it forms a Markov triplet if and only if it saturates the strong subadditivity of von Neumann entropy with equality. We define the following auxiliary density matrices on the matrix algebra $M_{2n}(\mathbb{C})$

$$\rho_Q := \frac{1}{n}\begin{bmatrix} Q & 0 \\ 0 & \mathbf{1_n} - Q \end{bmatrix}, \tag{5.17}$$

$$\rho_{Q_{12}} := \frac{1}{n}\begin{bmatrix} W_{12} & 0 \\ 0 & \mathbf{1_n} - W_{12} \end{bmatrix}, \tag{5.18}$$

$$\rho_{Q_{23}} := \frac{1}{n}\begin{bmatrix} W_{23} & 0 \\ 0 & \mathbf{1_n} - W_{23} \end{bmatrix}, \tag{5.19}$$

and finally

$$\rho_{Q_2} := \frac{1}{n}\begin{bmatrix} W_2 & 0 \\ 0 & \mathbf{1_n} - W_2 \end{bmatrix}, \tag{5.20}$$

where $\mathbf{1_n}$ is the identity matrix in $M_n(\mathbb{C})$ and we have used the following notations for the block matrices:

$$W_{12} = \begin{bmatrix} A & X & 0 \\ X^* & B & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1_m} \end{bmatrix} = \begin{bmatrix} Q_{12} & 0 \\ 0 & \frac{1}{2}\mathbf{1_m} \end{bmatrix}, \tag{5.21}$$

$$W_{23} = \begin{bmatrix} \frac{1}{2}\mathbf{1_k} & 0 & 0 \\ 0 & B & Y \\ 0 & Y^* & C \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{1_k} & 0 \\ 0 & Q_{23} \end{bmatrix}, \tag{5.22}$$

and finally

$$W_2 = \begin{bmatrix} \frac{1}{2}\mathbf{1_k} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1_m} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{1_k} & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1_m} \end{bmatrix}. \tag{5.23}$$

One can check that $\rho_{Q_{12}}$, $\rho_{Q_{23}}$ and ρ_{Q_2} are the reduced density matrices of the state ρ_Q of the composite system $\mathcal{B} \equiv M_{2n}(\mathbb{C})$ for the subalgebras

$\mathcal{B}_{12} \equiv M_{k+l}(\mathbb{C}) \oplus \mathbf{1_m} \oplus M_{k+l}(\mathbb{C}) \oplus \mathbf{1_m}$, $\mathcal{B}_{23} \equiv \mathbf{1_k} \oplus M_{l+m}(\mathbb{C}) \oplus \mathbf{1_k} \oplus M_{l+m}(\mathbb{C})$ and $\mathcal{B}_2 \equiv \mathbf{1_k} \oplus M_l(\mathbb{C}) \oplus \mathbf{1_m} \oplus \mathbf{1_k} \oplus M_l(\mathbb{C}) \oplus \mathbf{1_m}$, respectively. A simple computation shows that the von Neumann entropy of these density matrices can be expressed by the von Neumann entropy of the corresponding quasi-free states, namely

$$S(\rho_Q) = \frac{S(\omega_Q)}{n} + \log n, \tag{5.24}$$

$$S(\rho_{Q_{12}}) = \frac{S(\omega_{Q_{12}})}{n} + \log n + \frac{m}{n}\log 2, \tag{5.25}$$

$$S(\rho_{Q_{23}}) = \frac{S(\omega_{Q_{23}})}{n} + \log n + \frac{k}{n}\log 2, \tag{5.26}$$

$$S(\rho_{Q_2}) = \frac{S(\omega_{Q_2})}{n} + \log n + \frac{k+m}{n}\log 2. \tag{5.27}$$

With the help of these formulas the strong subadditivity of the von Neumann entropy of the quasi-free states is given by

$$\begin{aligned} &S(\omega_{Q_{12}}) + S(\omega_{Q_{23}}) - S(\omega_Q) - S(\omega_{Q_2}) \\ &\qquad = n\left\{S(\rho_{Q_{12}}) + S(\rho_{Q_{23}}) - S(\rho_Q) - S(\rho_{Q_2})\right\} \ge 0. \end{aligned}$$

So we have got equality for the quasi-free states if and only if we saturate the equality for the auxiliary states defined above. Remember that Theorem 2.6.8 gives an equivalent condition for the equlity in terms of density matrices. Our goal is to find the necessary and sufficient conditions for the matrix Q for which this equivalent condition holds. By the analytic continuation the (i) part of the Theorem 2.6.8 also holds for $t = -i$, i.e.

$$\rho_Q \rho_{Q_{23}}^{-1} = \rho_{Q_{12}} \rho_{Q_2}^{-1} \tag{5.28}$$

is a necessary condition for the strong additivity of the von Neumann entropy. If we like to compute the inverse of a block-matrix, the following formula is very useful.

$$\begin{aligned} &\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \\ &= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \end{aligned} \tag{5.29}$$

if A and D are square matrices and A is invertible. Its checking is a simple multiplication, but its constructive proof based on the Schur complement can be found in several books on linear algebra [66]. With the help of (5.29), we have for the inverse

$$\rho_{Q_{23}}^{-1} = n \begin{bmatrix} W_{23}^{-1} & 0 \\ 0 & (\mathbf{1_n} - W_{23})^{-1} \end{bmatrix}, \tag{5.30}$$

where

$$W_{23}^{-1} = \begin{bmatrix} 2\mathbf{1_k} & 0 & 0 \\ 0 & B^{-1} + B^{-1}YR^{-1}Y^*B^{-1} & -B^{-1}YR^{-1} \\ 0 & -R^{-1}Y^*B^{-1} & R^{-1} \end{bmatrix}.$$

We have used the abbreviation $R = C - Y^*B^{-1}Y = R^*$ for simplicity. Similarly

$$\begin{aligned} &(\mathbf{1_n} - W_{23})^{-1} \\ &= \begin{bmatrix} 2\mathbf{1_k} & 0 & 0 \\ 0 & (\mathbf{1_l} - B)^{-1} + (\mathbf{1_l} - B)^{-1}YS^{-1}Y^*(\mathbf{1_l} - B)^{-1} & -(\mathbf{1_l} - B)^{-1}YS^{-1} \\ 0 & -S^{-1}Y^*(\mathbf{1_l} - B)^{-1} & S^{-1} \end{bmatrix}, \end{aligned}$$

with $S = (\mathbf{1_m} - C) - Y^*(\mathbf{1_m} - B)^{-1}Y = S^*$. The inverse of ρ_{Q_2} is much more simple, it is a block diagonal matrix in the form:

$$\rho_{Q_2}^{-1} = n\mathrm{Diag}\left(2\mathbf{1_k}, B^{-1}, 2\mathbf{1_m}, 2\mathbf{1_k}, (\mathbf{1_l} - B)^{-1}, 2\mathbf{1_m}\right). \tag{5.31}$$

Substituting our matrices into the equation (5.28), we get that the equality holds if and only if $Y = 0$ and $Z = 0$, so these conditions are necessary for get equality in the strong subadditivity of von Neumann entropy. If we consider the (i) part of the Theorem 2.6.8 for $t = i$ we get an other necessary condition for the equality, namely

$$\rho_{Q_{23}}\rho_{Q_2}^{-1} = \rho_Q\rho_{Q_{12}}^{-1}. \tag{5.32}$$

With the help of (5.29) we can compute the inverses again as we have done above, and after the substituting we get that the equation (5.32) holds if and only if $X = 0$ and $Z = 0$. So Q must have the following block-diagonal form:

$$Q = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}, \tag{5.33}$$

i.e. the quasi-free state is a product state according the Theorem 4.3.3. □

Since a quasi-free state is translation-invariant if and only if its symbol is a Töplitz matrix, as a consequence of the Theorem 5.2.1 we get that a translation-invariant quasi-free state is Markov state if and only if its symbol is some constant times the identity matrix. This result can be found in [63].

Chapter 6

CCR algebras and quasi-free states

6.1 Introduction to Weyl unitaries

As we have seen in the case of CAR algebra the anti-symmetry of the wave function had a deep consequence, namely the Pauli principle: It is impossible to create two fermions in the same state. The main qualitative difference between fermions and bosons is the absence of a Pauli principle for the latter particles. There is no bound on the number of particles which can occupy a given state. Mathematically this is reflected by the unboundedness of the Bose annihilation and creation operators. This unboundedness leads to a large number of technical difficulties which are absent for fermions. These problems can be partially avoided by consideration of bounded functions of the annihilation and creation operators. This idea yields the Weyl operators. As an introduction, in this section the basis of Hermite functions of the Hilbert space $L^2(\mathbb{R})$ is described in details, the creation, annihilation operators and the Weyl unitaries are constructed.

In the Hilbert space formulation of quantum mechanics one considers the abstract self-adjoint operators $Q_1, Q_2, \dots, Q_n, P_1, P_2, \dots, P_n$ acting on a Hilbert space $\mathcal{H}$ and satisfying the **Heisenberg commutation relations** for n degrees of freedom:

$$
\begin{aligned}
[Q_i, Q_j] &= [P_i, P_j] = 0, \\
[Q_i, P_j] &= \mathrm{i}\delta_{ij}\mathbf{1}
\end{aligned}
$$

for all $i, j = 1, 2, \dots, n$, where any pair of operators A and B on $\mathcal{H}$ $[A, B] = AB - BA$ for the commutator. There is no essential difference between the case $n = 1$ and the general n, therefore we restrict ourself for one degree of freedom. Then Q is associated to the **position** and P to the **momentum** of a single particle. Since the early days of quantum mechanics it has been the problem to find concrete operators satisfying the Heisenberg commutation relations. If one takes the complex Hilbert space $L^2(\mathbb{R})$, Q as the multiplication operator by the variable, i.e.

$$(Qf)(x) = xf(x), \tag{6.1}$$

and P the differentiation, i.e.

$$(Pf)(x) = -\mathrm{i}f'(x), \tag{6.2}$$

then these operators form a representation of the Heisenberg commutation relations on $L^2(\mathbb{R})$. This representation is called the **Schrödinger representation**. The **Hermite polynomials**

$$H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad (n = 0, 1, \dots) \tag{6.3}$$

are orthogonal in the Hilbert space $L^2(\mathbb{R}, e^{-x^2}\,dx)$, they satisfy the recursion

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \tag{6.4}$$

and the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{6.5}$$

The normalized Hermite polynomials

$$\tilde{H}_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) \tag{6.6}$$

form an orthonormal basis. From this basis of $L^2(\mathbb{R}, e^{-x^2}\,dx)$, we can get easily a basis in $L^2(\mathbb{R})$:

$$\varphi_n(x) := e^{-x^2/2} \tilde{H}_n(x)\,. \tag{6.7}$$

These are called **Hermite functions**. In terms of the Hermite functions equation (6.4) becomes

$$x\varphi_n(x) = \frac{\sqrt{n}\varphi_{n-1}(x) + \sqrt{n+1}\varphi_{n+1}(x)}{\sqrt{2}}. \tag{6.8}$$

If the operators a and a^+ are defined as

$$a\varphi_n = \sqrt{n}\varphi_{n-1}, \qquad a^+\varphi_n = \sqrt{n+1}\varphi_{n+1} \tag{6.9}$$

with $a\varphi_0 = 0$, then in accordance with (6.8), the position operator Q is given by

$$Q = \frac{1}{\sqrt{2}}(a + a^+). \tag{6.10}$$

From the equation

$$\frac{\partial}{\partial x}\Big(H_n(x)e^{-x^2/2}\Big) = H_n'(x)e^{-x^2/2} - xH_n(x)e^{-x^2/2},$$

one can obtain

$$P\varphi_n := -\mathrm{i}\varphi_n' = \frac{\sqrt{n}\varphi_{n-1} - \sqrt{n+1}\varphi_{n+1}}{\mathrm{i}\sqrt{2}}, \tag{6.11}$$

that is

$$P = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a) \tag{6.12}$$

for the momontum operator. Therefore,

$$a = \frac{1}{\sqrt{2}}(Q + \mathrm{i}P), \qquad a^+ = \frac{1}{\sqrt{2}}(Q - \mathrm{i}P).$$

Lemma 6.1.1 *For $z \in \mathbb{C}$ the identity*

$$e(z) := \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\varphi_n(x) = \pi^{-1/4}\exp\Big(-\frac{z^2 + x^2}{2}\Big)\exp(zx\sqrt{2})$$

holds. Moreover,

$$e(z) = e^{za^+}\varphi_0, \qquad \|e(z)\| = e^{|z|^2/2}.$$

Proof. The identity can be deduced from the generator function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}H_n(x) = \exp(2xt - t^2) \tag{6.13}$$

of the Hermite polynomials. □

The above $e(z)$ is called **exponential vector**. For $z \in \mathbb{C}$, the operator $\mathrm{i}(za - \bar{z}a^+)$ is defined originally on the linear combinations of the basis vectors φ_n and it is a symmetric operator. It can be proven that its closure is self-adjoint, therefore $\exp(za - \bar{z}a^+)$ becomes a unitary.

$$W(z) := e^{za - \bar{z}a^+} \tag{6.14}$$

is called **Weyl unitary**. Note that

$$W(z) = \exp \mathrm{i}\sqrt{2}(\alpha P + \beta Q)$$

if $z = \alpha + \mathrm{i}\beta$. Multiple use of the identity

$$e^{\mathrm{i}(tQ+uP)} = \exp(\mathrm{i}tu/2)e^{\mathrm{i}tQ}e^{\mathrm{i}uP} = \exp(-\mathrm{i}tu/2)e^{\mathrm{i}uP}e^{\mathrm{i}tQ} \qquad (6.15)$$

gives the following result.

Theorem 6.1.1

$$W(z)W(z') = W(z+z')\exp(\mathrm{i}\,\mathrm{Im}(\bar{z}z'))$$

for $z, z' \in \mathbb{C}$.

With straightforward computation one gets

Lemma 6.1.2

$$e^{za-\bar{z}a^+}\varphi_0 = e^{-|z|^2/2}e^{za^+}\varphi_0 = \frac{e(z)}{\|e(z)\|}.$$

The functions

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k(n+\alpha)!}{k!(n-k)!(\alpha+k)!}x^k \qquad (\alpha > -1) \qquad (6.16)$$

are called **associated Laguerre polynomials**. We write simply $L_n(x)$ for $\alpha = 0$.

Theorem 6.1.2 *For* $n \geq m$

$$\langle \varphi_m, W(z)\varphi_n \rangle = e^{-|z|^2/2}\sqrt{\frac{m!}{n!}}z^{n-m}L_m^{n-m}(|z|^2)$$

holds.

Proof. First note that definition (6.9) implies

$$a^k(a^+)^n\varphi_0 = \begin{cases} \frac{n!}{(n-k)!}(a^+)^{n-k}\varphi_0 & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases} \qquad (6.17)$$

If $[A,B]$ commutes with A and B, then the formula $e^A e^B = e^{[A,B]/2} e^{A+B}$ holds. Since $[-\bar{z}a^+, za] = |z|^2 I$, we can write

$$\begin{aligned} W(z)\varphi_n &= e^{za-\bar{z}a^+}\varphi_n = e^{-|z|^2/2} e^{-\bar{z}a^+} e^{za}\varphi_n \\ &= \frac{e^{-|z|^2/2} e^{-\bar{z}a^+}}{\sqrt{n!}} \sum_{k=0}^{\infty} \frac{z^k}{k!} a^k (a^+)^n \varphi_0 \\ &= \frac{e^{-|z|^2/2} e^{-\bar{z}a^+}}{\sqrt{n!}} \sum_{k=0}^{n} \frac{z^k n!}{k!(n-k)!} (a^+)^{n-k} \varphi_0. \end{aligned}$$

Now we can compute the matrix elements:

$$\begin{aligned} \langle \varphi_m | W(z)\varphi_n\rangle &= \frac{e^{-|z|^2/2}}{\sqrt{m!\,n!}} \sum_{k=0}^{n} \sum_{\ell=0}^{\infty} \frac{(-\bar{z})^\ell z^k n!}{\ell!\,k!\,(n-k)!} \langle (a^+)^m \varphi_0, (a^+)^{n-k+\ell}\varphi_0\rangle \\ &= \frac{e^{-|z|^2/2}}{\sqrt{m!\,n!}} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \frac{(-\bar{z})^\ell z^k n!\,m!}{\ell!\,k!\,(n-k)!(m-\ell)!} \langle (a^+)^{m-\ell} \varphi_0, (a^+)^{n-k}\varphi_0\rangle \\ &= e^{-|z|^2/2} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \frac{(-\bar{z})^\ell z^k}{\ell!\,k!} \sqrt{\frac{m!\,n!}{(n-k)!\,(m-\ell)!}} \langle \varphi_{m-\ell}, \varphi_{n-k}\rangle \\ &= e^{-|z|^2/2} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \frac{(-\bar{z})^\ell z^k}{\ell!\,k!} \sqrt{\frac{m!\,n!}{(n-k)!\,(m-\ell)!}} \delta_{m-l,n-k}, \end{aligned}$$

where $\delta_{k,\ell}$ denotes the Kronecker symbol. For $n \geq m$, we get non-vanishing elements if and only if $k = n - m + \ell$, where $m - n \leq k \leq n$ and by the formula (6.16) we obtain

$$\begin{aligned} \langle \varphi_m, W(z)\varphi_n\rangle &= e^{-|z|^2/2} \sqrt{\frac{m!}{n!}} \sum_{l=0}^{m} \frac{(-1)^\ell |z|^{2\ell} z^{n-m} n!}{\ell!(m-l)!(n-m+l)!} \\ &= e^{-|z|^2/2} \sqrt{\frac{m!}{n!}} z^{n-m} L_m^{n-m}(|z|^2), \end{aligned}$$

as we stated. □

Note that the case $m \geq n$ can be read out from the Theorem 6.1.2, since

$$\langle \varphi_m, W(z)\varphi_n\rangle = \overline{\langle \varphi_n, W(-z)\varphi_m\rangle}.$$

The case $m = n$ involves the Laguerre polynomials. The analogue of (6.13) is the formula

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} \exp\Big(-\frac{xt}{1-t}\Big) \tag{6.18}$$

which holds for $|t| < 1$ and $x \in \mathbb{R}^+$. This formula is used to obtain

$$\sum_{n=0}^{\infty} \mu^n (1-\mu) \langle \varphi_n, W(z)\varphi_n \rangle = \exp\Big(-\frac{|z|^2}{2} \frac{1+\mu}{1-\mu} \Big) \tag{6.19}$$

for $0 < \mu < 1$. Note that

$$D = \sum_{n=0}^{\infty} \mu^n (1-\mu) |\varphi_n\rangle\langle\varphi_n| \tag{6.20}$$

is a statistical operator (in spectral decomposition). Remark, that in the corresponding state the self-adjoint operator

$$\frac{za - \bar{z}a^+}{\mathrm{i}} = (-\mathrm{i}z)a + \overline{(-\mathrm{i}z)}a^+$$

has Gaussian distribution.

6.2 The symmetric Fock space

We have already introduced the anti-symmetric Fock space in connection with the fermionic systems. In this section we turn to the bosonic case. Let $\mathcal{H}$ be a Hilbert space. If π is a permutation of the numbers $\{1, 2, \dots, n\}$, then on the n-fold tensor product $\mathcal{H}^{n\otimes} := \mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$ we have a unitary U_π such that

$$U_\pi(f_1 \otimes f_2 \otimes \dots \otimes f_n) = f_{\pi(1)} \otimes f_{\pi(2)} \otimes \dots \otimes f_{\pi(n)}.$$

The operator

$$P_s(f_1 \otimes f_2 \otimes \dots \otimes f_n) := \frac{1}{n!} \sum_\pi f_{\pi(1)} \otimes f_{\pi(2)} \otimes \dots \otimes f_{\pi(n)}$$

is a projection onto the symmetric subspace

$$\mathcal{H}^{n\vee} := \{ g \in \mathcal{H}^{n\otimes} : U_\pi g = g \text{ for every } \pi \}$$

which is the linear span of the vectors

$$|f_1, f_2, \dots, f_n\rangle \equiv f_1 \vee f_2 \vee \dots \vee f_n := \frac{1}{\sqrt{n!}} \sum_\pi f_{\pi(1)} \otimes f_{\pi(2)} \otimes \dots \otimes f_{\pi(n)} ,$$

where $f_1, f_2, \dots, f_n \in \mathcal{H}$. Obviously,

$$f_1 \vee f_2 \vee \dots \vee f_n = f_{\pi(1)} \vee f_{\pi(2)} \vee \dots \vee f_{\pi(n)}$$

for any permutation π. Assume that $e_1, e_2, \dots, e_m$ is a basis in $\mathcal{H}$. When we consider a vector

$$e_{i(1)} \vee e_{i(2)} \vee \dots \vee e_{i(n)}$$

in the symmetric tensor power $\mathcal{H}^{n\vee}$, we may assume that $1 \le i(1) \le i(2) \le \dots \le i(n) \le m$. A vector e_t may appear several times, assume that its multiplicity is r_t, that is, $r_t := \{\ell : i(\ell) = t\}$. The norm of the vector is $\sqrt{r_1!\, r_2! \dots r_m!}$ and

$$\left\{ \frac{1}{\sqrt{r_1!\, r_2! \dots r_m!}}\, e_{i(1)} \vee e_{i(2)} \vee \dots \vee e_{i(n)} \ : \ 1 \le i(1) \le i(2) \le \dots \le i(n) \le m \right\} \tag{6.21}$$

is an orthonormal basis in $\mathcal{H}^{n\vee}$. Another notation is

$$|e_1^{t_1}, e_2^{t_2}, \dots, e_m^{t_m}\rangle \equiv e_{i(1)} \vee e_{i(2)} \vee \dots \vee e_{i(n)}.$$

The **symmetric Fock space** is the subspace of the whole Fock space $\mathcal{F}(\mathcal{H})$, spanned by the symmetric vectors, i.e.

$$\mathcal{F}_s(\mathcal{H}) := P_s \mathcal{F}(\mathcal{H}) = \oplus_{n \ge 0} \mathcal{H}^{n\vee}$$

where $\mathcal{H}^{0\vee} \equiv \mathbb{C}$. The element $\mathbf{1} \equiv \Phi \in \mathcal{H}^{0\vee}$ is called the **vacuum vector** and in this spirit the summand $\mathcal{H}^{n\vee}$ is called the n-particle subspace. Since $\mathcal{H}^{1\vee}$ is identical with $\mathcal{H}$, the Hilbert space $\mathcal{F}_s(\mathcal{H})$ is an extension of $\mathcal{H}$. The union of the vectors (6.21) (for every n) is a basis of the Fock space.

Lemma 6.2.1 *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then $\mathcal{F}_s(\mathcal{H}) = \mathcal{F}_s(\mathcal{H}_1) \otimes \mathcal{F}_s(\mathcal{H}_2)$.*

Proof. It is enough to see that

$$(\mathcal{H}_1 \oplus \mathcal{H}_2)^{n\vee} = \mathcal{H}_1^{n\vee} \oplus (\mathcal{H}_1^{(n-1)\vee} \otimes \mathcal{H}_2) \oplus \dots (\mathcal{H}_1 \otimes \mathcal{H}_2^{(n-1)\vee}) \oplus \mathcal{H}_2^{n\vee}.$$

If $e_1, e_2, \dots, e_m$ is a basis in $\mathcal{H}_1$ and $f_1, f_2, \dots, f_k$ is a basis in $\mathcal{H}_2$, then the (non-normalized) basis vector

$$e_{i(1)} \vee e_{i(2)} \vee \dots \vee e_{i(t)} \vee f_{j(1)} \vee f_{j(2)} \vee \dots \vee f_{j(n-t)}$$

can be identified with

$$e_{i(1)} \vee e_{i(2)} \vee \dots \vee e_{i(t)} \otimes f_{j(1)} \vee f_{j(2)} \vee \dots \vee f_{j(n-t)}$$

which is a basis vector in $\mathcal{H}_1^{t\vee} \otimes \mathcal{H}_2^{(n-t)\vee}$. □

For $f \in \mathcal{H}$ the **creation operator** $a^+(f)$ is defined as

$$a^+(f)|f_1, f_2, \dots, f_n\rangle = |f, f_1, f_2, \dots, f_n\rangle. \tag{6.22}$$

$a^+(f)$ is linear in the variable f and it maps the n-particle subspace into the $(n+1)$-particle subspace. Its adjoint is the **annihilation operator** which acts as

$$a(f)|f_1, f_2, \ldots, f_n\rangle = \sum_{i=1}^{n} \langle f, f_i\rangle |f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n\rangle. \tag{6.23}$$

One computes straightforwardly that the creation and the annihilation operators satisfy the so-called **canonical commutation relations**, briefly the **CCR**, i.e.

$$\begin{aligned} [a(f), a(g)] &= [a^+(f), a^+(g)] = 0 && (6.24) \\ [a(f), a^+(g) &= \langle f|g\rangle \mathbf{1}. && (6.25) \end{aligned}$$

Althoug there is a superficial similarity between the algebraic rules of CAR (4.29) and CCR case, the properties of the respective operators are radically different. In applications to physics these differences are thought to be at the root of the fundamentally disparate behaviours of Bose and Fermi systems at low temperature.

The procedure of second quantization is similar to the anti-symmetric case, but there is a few differencies. Consider $U \in B(\mathcal{H})$ acting on the one-particle space, we can extend it to the symmetric Fock space in a natural way. Denote

$$U_n|f_1, f_2, \ldots, f_n\rangle = |Uf_1, Uf_2, \ldots, Uf_n\rangle \tag{6.26}$$

and

$$\Gamma(U) := \bigoplus_{n\geq 0} U_n. \tag{6.27}$$

This operator $\Gamma(U)$ is called the **second quantization** of U, as in the fermionic case, but one must be careful even if U is a bounded operator, $\Gamma(U)$ is not bounded in general. Indeed, if $||U|| > 1$, then $\Gamma(U)$ is not bounded. It is easy to see that

Lemma 6.2.2

$$\Gamma(U_1U_2) = \Gamma(U_1)\Gamma(U_2) \qquad \textit{and} \qquad \Gamma(U^*) = \Gamma(U)^*,$$

moreover

$$\Gamma(U)^s = \Gamma(U^s). \tag{6.28}$$

In particular if U is a unitary, then $\Gamma(U)$ is unitary as well. Moreover, if $U(t)$ is a strongly continuous one-parameter group of unitary operators, then so is $\Gamma(U(t))$. In other words, if $U(t) = e^{itA}$ for some self-adjoint operator A, then

$\Gamma(U(t)) = e^{it\Delta(A)}$ for some self-adjoint operator $\Delta(A)$ called **differential second quantization of** A. From this reason sometimes it is denoted by $\mathrm{d}\Gamma(A)$ in the literature. It easy to see, that

$$\Delta(A) \equiv \mathrm{d}\Gamma(A) = \bigoplus_{n\geq 0} A_n, \tag{6.29}$$

where

$$A_n|f_1, f_2, \ldots, f_n\rangle = \sum_{i=1}^{n} |f_1, \ldots, f_{i-1}, Af_i, f_{i+1}, \ldots, f_n\rangle. \tag{6.30}$$

The next lemma can be shown by simple computation.

Lemma 6.2.3 *For $f, g \in \mathcal{H}$, we have*

$$\mathrm{d}\Gamma(|f\rangle\langle g|) = a^+(f)a(g).$$

We note that the statistical operator (6.20) is $(1-\mu)\Gamma(\mu)$ in the case of a one-dimensional $\mathcal{H}$.

Example 6.2.1 *If $\mathcal{H} = \mathbb{C}$, then $\mathcal{F}_s(\mathcal{H}) \equiv l^2(\mathbb{N})$ which is isomorph with $L^2(\mathbb{R})$. Then there is only one differential second quantization:*

$$\mathrm{d}\Gamma(\mathbf{1}) = a^+a = \frac{1}{2}(Q^2 + P^2),$$

that is the Hamiltonian of the one dimensional harmonic oscillator.

Similarly to the fermionic case, if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $U = U_1 \oplus U_2$, then $\Gamma(U_1 \oplus U_2) = \Gamma(U_1) \otimes \Gamma(U_2)$ holds. The following result will be very useful in our further investigations.

Lemma 6.2.4 *Suppose that* $\dim\mathcal{H} = N$ *and let $\{\lambda_i\}$ be the eigenvalues of the operator U with multiplicities. Then*

$$Tr\,\Gamma(U) = Det(\mathbf{1} - U)^{-1}. \tag{6.31}$$

Proof.

$$\begin{aligned}
\mathrm{Tr}\,\Gamma(U) &= \textstyle\sum_{n\geq 0} \displaystyle\sum_{m_1+m_2+\ldots m_N = n} \lambda_1^{m_1}\lambda_2^{m_2}\ldots\lambda_N^{m_N} \\
&= \prod_{i=1}^{N}(1-\lambda_i)^{-1} \\
&= \exp\{-\mathrm{Tr}\,\log(\mathbf{1} - U)\} \\
&= \mathrm{Det}\,(\mathbf{1} - U)^{-1},
\end{aligned}$$

where we have used, that Tr $\log a = \log \mathrm{Det}\, A$. Note that

$$\mathrm{Tr}\,\Gamma(U) = \exp\{-\mathrm{Tr}\ \log(\mathbf{1} - U)\} \tag{6.32}$$

holds even if $\mathcal{H}$ is not finite dimensional. □

In the next section we give the abstract definition of the CCR algebra.

6.3 The CCR algebra

Next we characterize the abstract properties of the C*-algebra generated by the Weyl operators, mentioned above. We remark that the monographs [16, 55] contain a very extensive outline of this topic. Let H be a real linear space equipped with a nondegenerate symplectic bilinear form σ, i.e. σ is a map from $H \times H$ into $\mathbb{R}$ such that

$$\sigma(f, g) = -\sigma(g, f)$$

for all $f, g \in H$ and if

$$\sigma(f, g) = 0$$

for all $\in H$, then $g = 0$. The pair (H, σ) will be refered to as a **symplectic space**.

Definition 6.3.1 *Let (H, σ) be a symplectic space. The C*-algebra of the canonical commutation relation over (H, σ), briefly **CCR algebra**, written as* CCR(H, σ)*, is a C*-algebra generated by elements $\{W(f) : f \in H\}$ such that*

$$W(f_1)W(f_2) = W(f_1 + f_2) \exp(\mathrm{i}\,\sigma(f_1, f_2)), \tag{6.33}$$

$$W(-f) = W(f)^* \tag{6.34}$$

hold for $f_1, f_2, f \in H$.

Condition (6.33) shows that $W(f)W(0) = W(0)W(f) = W(f)$ for all $f \in H$, hence $W(0)$ is the unit of the algebra and it follows that $W(f)$ is unitary for every $f \in H$. The operators $W(f)$ are called **Weyl unitaries**. More generally, if $\mathcal{H}$ is a complex pre-Hilbert space then σ given by

$$\sigma(f_1, f_2) := \mathrm{Im}\langle f_1, f_2\rangle$$

is a nondegenerate symplectic form on the real linear space $\mathcal{H}$. Moreover in finite dimension this is the typical way to define a symplectic space, hence the dimension of a nondegenerate symplectic space is even or infinite. We will follow this line and for the most part we will consider a finite dimensional

Hilbert space $\mathcal{H}$, with $\sigma(f_1, f_2) := \text{Im}\langle f_1, f_2\rangle$ and the C*-algebra generated by the Weyl unitaries is denoted by CCR($\mathcal{H}$). The relation (6.33) shows that $W(f_1)$ and $W(f_2)$ commute if f_1 and f_2 are orthogonal. Therefore for an n-dimensional $\mathcal{H}$, the algebra CCR($\mathcal{H}$) is an n-fold tensor product

$$\text{CCR}(\mathbb{C}) \otimes \ldots \otimes \text{CCR}(\mathbb{C}).$$

Since $W(tf)W(sf) = W((t+s)f)$ for $t, s \in \mathbb{R}$, the mapping $t \mapsto W(tf)$ is a one-parameter unitary group which is not norm continuous since $\|W(f_1) - W(f_2)\| \geq \sqrt{2}$ when $f_1 \neq f_2$ [55]. The C*-algebra CCR($\mathcal{H}$) has a very natural state

$$\omega(W(f)) := \exp\big(-\|f\|^2/2\big) \tag{6.35}$$

which is called **Fock state**. The GNS-representation of CCR($\mathcal{H}$) is called **Fock representation** and it leads to the the Fock space $\mathcal{F}_s(\mathcal{H})$ with cyclic vector Φ. Since ω is a product state, the GNS Hilbert space is a tensor product. (This is another argument to justify Lemma 6.2.1.) We shall identify the abstract unitary $W(f)$ with the representing unitary acting on the tensor product GNS-space $\mathcal{F}_s(\mathcal{H})$.The map

$$t \mapsto \pi_\Phi(W(tf))$$

is a so-continuous 1-parameter group of unitaries, and according to the Stone theorem we have

$$W(tf) = \exp(\text{i}tB(f)) \quad \text{and} \quad \frac{\partial}{\partial t}\Big|_{t=0} W(tf) = \text{i}B(f)$$

for a self-adjoint operator $B(f)$, called **field operator**. The distribution of a field operator is Gaussian with respect to the Fock state. Let introduce the operators

$$B^{\pm}(f) = \frac{1}{2}(B(f) \mp \text{i}B(if)).$$

Then

$$[B^-(f), B^+(g)] = \langle f, g\rangle I \tag{6.36}$$

is the canonical commutation relation for the **creation operator** $B^+(g)$ and the **annihilation operator** $B^-(f)$ and

$$[B^-(f), B^-(g)] = [B^+(f), B^+(g)] = 0 \tag{6.37}$$

are also hold. The following relations outstanding in the Fock representation account for the denomination 'creation' and 'annihilation'

$$B^+(f)|f_1, f_2, \ldots, f_n\rangle = |f, f_1, f_2, \ldots, f_n\rangle \tag{6.38}$$

and

$$B(f)|f_1, f_2, \dots, f_n\rangle = \sum_{i=1}^{n} \langle f, f_i\rangle |f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n\rangle. \tag{6.39}$$

The physical interpretation of these operators is the usual one. Let $\Phi = (1, 0, 0, \dots)$ be the so called **vacuum vector**, corresponding to the zero-particle state. The vectors

$$|f_1, f_2, \dots, f_n\rangle = (n!)^{-1/2} B^+(f_1) \dots B^+(f_n)\Phi$$

are n-particle states which arise from successive "creation" of particles (especially **bosons**) in the states $f_n, f_{n-1}, \dots, f_1$ and (6.38) expresses the creation of a plus particle in the state f. Similarly, as (6.39) shows, $B^-(f)$ reduce the number of particles, i.e. they annihilate particles. It is important to remark, that $B^+(f)$ and $B^-(f)$ are never bounded and this is the reason of the absence of a **Pauli principle**, i.e. in the bosonic case (in contrast to the fermionic one) there is no bound on the number of particles which can occupy a given state. It is useful to know the relation of the second quantization of an operator U and the creation operator. It is easy to check, that

$$\Gamma(U)B^+(f) = B^+(Uf)\Gamma(U). \tag{6.40}$$

Note that in the 1-dimensional case $\mathcal{H} = \mathbb{C}$

$$W(z) = \exp \mathrm{i}(a(z) + a^+(z)),$$

where $a^+(z) = \mathrm{i}\bar{z}a^+$, and we get back (6.14), our 'original' Weyl unitary. Similarly to the CAR algebraic case, in the next section we attend to the non-commutative analogue of the Gaussian measure.

6.4 Quasi-free states

The Fock state (6.35) can be generalized by choosing a positive operator $A \in B(\mathcal{H})$:

$$\omega_A(W(f)) := \exp\big(-\|f\|^2/2 - \langle f, Af\rangle\big). \tag{6.41}$$

This state, as we will see, can be ragarded as the non-commutative analogue of the Gaussian measure and so is called **Gaussian** or **quasi-free state** with the **symbol** A. Indeed, by derivation we get

$$\omega_A(B(f)B(g)) = -\mathrm{i}\sigma(f,g) + \frac{1}{2}\Big(\langle f, (I+2A)g\rangle + \langle g, (I+2A)f\rangle\Big),$$

and the value of a quasi-free state ω_A on any polynomial of field operators is completely determined by these two-point functions [16, 55]:

$$\omega_A(B(f_n)B(f_{n-1})\dots B(f_1)) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sum\prod \omega_A(B(f_{k_m})B(f_{j_m})) & \text{if } n \text{ is even,} \end{cases} \tag{6.42}$$

where the summation is over all partitions $\{H_1, H_2, \dots, H_{n/2}\}$ of $\{1, 2, \dots, n\}$ such that $H_m = \{j_m, k_m\}$ with $j_m < k_m$, $(m = 1, 2, \dots, n/2)$. Remember that in the classical probability we met something similar in the case of Gaussian measure: it leads to a characteristic function which is the exponential of a quadratic form. Its logarithm is therefore a quadratic polynomial, and all correlations beyond the second order vanish. Moreover, for the creation and the annihilation operator we have

$$\omega_A(B^+(f)B^-(g)) = \langle g, Af\rangle\,. \tag{6.43}$$

It was shown in [67] that if we have a state of the CCR algebra, and all correlation functions vanish beyond some order, then the state is necessarily quasi-free. This is the quantum analogue of the theorem of *J. Marcinkiewicz* [40]. We compute the statistical operator of a quasi-free state.

Lemma 6.4.1 *The statistical operator of a quasi-free state ω_A in the Fock representation is given by*

$$\rho_A = \frac{\Gamma(A(\mathbf{1}+A)^{-1}))}{Tr\Gamma(A(\mathbf{1}+A)^{-1}))}. \tag{6.44}$$

Proof. Since ω_A is normal, there exists a positive trace class operator S which for

$$\rho_A = \frac{\Gamma(S)}{\mathrm{Tr}\,\Gamma(S)}. \tag{6.45}$$

It follows from (6.43) that

$$\begin{aligned} \omega_A(B^+(f)B^-(g)) &= \frac{1}{\mathrm{Tr}\,\Gamma(S)}\mathrm{Tr}\,\big(\Gamma(S)B^+(f)B^-(g)\big) \\ &= \frac{1}{\mathrm{Tr}\,\Gamma(S)}\mathrm{Tr}\,\big(B^-(g)B^+(Sf)\Gamma(S)\big) \\ &= \langle g|Sf\rangle + \omega_A\big(B^+(Sf)B^-(g)\big) = \langle g|Af\rangle, \end{aligned}$$

for all f and g, where we have use the trace property and (6.40) in the second line and (6.36) in the third one. By choosing $f = (\mathbf{1}-S)^{-1}g$, we arrive to

$$\omega_A(B^+(g)B^-(g)) = \langle g|S(\mathbf{1}-S)^{-1}g\rangle, \tag{6.46}$$

which gives

$$S = \frac{A}{\mathbf{1} + A}$$

since (6.43) as we stated. □

Example 6.4.1 *Assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and write the positive mapping $A \in B(\mathcal{H})$ in the form of block matrix:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If $f \in \mathcal{H}_1$, then

$$\omega_A(W(f \oplus 0)) = \exp\big(- \|f\|^2/2 - \langle f, A_{11} f\rangle\big).$$

Therefore the restriction of the quasi-free state ω_A to $\mathrm{CCR}(\mathcal{H}_1)$ is the quasi-free state $\omega_{A_{11}}$.

If the spectral decomposition of $0 \le A \in B(\mathcal{H})$ is

$$A = \sum_{i=1}^{m} \lambda_i |e_i\rangle\langle e_i|, \tag{6.47}$$

then

$$\Gamma(A(I+A)^{-1})|e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}\rangle = \prod_{i=1}^{m} \left(\frac{\lambda_i}{1+\lambda_i}\right)^{r_i} |e_1^{r_1}, e_2^{r_2}, \dots, e_m^{r_m}\rangle$$

and as we have seen above we have

$$D_A = \frac{1}{C}\Gamma(A(\mathbf{1}+A)^{-1})\,, \quad \text{where} \quad C = \mathrm{Tr}\,\Gamma(A(\mathbf{1}+A)^{-1}). \tag{6.48}$$

This leads to the following result.

Theorem 6.4.1 *Let ω_A and ω_B be quasi-free state of $\mathrm{CCR}(\mathcal{H})$ which correspond to the operators $0 \le A, B \in B(\mathcal{H})$. Their Connes cocycle is*

$$[D\omega_A, D\omega_B]_t = u_t \Gamma\Big((A(\mathbf{1}+A)^{-1})^{\mathrm{i}t}(B(\mathbf{1}+B)^{-1})^{-\mathrm{i}t}\Big) \tag{6.49}$$

where

$$u_t = \Big(Tr\,\Gamma(A(\mathbf{1}+A)^{-1})\Big)^{-\mathrm{i}t}\Big(Tr\,\Gamma(B(\mathbf{1}+B)^{-1})\Big)^{\mathrm{i}t}.$$

In the nex section we compute some entropy related quantities in connection with quasi-free states.

6.4.1 Entropy related quantities of quasi-free states

The following example help us to deduce the von Neumann entropy of a quasi-free state.

Example 6.4.2 *Assume that $\mathcal{H}$ is one-dimensional and let $A = \lambda > 0$. We can read out from formulas (6.19) and (6.20) that the statistical operator of ω_λ in the Fock representation is*

$$D_\lambda = \sum_{n=0}^{\infty} \frac{1}{1+\lambda}\Big(\frac{\lambda}{1+\lambda}\Big)^n |\varphi_n\rangle\langle\varphi_n|. \tag{6.50}$$

Moreover,

$$\omega_\lambda(a^+a) = \lambda, \tag{6.51}$$

and it is not hard to see, that ω_λ is quasi-free. One can easily compute the von Neumann entropy of the state ω_λ from the eigenvalues of the statistical operator D_λ:

$$S(\omega_\lambda) = \kappa(\lambda), \tag{6.52}$$

where

$$\kappa(t) = -t\log t + (t+1)\log(t+1). \tag{6.53}$$

The case of finite dimensional $\mathcal{H}$ can be reduced to the one-dimensional by the spectral decomposition of the operator A.

Theorem 6.4.3 *Assume that the spectral decomposition of $0 \le A \in B(\mathcal{H})$ is*

$$A = \sum_{i=1}^{m} \lambda_i |e_i\rangle\langle e_i|. \tag{6.54}$$

Then the statistical operator of the quasi-free state ω_A in the Fock representation is

$$D_A = \left(\prod_{i=1}^{m} \frac{1}{1+\lambda_i}\right) \sum_{r_j} \left(\prod_{i=1}^{m} \Big(\frac{\lambda_i}{1+\lambda_i}\Big)^{r_i} \frac{1}{r_i!}\right) |e_1^{r_1}, \dots, e_m^{r_m}\rangle\langle e_1^{r_1}, \dots, e_m^{r_m}|, \tag{6.55}$$

where summation is over $n = 0, 1, 2\dots$ and the decompositions $n = r_1 + r_2 + \dots + r_m$. Moreover, the von Neumann entropy is

$$S(\omega_A) = Tr\,\kappa(A), \quad \text{where} \quad \kappa(t) = -t\log t + (t+1)\log(t+1). \tag{6.56}$$

Proof. The basic idea is the decomposition

$$\omega_A = \omega_{\lambda_1} \otimes \omega_{\lambda_2} \otimes \ldots \otimes \omega_{\lambda_m} \tag{6.57}$$

when the space $\mathcal{H}$ is decomposed into the direct sum of the one-dimensional subspaces $\mathbb{C}|e_i\rangle$ and $\mathcal{F}(\mathcal{H})$ and $\mathrm{CCR}(\mathcal{H})$ become tensor product. The statistical operator of ω_{λ_i} is

$$D_{\lambda_i} = \sum_{r_i=0}^{\infty} \frac{1}{1+\lambda_i}\left(\frac{\lambda_i}{1+\lambda_i}\right)^{r_i} \frac{1}{r_i!}|e_i^{r_i}\rangle\langle e_i^{r_i}|$$

the tensor product is exactly the stated matrix. The von Neumann entropy is deduced from (6.52) and (6.57). □

As we have seen, the von Neumann entropy and the relative entropy were defined originally for statistical operators:

$$S(\rho) = -\mathrm{Tr}\,\rho\log\rho, \quad S(\rho\|\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma),$$

where $\mathrm{supp}\rho \subset \mathrm{supp}\sigma$ was supposed. Kosaki's formula can be used to define relative entropy of states of a C*-algebra $\mathcal{A}$:

$$S(\varphi\|\omega) = \sup_n \sup \left\{ \varphi(I)\log n - \int_{1/n}^{\infty} \left(\varphi(y(t)^* y(t)) + t^{-1}\omega(x(t)x(t)^*)\right)\frac{\mathrm{d}t}{t} \right\},$$

where the first sup is taken over all natural numbers n, the second one is over all step functions $x : (1/n, \infty) \to \mathcal{A}$ with finite range and $y(t) = I - x(t)$ [35]. The von Neumann entropy can be defined via the relative entropy:

$$S(\varphi) = \sup\left\{ \sum_i \lambda_i S(\varphi_i\|\varphi) : \sum_i \lambda_i\varphi_i = \varphi \right\}.$$

Here the supremum is over all decompositions of φ into finite (or equivalently countable) convex combinations of other states.

In our situation ω_A is a Gaussian state of the CCR-algebra which has a normal extension $\bar{\omega}_A$ in the Fock representation and so $S(\omega_A) = S(\bar{\omega}_A)$. If the state ψ does not have a normal extension, then $S(\psi\|\omega_A) = +\infty$. When $\bar{\psi}$ is the normal extension, then $S(\psi\|\omega_A) = S(\bar{\psi}\|\bar{\omega}_A)$, see Chapters 5 and 6 in [52] about the details. It is a consequence that we can work in the Fock representation. Assume that ψ is a state of $CCR(\mathcal{H})$. If

$$C_\psi(f,g) := \psi(B^+(f)B^-(g))$$

can be defined, then it will be called **2-point function** of ψ. A positive operator T, defined by

$$\langle g|Tf\rangle = C_\psi(f,g), \tag{6.58}$$

is called the **2-point operator** of ψ. We want to compute the relative entropy of a state ψ and a Gaussian state ω_A.

Theorem 6.4.4 *Consider a state ψ on the algebra $CCR(\mathcal{H})$ with a 2-point operator T. Then its relative entropy with respect to the quasi-free state ω_A is given by*

$$S(\psi\|\omega_A) = -S(\psi) - \, TrT \log A(I+A)^{-1} + Tr \log(I+A). \tag{6.59}$$

Proof. Since

$$S(\psi||\omega_A) = -S(\psi) - \psi(\log \rho_A), \tag{6.60}$$

the point is the computation of the term $\psi(\log \rho_A)$, where ρ_A is the statistical operator of ω_A given in (6.44). Supposing the decomposition

$$A = \sum_i \lambda_i |f_i\rangle\langle f_i|,$$

we have

$$\log \Gamma(A(I+A)^{-1}) = \sum_i \log \frac{\lambda_i}{1+\lambda_i} B^+(f_i)B^-(f_i) \tag{6.61}$$

and

$$\begin{aligned}
\psi(\log D_A) &= \psi(\log \Gamma(A(I+A)^{-1}))) - \log \mathrm{Tr}\,\Gamma(A(I+A)^{-1}) \\
&= \sum_i \log \frac{\lambda_i}{1+\lambda_i} \psi(B^+(f_i)B^-(f_i)) - \mathrm{Tr}\,\log(I+A) \\
&= \sum_i \log \frac{\lambda_i}{1+\lambda_i} \langle f_i, Tf_i\rangle - \mathrm{Tr}\,\log(I+A) \\
&= \mathrm{Tr}\, T \log A(I+A)^{-1} - \mathrm{Tr}\,\log(I+A),
\end{aligned}$$

where we have used, that

$$\log \mathrm{Tr}\,\Gamma(A(I+A)^{-1}) = \mathrm{Tr}\,\log(I+A), \tag{6.62}$$

in accordance with (6.32). Substituting into (6.60) the proof is complete. □
We show that the quasi-free state ω_A has the largest entropy among states with 2-point operator A.

Theorem 6.4.2 *Let ψ be a state of* CCR($\mathcal{H}$) *such that its 2-point function is $\psi(B^+(f)B^-(g)) = \langle g, Af\rangle$ $(f, g \in \mathcal{H})$ for a positive operator $A \in B(\mathcal{H})$. Then $S(\psi) \le S(\omega_A)$ and equality implies $\psi = \omega_A$.*

Proof. With substitution $A = T$ into (6.59) and using the non-negativeness of the relative entropy we get

$$-S(\psi) + S(\omega_A) \geq 0.$$

□

On the other hand, the relative entropy of the Gaussian states ω_B and ω_A is also immediate:

$$S(\omega_B \| \omega_A) = \operatorname{Tr} B(\log B - \log A) - \operatorname{Tr}(I+B)(\log(I+B) - \log(I+A)). \quad (6.63)$$

Chapter 7

Markov states on CCR algebras

7.1 Gaussian Markov triplets

As we have seen before, the Markov property invented by Accardi in the non-commutative (or quantum probabilistic) setting is based on a completely positive, identity preserving map, so-called quasi-conditional expectation and it was formulated in the tensor product of matrix algebras. A state of a tensor product system is Markovian if and only if the von Neumann entropy increase is constant. We also showed the strong subadditive property of the von Neumann entropy which plays an important role in the investigations of quantum system's correlations. The above mentioned constant increase of the von Neumann entropy is the same as the equality for the strong subadditivity of von Neumann entropy. Assume that the Hilbert space $\mathcal{H}$ has the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Then

$$\mathrm{CCR}(\mathcal{H}) = \mathrm{CCR}(\mathcal{H}_1) \otimes \mathrm{CCR}(\mathcal{H}_2) \otimes \mathrm{CCR}(\mathcal{H}_3)$$

and the equality in the strong subadditivity of the von Neumann entropy can be the definition of the Markov property [57]:

Definition 7.1.1 *Assume that φ_{123} is a state on the composite system $\mathrm{CCR}(\mathcal{H}_1) \otimes \mathrm{CCR}(\mathcal{H}_2) \otimes \mathrm{CCR}(\mathcal{H}_3)$. Denote by $\varphi_{12}, \varphi_{23}$ the restriction to the first two and to the second and third factors, similarly φ_2 is the restriction to the second factor. We say that φ_{123} form a* ***Markov triplet*** *if*

$$S(\varphi_{123}) - S(\varphi_{12}) = S(\varphi_{23}) - S(\varphi_2),$$

where S denotes the von Neumann entropy.

We have seen, that if φ_{123} is quasi-free (Gaussian), then it is given by a positive operator (corresponding to the 2-point function) and the main goal of this section is to describe the Markov property in terms of this operator. The technique will be use block matrix methods and linear analysis. The paper [32] studies this problem under the finite dimensional condition, and the genearal case is contained in [61].

Consider a Gaussian state $\omega_A \equiv \omega_{123}$, where A is a positive operator acting on $\mathcal{H}$. This operator has the block-matrix form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix}.$$

and we set

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}, \qquad C = \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{bmatrix}.$$

In connection with the strong subadditivity of the von Neumann entropy, the definition of the Markov property is

$$\operatorname{Tr}\kappa(A) + \operatorname{Tr}\kappa(A_{22}) = \operatorname{Tr}\kappa(B) + \operatorname{Tr}\kappa(C), \tag{7.1}$$

in accordance with (6.56). Our aim is to characterize the Markov property in terms of the block-matrix A. Denote by P_i the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_i$, $1 \le i \le 3$. Of course, $P_1 + P_2 + P_3 = I$ and we use also the notation $P_{12} := P_1 + P_2$ and $P_{23} := P_2 + P_3$.

Theorem 7.1.1 *Assume that $A \in B(\mathcal{H})$ is a positive invertible operator and the Gaussian state $\omega_A \equiv \omega_{123}$ on* CCR($\mathcal{H}$) *has finite von Neumann entropy. Then the following conditions are equivalent.*

(a) $S(\omega_{123}) + S(\omega_2) = S(\omega_{12}) + S(\omega_{23})$

(b) $Tr\,\kappa(A) + Tr\,\kappa(P_2AP_2) = Tr\,\kappa(P_{12}AP_{12}) + Tr\,\kappa(P_{23}AP_{23})$

(c) There is a projection $P \in B(\mathcal{H})$ such that $P_1 \le P \le P_1 + P_2$ and $PA = AP$.

Proof. (a) and (b) are different only in notation. Condition (c) tells that the matrix A has a special form:

$$A = \begin{bmatrix} A_{11} & \begin{bmatrix} a & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} a^* \\ 0 \end{bmatrix} & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} & \begin{bmatrix} 0 \\ b \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & b^* \end{bmatrix} & A_{33} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & a \\ a^* & c \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} d & b \\ b^* & A_{33} \end{bmatrix} \end{bmatrix}, \tag{7.2}$$

where the parameters a, b, c, d (and 0) are operators. This is a block diagonal matrix, $A = \mathrm{Diag}\,(A_1, A_2)$,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

and the projection P is

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

in this setting. The Hilbert space $\mathcal{H}_2$ is decomposed as $\mathcal{H}_2^L \oplus \mathcal{H}_2^R$, where $\mathcal{H}_2^L$ is the range of the projection PP_2. Therefore,

$$\mathrm{CCR}(\mathcal{H}) = \mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L) \otimes \mathrm{CCR}(\mathcal{H}_2^R \oplus \mathcal{H}_3) \tag{7.3}$$

and ω_{123} becomes a product state $\omega_L \otimes \omega_R$. This shows that the implication $(c) \Rightarrow (a)$ is obvious. The essential part is the proof $(a) \Rightarrow (c)$ which is based on Theorem 1.3.2, namely

$$\mathrm{Tr}\ \log(A) + \mathrm{Tr}\ \log(A_{22}) \leq \mathrm{Tr}\ \log(B) + \mathrm{Tr}\ \log(C). \tag{7.4}$$

The necessary and sufficient condition for equality is $A_{13} = A_{12}A_{22}^{-1}A_{23}$. Lemma 2.7.1 says that the function κ is operator monotone and the integral representation

$$\kappa(x) = \int_1^\infty t^{-2} \log(tx+1)\, dt \tag{7.5}$$

also implies the inequality

$$\mathrm{Tr}\, \kappa(A) + \mathrm{Tr}\, \kappa(A_{22}) \leq \mathrm{Tr}\, \kappa(B) + \mathrm{Tr}\, \kappa(C). \tag{7.6}$$

The equality holds if and only if

$$tA_{13} = tA_{12}(tA_{22} + I)^{-1} tA_{23}$$

for almost every $t > 1$. The continuity gives that actually for every $t > 1$ we have

$$A_{13} = A_{12}(A_{22} + t^{-1}I)^{-1}A_{23}.$$

The right-hand-side, $A_{12}(A_{22} + zI)^{-1}A_{23}$, is an analytic function on $\{z \in \mathbb{C} : \mathrm{Re}z > 0\}$, therefore we have

$$A_{13} = 0 = A_{12}(A_{22} + sI)^{-1}A_{23} \qquad (s \in \mathbb{R}^+),$$

as the $s \to \infty$ case shows. Since $A_{12}s(A_{22}+sI)^{-1}A_{23} \to A_{12}A_{23}$ as $s \to \infty$, we have also $0 = A_{12}A_{23}$. The latter condition means that $\mathrm{Rng}\, A_{23} \subset \mathrm{Ker}\, A_{12}$, or equivalently $(\mathrm{Ker}\, A_{12})^\perp \subset \mathrm{Ker}\, A_{23}^*$.

The linear combinations of the functions $x \mapsto 1/(s+x)$ form an algebra and due to the Stone-Weierstrass theorem $A_{12}g(A_{22})A_{23} = 0$ for any continuous function g.

We want to show that the equality implies the structure (7.2) of the operator A. We have $A_{23} : \mathcal{H}_3 \to \mathcal{H}_2$ and $A_{12} : \mathcal{H}_2 \to \mathcal{H}_1$. To show the structure (7.2), we have to find a subspace $H \subset \mathcal{H}_2$ such that

$$A_{22}H \subset H, \quad H^\perp \subset \operatorname{Ker} A_{12}, \quad H \subset \operatorname{Ker} A_{23}^*,$$

or alternatively $(H^\perp =)K \subset \mathcal{H}_2$ should be an invariant subspace of A_{22} such that

$$\operatorname{Rng} A_{23} \subset K \subset \operatorname{Ker} A_{12}.$$

Let

$$K := \Big\{ \sum_i A_{22}^{n_i} A_{23} x_i \; : \; x_i \in \mathcal{H}_3, n_i \in \mathbb{Z}^+ \Big\}$$

be a set of finite sums. It is a subspace of $\mathcal{H}_2$. The property $\operatorname{Rng} A_{23} \subset K$ and the invariance under A_{22} are obvious. Since

$$A_{12}A_{22}^n A_{23} x = 0,$$

$K \subset \operatorname{Ker} A_{12}$ also follows. The proof is complete. □

We can compare the structure of a Markov state on the CCR-algebra with the tensor product of full matrix algebras investigated in Theorem 3.2.2. In the case $M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$, the middle factor contains a decomposition

$$\oplus_i \left(\mathcal{B}_i^L \otimes \mathcal{B}_i^R \right) \tag{7.7}$$

and the Markov state has the form $\sum_i p_i \, \psi_i \otimes \varphi_i$, where ψ_i is a state of $M_k(\mathbb{C}) \otimes \mathcal{B}_i^L$ and φ_i is a state of $\mathcal{B}_i^R \otimes M_k(\mathbb{C})$ [29]. The CCR situation is similar, but we do not have direct sum as (7.7), but only tensor product decomposition.

Now we discuss the minimization of the relative entropy with respect a quasi-free state on CCR algebras under some conditions. We show that the minimizer is Markovian similarly to the classical probabilistic case [14].

Theorem 7.1.2 *Let $\omega \equiv \omega_A$ be a Markovian a Gaussian state on the CCR-algebra $\mathrm{CCR}(\mathcal{H})$ and let ψ_1 be a state of $\mathrm{CCR}(\mathcal{H}_1)$ with a 2-point function. If ψ is the state minimizing the relative entropy $S(\psi||\omega_A)$ under the constraint that $\psi|\mathrm{CCR}(\mathcal{H}_1) = \psi_1$ is fixed, then ψ is a Markov state.*

Proof. We have the tensor product structure $\omega = \omega_L \otimes \omega_R$ on (7.3). Due to the monotonicity of the relative entropy

$$S(\psi||\omega) \geq S(\psi \,|\, \mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L) \;||\; \omega|\mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L)),$$

holds and it is enough to minimize the right-hand-side. The right-hand-side can be finite, for example if ψ is Gaussian, therefore, the minimizer is uniquely exists. If the state ψ' is the minimizer, then $\psi = \psi' \otimes \omega_R$ is the minimizer on $\mathrm{CCR}(\mathcal{H})$ due to the conditional expectation property, see Chapter 2 in [52]. From the product structure the Markov property follows. □

Note that the minimizer Markovian state ψ has the same conditional expectation than the given state ω. In the probabilistic case the similar statement is well-known, see [14], for example.

Theorem 7.1.3 *Let $\omega \equiv \omega_A$ be a Markovian quasi-free state on the CCR-algebra $\mathrm{CCR}(\mathcal{H})$. There exists a state ψ which is minimizing the relative entropy $S(\psi||\omega_A)$ under the constraint that $\psi|\mathcal{A}_1$ has a fixed 2-point operator. Moreover, ψ is a Markov state.*

Proof. Let T be the 2-point operator of $\psi|\mathcal{A}_1 = \varphi$ and assume that ω_A is determined by the matrix (7.2). Similarly to the proof of the previous theorem, we have to concentrate first to the restrictions of ψ and ω_A to $\mathcal{A}_L := \mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L)$. Here they have the block matrices

$$\varphi : \begin{bmatrix} T & u \\ u^* & v \end{bmatrix} \quad \text{and} \quad \omega_A|\mathcal{A}_L : \begin{bmatrix} A_{11} & a \\ a^* & c \end{bmatrix}.$$

The unknown entries u and v of the first matrix are uniquely determined by the minimization of $S(\varphi \,\|\, \omega_A|\mathcal{A}_L)$. When φ is obtained, ψ has the matrix

$$\begin{bmatrix} T & u & 0 & 0 \\ u^* & v & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & b^* & A_{33} \end{bmatrix}.$$

So $\psi = \varphi \otimes \omega_R$. From the product structure the Markov property follows. □

7.2 Connection to classical Gaussians

We want to compare the classical Gaussian situation with the CCR setting. For the sake of simplicity in this subsection we assume that the Hilbert space $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ are all k-dimensional.

Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ be vector-valued random variables with Gaussian joint probability distribution

$$\sqrt{\frac{\mathrm{Det}\, M}{(2\pi)^{3k}}} \exp\Big(-\tfrac{1}{2}\langle \mathbf{x}, M\mathbf{x}\rangle \Big), \tag{7.8}$$

where $M \in M_{3k}(\mathbb{C})$ is positive definite matrix. Theorem 1.3.1 says that the triplet $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ has the Markov property if and only if the covariance matrix $S = M^{-1}$ of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ is of the form

$$\begin{bmatrix} S_{11} & S_{12} & S_{12}S_{22}^{-1}S_{23} \\ S_{12}^* & S_{22} & S_{23} \\ S_{23}^*S_{22}^{-1}S_{12}^* & S_{23}^* & S_{33} \end{bmatrix}, \tag{7.9}$$

that is

$$S_{13} = S_{12}S_{22}^{-1}S_{23}, \tag{7.10}$$

see. To show some analogy between the classical Gaussian and the CCR Gaussian case, we formulate a somewhat similar description to (7.10) in the CCR setting.

Theorem 7.2.1 *The block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

gives a quasi-free state with the Markov property if and only if

$$A_{13} = A_{12}f(A_{22})A_{23} \tag{7.11}$$

for any continuous function $f : \mathbb{R} \to \mathbb{R}$.

Proof. If the Markov property holds, then A has the form of (7.2) and we have

$$A_{12}f(A_{22})A_{23} = \begin{bmatrix} a & 0 \end{bmatrix} \begin{bmatrix} f(c) & 0 \\ 0 & f(d) \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = 0 = A_{13}.$$

The converse is part of the proof of Theorem 7.1.1. □

We choose unit vectors e_j, $1 \le j \le 3k$ such that

$$e_{(i-1)k+r} \in \mathcal{H}_i, \qquad 1 \le i \le 3, \quad 0 \le r \le k-1 \tag{7.12}$$

and

$$\langle e_t, e_u \rangle \text{ is real for any } 1 \le t, u \le 3k. \tag{7.13}$$

In the Fock representation the Weyl unitaries $W(te_j) = \exp(tiB(e_j))$ commute and give the (unbounded) field operators $B(e_j)$. It follows from [9, 33] that the classical (multi-valued) Gaussian triplet

$$(B(e_1), \dots, B(e_k)), \quad (B(e_{k+1}), \dots, B(e_{2k})), \quad (B(e_{2k+1}), \dots, B(e_{3k})) \tag{7.14}$$

is Markovian if and only if

$$(I+2A)_{13} = (I+2A)_{12}(I+2A)_{22}^{-1}(I+2A)_{23} \tag{7.15}$$

which means that $(1,3)$ element of $(I+2A)^{-1}$ is 0. If the quasi-free state induced by A gives a Markov triplet, then (7.15) is true and the classical Markov property of (7.14) follows. The converse is not true. However, if for every λA $(\lambda > 0)$ the classical Markov property is true, then from (7.15) we have

$$A_{13} = A_{12}(I/(2\lambda)+A)_{22}^{-1}A_{23}$$

and the Markovianity of the quasi-free state follows.

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