Stabilization of sampled-data nonlinear systems by receding horizon control via discrete-time approximations

Ahmed Mohamed Ahmed Elaiw

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Abstract

This thesis is a systematic presentation of our results in the field of stabilization of sampled data nonlinear systems by receding horizon control. The results presented in the thesis were published in [17], [18], [19], [20], [21], [22], [23], [41] and [42]. The research reported in this thesis was carried out at the Budapest University of Technology and Economics, Mathematical Institute, Department of Differential Equations.

The stabilization problem of nonlinear systems has received considerable attention in recent decades. The use of digital computers in the implementation of the controllers necessitated the investigation of sampled-data systems. However, the research activity has only turned to the systematic study of the stabilization of continuous-time system with piecewise constant controllers on the basis of approximate discrete-time models in the last few years. In our work, the receding horizon (or model predictive) control method is applied for the construction of stabilizing feedback. It is shown that the use of small sampling parameters alone doesn't guarantee the stability of a closed-loop exact system. A necessary and sufficient condition for the existence of stabilizing state-feedback controller is presented. A suitable version of the receding horizon control is presented on the basis of the solution of Bolza-type optimal control problems for the parametrized family of approximate discrete-time models. We investigate both situations when the sampling period T is fixed and the integration parameter h can be chosen to be arbitrarily small, and when these two parameters coincide but can be adjusted arbitrarily. Sufficient conditions are established which guarantee that the controller that renders the origin to be asymptotically stable for the approximate model also stabilizes the exact discrete-time model for sufficiently small integration and/or sampling parameters. These conditions concern directly the data of the problem and the design parameters of the method, but not the results of the design procedure. From practical point of view, it is important to know whether the basin of attraction is sufficiently large when some stabilizing controller is applied. In the case of fixed sampling periods we can only prove local result. However, in the case of an adjustable sampling parameter, we show that the basin of attraction contains any compact subset of the set of initial points which are practically asymptotically controllable to the origin with piecewise constant controllers. (Thus, if the controllability property is semiglobal, then semiglobal practical asymptotic stability is achieved, as well.) The effectiveness of the method is illustrated by simulation examples.

A multirate version of the receding horizon algorithm for the stabilization of sampled-data nonlinear systems is presented. The computations are based on discrete-time approximate models. "Low measurement rate" is assumed, and the presence of measurement and computational delays are taken into account. In this approach, we investigate the case when the sampling period T is fixed and h can be chosen to be arbitrarily small. It is shown that, if the occurring delays are not taken into account, then instability of the closed-loop may occur, but under reasonable assumptions, the proposed algorithm gives a closed-loop system which is semiglobally practically asymptotically stable about the origin. In the second part of the work, the proposed methods are applied to recently developed models of the interaction of the HIV virus and the immune system of the human body. Two kinds of four-dimensional models are considered, in which the drug dose is considered as a control input and the goal is to stabilize the system around the uninfected steady state. Both the one-step and the $\ell-$step versions of the receding horizon control method are used to determine the treatment schedules. Simulation results are discussed.

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Notation and abbreviation

$\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$	The sets of real, natural and integer numbers
$\mathbb{R}^n$	The sets of all n-tuples (vectors) of real numbers
$\forall$	Universal quantifier
$a \in A$	a is an element of set A
$A \subset B$	A is a subset of B
$\dot{x}$	The first derivative of x w.r.t. time t
$\|x\|$	Euclidean norm of x
$\mathcal{B}_\Delta$	Closed ball with radius Δ and center 0
A'	Transpose of matrix or vector A
A^{-1}	Inverse of matrix A
$a \circ b$	Composite operation of a and b
$a \equiv b$	a is equivalent to b
A/D	Analog to digital
D/A	Digital to analog
RHC	Receding horizon control
MPC	Model predictive control
PAC	Practical asymptotic controllability
AC	Asymptotic controllability
PAS	Practical asymptotic stability
AS	Asymptotic stability
HIV	Human Immunodeficiency Virus
AIDS	Acquired Immune Deficiency Syndrome
RTI	Reverse transcriptase inhibitors
PI	Protease inhibitors
HAART	Highly active antiretroviral therapies

Contents

Chapter 1

Introduction

This thesis explores important issues in receding horizon control design of nonlinear sampled data systems. While sampled-data nonlinear control systems are commonly found in real applications, the design tools for such systems are still limited. The research covered in this thesis contributes theoretical results for the receding horizon control design of sampled-data nonlinear systems. The thesis also present applications of the results obtained.

1.1 Sampled-data systems

The stabilization problem of nonlinear systems has received considerable attention in the last decades. The use of digital computers in the implementation of the controllers necessitated the investigation of sampled-data systems. In this problem, a continuous-time plant is typically controlled by a discrete-time feedback algorithm. By controlling a continuous-time plant using a digital controller that operates in a discrete-time environment, we form a sampled-data system. A sampled-data control system therefore consists of a continuous-time plant/process controlled by a digital controller, as a digital computer providing the control action. Consequently, a sampled-data control system is often referred to as computer-controlled system. A general configuration of a sampled-data control system is given schematically in Figure 1 [4].

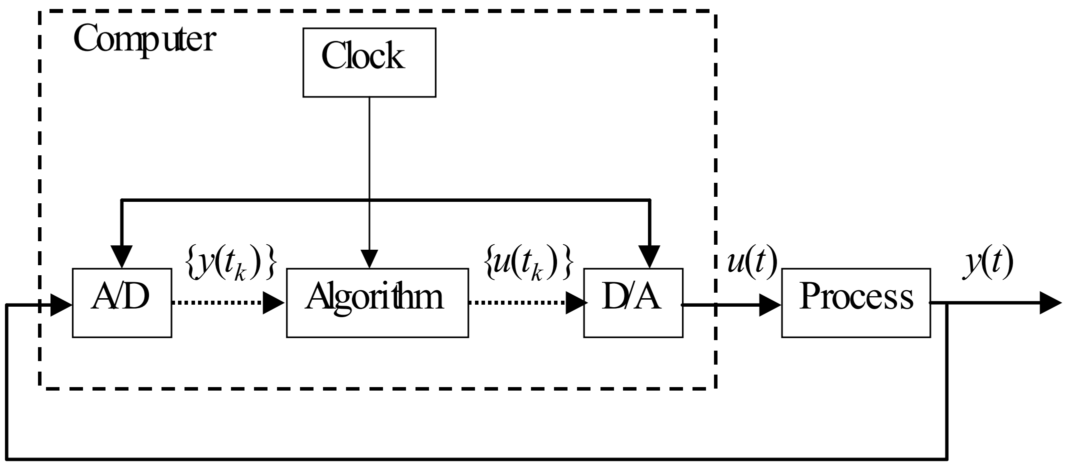

Fig. 1. General sampled-data control system configuration.

In Figure 1, the output from the process $y(t)$ is a continuous-time signal. The output is converted into digital form by the analog-to-digital (A/D) converter. The conversion is done at the sampling time, t_k. The control algorithm interprets the converted signal, $\{y(t_k)\}$, to calculate the required control sequence, $\{u(t_k)\}$. This sequence is converted to an analog signal by a digital-to-analog (D/A) converter. All these events are synchronized by a real-time clock in the computer. The digital computer operates sequentially in time, and each operation takes some time. The D/A converter must, however, produce a continuous-time signal. The simplest way to do this is to keep the control signal constant between each conversion time. In this case the system runs open-loop in the time between the sampling instants because the control signal is constant, irrespective of the value of the plant output $y(t)$. In practice, the A/D and D/A converters can be parts of the computer or may be built as separate units.

1.1.1 Emulation design

One way to design a digital controller is to design a continuous-time controller based on the continuous-time plant model and then discretize it using fast sampling for digital implementation (see e.g., [7], [59], [84], [90], [102]). Also, this approach is proposed in connection with the receding horizon method among the others in [9], [10], [29], [30], [31], [33], [47]. However, some difficulty may arise during the application of this method:

1) because of hardware limitations, it may be impossible to reduce the sampling period to a sufficiently small value that ensures the desired performance of the system. A set of comparative examples in [49] illustrate this point. These examples present comparison of various methods of discretizing a continuous-time controller with the variation of the sampling periods. It is shown that stabilization is in general achieved only when very fast sampling (i.e. very small sampling periods) is used. Moreover, the system may lose the stability when a large sampling period is used;

2) the exact solution of the nonlinear continuous-time model is typically unknown, therefore an approximation procedure is unavoidable;

3) it may be difficult to implement an arbitrarily time-varying control function.

In this method, the sampling is completely ignored at the controller design step, therefore we expect that if the sampling is taken into account in the design, better performance can be achieved. In such cases, a better alternative is a direct discrete-time design which is based on the discrete-time model of the plant.

1.1.2 Direct discrete-time design

The second way to design a digital controller is to discretize the continuous-time plant model and design a controller on the basis of the discrete-time model. In [60] it is shown that the discrete-time controller obtained by direct discrete-time design outperforms the emulation controller. An example presented in [77] has exhibited similar phenomenon, where the direct discrete-time controller consistently shows a larger domain of attraction than the emulation controller. Also in [98], a direct discrete-time controller guarantees asymptotic stability of the closed-loop system that is not achieved by the emulation controller for a two-link manipulator system with Slotine and Lie controller. Therefore, if the sampling is considered from the beginning of the design process, better performance can be achieved. Moreover, larger sampling periods may be applied to the controller obtained using a direct discrete-time design.

While for linear systems we can in principle compute the exact discrete-time model of the plant, this is not the case for nonlinear systems. Computing the exact model involves solving an initial value problem. In the case of nonlinear systems, this involves solving analytically a nonlinear differential equation over one sampling interval, which is impossible in general. Instead, various numerical algorithms are used to approximate the solutions. As a result, the controller design can be carried out by means of an approximate discrete-time model. This has motivated research on controller design via approximate discrete-time models for sampled-data nonlinear systems ([16], [34], [67]). A drawback of these early results was their limited applicability: the studies investigated a particular class of plant models, a particular approximate discrete-time model (usually Euler) and a particular controller.

The issue in direct discrete-time design does not stop as we use an approximate model to design a controller that will be implemented to control the original continuous-time plant. Numerical approximation we use to obtain the approximate model of the plant will certainly cause discrepancy between the exact and the approximate models of the plant. This inaccuracy of modeling can lead the design to fail. It can happen that the controller obtained using the approximation-based design stabilizes the approximate model of the closed-loop system, but destabilizes the exact model of the system for all sampling periods. Therefore, there is a strong concern to build a framework of direct discrete-time design that provide conditions which guarantee that the final design objectives are still achieved even though an approximate model is used for the design.

A more general framework for stabilization of sampled-data nonlinear systems via their approximate discrete-time models that is applicable to general plant models, controllers and approximate discrete-time models was presented in the recent papers, ([74], [75], [78], [79], [80]). In [78] and [80] a systematic investigation of the connection between the exact and approximate discrete-time models is carried out. Moreover, results in [78] and [80] present a set of general sufficient conditions on the continuous-time plant model, the approximate discrete-time plant model, and the designed controller that guarantee that the same family of controllers that stabilizes the approximate discrete-time model also practically stabilizes the exact discrete-time model of the plant both for the cases of fixed sampling period T and varying integration parameter h [78] and for the case when these two parameters coincide [78] and [80].

The approximation-based direct discrete-time design procedure is to:

1) develop a parametrized family of approximate discrete-time models, where the family of approximate models approaches the exact model as the parameter (e.g., integration and/or sampling period) converges to zero;

2) design a corresponding family of discrete-time controllers (e.g., by receding horizon control method);

3) pick the modeling parameter small enough to guarantee stability of the exact nonlinear sampled-data system.

Results in [78] and [80] provide a framework for controller design via approximate discrete-time models, but they did not explain how the actual controller design can be carried out within this framework. Indeed, we can say that the results in [78] and [80] are prescriptive since they can be used to guide one when designing a controller based on an approximate discrete-time plant model. These results were further generalized in [74] and

[75] to deal respectively with integral versions of input-to-state stabilization and input-to-state stabilization of sampled-data nonlinear systems with disturbances. Controller design within this framework is also addressed in ([58], [76], [77] and [79]).

There are several ways to design controllers satisfying the conditions given in [78] and [80]. In [36], optimization-based methods are studied; the design is carried out either via an infinite horizon optimization problem or via an optimization problem over a finite horizon with varying length. To relax the computational burden needed in the case of infinite horizon optimization and in the case of optimization over a varying time interval, the application of the receding horizon method offers good vistas. In [36], it was pointed out that the results presented in that paper were not directly applicable for receding horizon control.

1.1.3 Multirate samplings

Both of the emulation and direct discrete-time approaches mentioned above are essentially single rate, i.e., the sampling rates of the control function and the state measurement coincide. Moreover, it is assumed that the measurement result and the corresponding controller are available instantaneously. The latter assumption is of course unrealistic and may be considered as one of the reasons why different rates of control and measurement samplings have to be taken into account: it is meaningless or impossible to perform a new measurement until the results of the previous one become available and worked up. Besides the measurement and computational delay, the nature of the problem may involve different measurement and control sampling rates (see e.g. [21]).

The notion of multirate sampled-data feedback (which was introduced to the best of our knowledge by [89]) is used in this thesis in this sense. Polushin and Marquez address the design of multirate controllers based on the knowledge of a continuous-time stabilizing feedback for the continuous-time model as well as on that of a discrete-time stabilizing feedback for the approximate model under the assumption of "low measurement rate" and in the presence of measurement delay. However, they did not explain how the actual controller design can be carried out.

1.2 Receding horizon control method

Receding horizon control (RHC), also known as model predictive control (MPC) has received much interest in the academic community in recent years, due to its capacity for handling constraints and obtaining a stabilizing state feedback controller. In addition, receding horizon control has several advantages: 1) it requires simpler computation algorithms than the more widely-known optimal control on an infinite horizon; 2) when a finite future command is available it presents good tracking performance, which is an important issue in industrial applications; 3) the receding horizon control presents a proper control strategy for time-varying systems. While the optimal control on the infinite horizon requires all future system parameters, which are unavailable in actual problems, the receding horizon control needs only finite future system parameters.

The receding horizon method obtains the feedback control by solving a finite horizon optimal control problem at each time instant using the current state of the plant (i.e., $x(t)$) as the initial state for the optimization and applying "the first part" of the optimal control.

Although receding horizon controllers of the first generation did not guarantee closed-loop stability even in the linear case [5] by, now there are several RHC schemes with guaranteed stability.

One way to enforce stability is to solve a Lagrange-type optimization problem subject to *zero terminal equality* constraint $x(t+N) = 0$, where N is the optimization horizon. This method has been established in ([54] and [55]), for time-varying linear systems. The method has been generalized for nonlinear systems in ([11], [70] and [71]), for continuous-time systems and in ([1] and [50]) for discrete-time systems. In [11] and [50], it is shown that, the value function (of a finite horizon optimal control problem) could be used as Lyapunov function to establish stability of the receding horizon control of nonlinear systems; thereafter, the value function was almost universally employed as a Lyapunov function for stability analysis of receding horizon control. In [70], a strong assumption is necessary to guarantee that the optimal value function is continuously differentiable, which is relaxed in [71] only for local Lipschitz continuity of the optimal value. The main advantages of a zero equality terminal constraint are straightforward application and conceptual simplicity. One disadvantage of a zero terminal constraint is that the predicted system state is forced to reach the origin in a finite time. This leads to feasibility problems for short horizon length, i.e., to small regions of attraction. From a computational point of view, solving nonlinear dynamic optimization problems with equality constraints is highly computationally intensive, and in many cases it is impossible to perform within a limited time.

As seen above, the zero terminal equality constraint may be unsatisfactory for performance and implementation issues. Hence, the idea of replacing the equality constraint with an inequality one, which is much easier to handle computationally, is proposed in ([12], [39], [72], [93]). The purpose of this approach is to force the state into a terminal constraint set X_f in a finite time i.e., $x(t+N) \in X_f$. To guarantee stability, they suggested a *dual-mode* receding horizon control scheme with a local linear state feedback controller inside the terminal region and a receding horizon controller outside the terminal region. Closed-loop control with this scheme is implemented by switching between the two controllers, depending on the states being inside or outside the terminal region. The main disadvantage of this scheme is that the switching from nonlinear control law to linear state feedback control is somewhat artificial and introduces certain discontinuities. Moreover, this approach cannot be used if the linearization of the system is not stabilizable.

Another important scheme which was developed in [5] used a terminal cost $g(.)$ to ensure closed-loop stability of unconstrained linear systems. There was no need to impose terminal constraints in this approach and the controller was computed off-line. In addition to these results, a very important result [91] was developed in the context of linear stable systems with input constraints. In this paper, the terminal cost is proposed to be an infinite horizon value function associated with zero control.

A different approach employs both a terminal constraint set X_f as well as a terminal cost $g(.)$. This variant of receding horizon control is the version attracting most attention in current research literature. In this method the terminal constraint $x(t+N) \in X_f$ is imposed in the optimization problem implicitly (see e.g., [14], [37], [38], [43], [47], [56], [66] and [85]) or explicitly (see e.g., [9], [10], [29], [30], [31] and [33]). The main difficulty in [37], and [56] is that the final state penalty function $g(.)$ has to satisfy a Hamilton-Jaccobi-Bellman-type inequality in the whole state space. In ([38], [43], [47] and [85]), the terminal constraint is omitted from the optimization problem and the horizon N is chosen to ensure automatic satisfaction of the terminal constraint. In these papers it is shown that the essential requirement for stability is that $g(.)$ be a control Lyapunov function in the neighborhood of the origin. The terminal constraint set is chosen as a level set of the terminal cost function i.e. $X_f = \{x : g(x) \leq \eta\}$. It is shown in ([38], [47], [85]) that, the domain of attraction contains a level set of the optimal value function $V_N(.)$. In [43], the nonlinear receding horizon control closed-loop system is shown to be infinite-horizon with the new definition of the terminal cost, provided that the terminal cost exactly captures the infinite-horizon optimal value in a neighborhood of the origin. Moreover, stability and optimality are proven for a set of initial states, which is invariant and approaches the set of all controllable states, as the prediction horizon N increases. In ([14], [66] and [86]), stability of the receding horizon control scheme is guaranteed by using a finite horizon optimization problem with a (possibly non-quadratic) terminal state penalty. The penalty is equal to the cost incurred over an infinite horizon by applying a (locally stabilizing) linear control law to the nonlinear system. However, the linear control law is never implemented but it is just used to compute the terminal state penalty. In these papers, a state inequality constraint is implicitly imposed.

Quasi-infinite horizon RHC is established in [8], [9], [27]. In this approach, the terminal cost and the terminal constraint set are obtained on the basis of a locally stabilizing linear control law considering a quadratic cost function. The terminal cost and the terminal constraint set are chosen as $g(x) = x'Px$ and $X_f = \{x : x'Px \leq \eta\}$, respectively. If the linearized system is stabilizable, then P and X_f can be computed off-line. However, the linear control law is never actually applied but it is just used to compute X_f and $g(.)$. If $g(.)$ and X_f are determined accordingly, the open-loop optimal trajectories found at each time instant approximate the optimal solution for the infinite horizon problem.

The disadvantage of the approaches proposed in [8], [9], [14], [27] and [66]) is that, if the linearization of the system around the origin cannot be stabilized then these approaches cannot be used.

When there are no constraints on the state, then the terminal constraint can be omitted from the optimization problem to reduce the computational burden in case of explicit terminal constraint. Moreover, a larger domain of attraction can be achieved by increasing the optimization horizon or by weighting the terminal cost (see [63]).

Results in [101] present a RHC scheme that deviates from conventional RHC schemes in that the control horizon is also a minimizer and the whole input sequence is implemented. In this scheme inequality contractive constraints are added so that the norm of the state vector is reduced by a prespecified factor before a new optimization begins. The stability is guaranteed by assuming the existence of the solution of the optimization problem at each time. However this is a very strong assumption and cannot be guaranteed in general (see [68]).

In nonlinear receding horizon control, several schemes exist that guarantee closed-loop stability but, the necessary on-line computation time is typically not taken into account. Even though recent developments in dynamic optimization have led to efficient numerical solution methods for the open-loop optimal control problem (see e.g., [28]), the solution time is often significant. Few works take the computational delay into account (see [10] and [29]). In these papers the proposed approaches are based on the continuous-time model (emulation design). In some applications, the necessary time to measure the state is significant and this causes measurement delay. Taking into account the resulting delays are thus of paramount interest. Otherwise the performance might degrade significantly or even instability of the closed-loop may occur.

A great majority of stabilization of nonlinear systems by the RHC method deal; 1) with continuous-time systems with sampling (see e.g., [9], [10], [29], [30], [31], [33], [47], [72]). In these papers, emulation designs are used which suffer from the drawbacks mentioned above; 2) with continuous-time systems without taking into account any sampling (see e.g., [11], [37], [38], [66], [70]). Although rigorous stability results for nonlinear RHC have been established in these papers, they are not applicable in practical implementations; 3) with discrete-time systems assuming that the exact discrete-time model of the plant is known (see e.g. [1], [14], [40], [43], [50], [64], [85]). However, as we have already pointed out, this is typically not true even if the continuous-time plant model is known exactly. We refer the reader to the excellent overview papers ([2], [15], [32], [57], [69]) and to the references therein. To the best of our knowledge, the only exceptions are the very recent works of [46], where the effect of the sampling and zero-order hold is considered assuming the existence of a global control Lyapunov function and of [65], where a sampled-data control is applied to the continuous-time system without taking into account any approximation in the plant model.

1.3 RHC for HIV/AIDS

One of the very important applications of the optimal control theory is to determine optimal treatment schedules of Acquired Immune Deficiency Syndrome (AIDS). The interaction of Human Immunodeficiency Virus (HIV) (which causes AIDS) with the immune system of the human body can be described by mathematical models. One way to design optimal treatment is to design an open-loop optimal controller by using Pontryagin's Maximum Principle (see [6], [13], [26] and [52]). However, some drawbacks may arise during the application of this method: 1) the optimization is performed over a finite time horizon, and no care is taken over the evolution of the process behind this time horizon; 2) the optimal controller is obtained as a continuous-time controller, in spite of the fact that continuous variation of the dose seems hard to apply in the real treatment of patients; 3) the optimal controller is given in an open-loop form and it does not deal with the changes that may happen in the system during the treatment.

To overcome the above-mentioned problems, the application of the receding horizon control method (RHC) based on the approximate discrete-time model seems to be obvious.

Last year, some papers [17], [19], [21], [23], [94], [100] appeared simultaneously (and independently of each other) with the application of receding horizon control method to stabilize AIDS models. In [94] and [100], the effect of the discretization of the continuous-time model on the stability analysis is completely ignored. Moreover, the sampling is completely ignored at the controller design step.

1.4 Basic definitions and background results

In this section we formulate some definitions and various important results from the literature that are required to prove results in this thesis.

Standard notation will be used throughout the thesis. The sets of real and natural numbers (including zero) are respectively denoted as $\mathbb{R}$ and $\mathbb{N}$. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

is of class-$\mathcal{K}$ if it is continuous and strictly increasing with $\sigma(0) = 0$. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and unbounded. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{L}$ if it is continuous and strictly decreasing to zero. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(., \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, .)$ is of class-$\mathcal{L}$ for each $s > 0$. The Euclidean norm of a vector x is denoted as $\|x\|$. In what follows, the notation $\mathcal{B}_\Delta = \{z \in \mathbb{R}^p : \|z\| \leq \Delta\}$ will be used both in $\mathbb{R}^n$ and $\mathbb{R}^m$.

The following Lemma states some important properties of class $\mathcal{K}$ and $\mathcal{KL}$ functions.

Lemma A [51] Let $\sigma_1(.)$ and $\sigma_2(.)$ be class-$\mathcal{K}$ functions on $\mathbb{R}_{\geq 0}$, $\sigma_3(.)$ and $\sigma_4(.)$ be class-$\mathcal{K}_\infty$ functions, and $\beta(.,.)$ be a class-$\mathcal{KL}$ function. Then

- $\sigma_1^{-1}(.)$ is defined on $\mathbb{R}_{\geq 0}$ and belongs to class-$\mathcal{K}$
- $\sigma_3^{-1}(.)$ is defined on $\mathbb{R}_{\geq 0}$ and belongs to class-$\mathcal{K}_\infty$
- $\sigma_1 \circ \sigma_2$ belongs to class-$\mathcal{K}$
- $\sigma_3 \circ \sigma_4$ belongs to class-$\mathcal{K}_\infty$
- $\sigma(r, s) = \sigma_1^{-1}(\beta(\sigma_2(r), s))$ belongs to class-$\mathcal{KL}$. □

Consider a continuous-time nonlinear plant

$$\dot{x}(t) = f(x(t), u(t)), \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $f(0, 0) = 0$, U is closed and $0 \in U$. The system is to be controlled digitally using piecewise constant control functions $u(t) = u(kT) =: u(k)$, if $t \in [kT, (k + 1)T)$, $k \in \mathbb{N}$, where $T > 0$ is the sampling period.

There are two approaches to perform discretization with respect to discretizing space: time discretization and state discretization. In this thesis, only time discretization will be used.

The exact discrete-time model of system (1.1) which describes the behavior of the system at the sampling instants kT, $k \in \mathbb{N}$, is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k)), \tag{1.2}$$

with given $u(k)$ and $x_0 = x(k)$, over the sampling interval $[kT, (k + 1)T]$. If we denote $\phi^E(t, x_0, \overline{u})$ the solution of the initial value problem (1.2) at time t with given $\overline{u} = u(k)$ and $x(k) = x_0$, then the exact discrete-time model of (1.1) can be written as

$$\begin{aligned} x_{k+1} &= x_k + \int_{kT}^{(k+1)T} f(x(s), u(k))ds \\ &=: F_T^E(x_k, u_k), \end{aligned} \tag{1.3}$$

where $F_T^E(x, u) := \phi^E(T, x, u)$. This model is well-defined when the continuous-time model (1.1) does not exhibit finite escape time within $[kT, (k + 1)T]$. The case of finite escape time will be discussed later.

We note that, since f is typically nonlinear, F_T^E in (1.3) is not known in most cases, therefore, the controller design can be carried out by means of an approximate discrete-time model. Techniques available for the time discretization are divided into one step and multi step approximations, while we limit our description only to the techniques belonging to the former. The approximate discrete-time model can be defined as

$$x_{k+1} = F_{T,h}^A(x_k, u_k), \tag{1.4}$$

where $T \in (0, T^*]$ is the sampling period with some upper bound $T^* > 0$ and $h \in (0, T]$ is a parameter for the accuracy of the approximate model, e.g., the integration step for some underlying numerical one-step approximation.

Concerning the parameters T and h, two cases are distinguished in the literature:

(i) $T = h$ and T can be adjusted (see e.g., [3], [36], [78] and [80]). The main motivation for using this approach is a reduced computational burden obtaining the approximate model. All of these results require fast sampling and in general they produce semiglobally practically stabilizing control feedback.

(ii) $T \neq h$. In this case T is fixed and the family of approximate discrete-time models is generated by a numerical integration method with adjustable integration step h. This approach is more realistic and usually produces better results, but the required numerical computations are more intensive, and moreover, semiglobal stabilization is not possible in general due to possible finite escape times between sampling instants (see e.g., [3], [36], [78] and [89]).

The map $F^A_{T,h}$ defining the approximate model is typically interpreted as a numerical approximation of F^E_T using some suitable numerical scheme. For instance $F^A_{T,h}$ may be constructed using multiple steps of a one-step Runge-Kutta scheme, Φ_{h_i} with integration step sizes h_i, $i = 1, ..., m$, satisfying $h_i \leq h$ and $\sum_{i=1}^m h_i = T$; i.e.,

$$x_0 = x, \quad x_{i+1} = \Phi_{h_i}(x_i, u), \quad F^A_{T,h}(x, u) = x_m.$$

Note that, for constant control function u, system (1.1) is an autonomous ODE, and hence all numerical schemes for autonomous ODEs are applicable; see e.g., [96] and [97] for a description of suitable numerical methods.

In the simplest case, Φ_{h_i} can be chosen as the Euler method $\Phi_{h_i}(x, u) = x + h_i f(x, u)$. For linear system

$$\dot{x} = Ax + Bu,$$

$$\Phi_h(x, u) = (I + hA)\, x + hBu$$

where $h = T/m$, then

$$F^A_{T,h} = (I + hA)^{T/h}\, x + A^{-1}\left((I + hA)^{T/h} - I\right) Bu.$$

As $h \to 0$, this $F^A_{T,h}$ converges to their exact model

$$F^E_T = e^{AT} x + A^{-1}\left(e^{AT} - I\right) Bu.$$

If $h = T$ then we obtain $F^A_{T,h} = (I + TA)\, x + TBu$. Note that, for any T, h, the numerical scheme $F^A_{T,h}$ is normally well-defined for all x, u because the computation of $F^A_{T,h}$ is usually based on finitely many evaluations of f only.

The problem is to define a state-feedback

$$u_k = u^A_{T,h}(x_k) \tag{1.5}$$

using the approximate model (1.4) which stabilizes the origin for the exact model (1.3) in an appropriate sense.

Definition 1 Let strictly positive real numbers (T, Δ', Δ'') be given. If there exists $h^* > 0$ such that

$$\sup_{\{x \in \mathcal{B}_{\Delta'},\, h \in (0,h^*]\}} \left\| u^A_{T,h}(x) \right\| \leq \Delta'',$$

then the family of controllers (1.5) is said to be (T, Δ', Δ'')-*uniformly bounded.*

In what follows, a stabilizing feedback will be constructed for the approximate model and conclusion about the stability of the closed-loop exact model is drawn on the basis of the closeness of solutions of the two models. This closeness will be characterized by the following definition. It guarantees that the error between solutions starting from the same initial state is small, over one step, relative to the size of the step.

Definition 2 Let a triplet of strictly positive real numbers (T, Δ', Δ'') be given and suppose that there exist $\gamma \in \mathcal{K}$ and $h^* > 0$ such that

$$(x, u) \in \mathcal{B}_{\Delta'} \times \mathcal{B}_{\Delta''}, \quad h \in (0, h^*] \Longrightarrow \left\| F^A_{T,h}(x,u) - F^E_T(x,u) \right\| \leq T\gamma(h), \tag{1.6}$$

then we say that the family $F^A_{T,h}$ is (T, Δ', Δ'')-*consistent* with F^E_T. Moreover, if $T = h$ and, for any pair of strictly positive numbers (Δ', Δ'') there exist $\gamma \in \mathcal{K}$ and $T^* > 0$ such that (1.6) holds true, then F^A_T is said to be *semiglobally consistent* with F^E_T.

The "consistency" property described in Definition 2 is an adaptation of the consistency property used in the numerical analysis literature (see e.g. [96] and [97]). We have to emphasize that this property can be checked without explicit knowledge of the exact discrete-time model. Sufficient checkable conditions for one-step consistency are given in the following lemma:

Lemma B [80]. If
1. (u_T, F^A_T) is one-step consistent with (u_T, F^{Euler}_T) where $F^{Euler}_T(x,u) := x + Tf(x,u)$,
2. for each of the positive numbers (Δ', Δ'') there exist $\rho \in \mathcal{K}$, $M > 0$, $T^* > 0$ such that for all $T \in (0, T^*]$, all $x, y \in \mathcal{B}_{\Delta'}$ and all $u_T \in \mathcal{B}_{\Delta''}$
a) $\|f(y, u_T(x))\| \leq M$,
b) $\|f(y, u_T(x)) - f(x, u_T(x))\| \leq \rho\left(\|y - x\|\right)$,
then (u_T, F^A_T) is one-step consistent with (u_T, F^E_T). $\square$

A special case where condition 2(b) of the lemma holds is when $f(.,u)$ is locally Lipschitz uniformly in u. Note that the function $u_T(x)$ does not need to be continuous for (u_T, F^A_T) to be one-step consistent with (u_T, F^E_T).

The stability property for the approximate discrete-time model can be characterized using the well known criterion of Lyapunov for asymptotic stability.

Definition 3 Let a pair of strictly positive real numbers (T, D), a family of functions $V_{T,h} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions σ_1, $\sigma_2 \in \mathcal{K}_\infty$ and a positive definite function $\sigma_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be given. Suppose that for any pair of strictly positive real numbers (δ_1, δ_2) with $\delta_2 < D$ there exist $h^* > 0$ and $c > 0$ such that for all $x \in \mathcal{B}_D$, $h \in (0, h^*]$, we have

$$\sigma_1(\|x\|) \leq V_{T,h}(x) \leq \sigma_2(\|x\|), \tag{1.7}$$

$$V_{T,h}(F^A_{T,h}(x, u^A_{T,h}(x))) - V_{T,h}(x) \leq -T\sigma_3(\|x\|), \tag{1.8}$$

and, for all $x_1, x_2 \in \mathcal{B}_D - \mathcal{B}_{\delta_2}$, with $\|x_1 - x_2\| \leq c$ we have

$$|V_{T,h}(x_1) - V_{T,h}(x_2)| \leq \delta_1, \tag{1.9}$$

then we say that the family (1.4), (1.5) is (T, D)-*stable with a continuous Lyapunov function.*

Lemma C [80]. The following statements are equivalent:

1. There exist $\beta \in \mathcal{KL}$ and $T^* > 0$ such that for each $T \in (0, T^*]$ the solution of the system

$$x(k+1) = F_T\left(x(k), u_T(x(k))\right), \tag{1.10}$$

satisfy:

$$\|\phi_T(k, x_0)\| \leq \beta(\|x_0\|, kT), \quad \forall x_0 \in \mathbb{R}^n, \ \ k \geq 0.$$

2. There exists $T^* > 0$, σ_1, $\sigma_2 \in \mathcal{K}_\infty$, $\sigma_3 \in \mathcal{K}$ and V_T and such that the family (u_T, F_T) satisfy

$$\sigma_1(\|x\|) \leq V_T(x) \leq \sigma_2(\|x\|),$$

$$V_{T,h}(F_T(x, u_T(x))) - V_T(x) \leq -T\sigma_3(\|x\|),$$

for all $T \in (0, T^*]$ and all $x \in \mathbb{R}^n$.

Theorem A ([36] and [78]) Suppose that there exists a triplet of strictly positive numbers (T, D, M) such that

(i) the family of the closed-loop systems $(F^A_{T,h},\ u^A_{T,h})$ is (T, D)-stable with a continuous Lyapunov function;

(ii) the family of controllers $u^A_{T,h}$ is (T, D, M)-uniformly bounded;

(iii) the family $F^A_{T,h}$ is (T, D, M)-consistent with F^E_T.

Then, there exists $\beta \in \mathcal{KL}$, $D_1 \in (0, D)$ and for any $\delta > 0$ there exists $h^* > 0$ such that for all $x_0 \in \mathcal{B}_{D_1}$ and $h \in (0, h^*]$, the solutions of the family $(F^E_T, u^A_{T,h})$ satisfy:

$$\|\phi^E_T(k, x_0)\| \leq \beta(\|x_0\|, kT) + \delta, \quad k \in \mathbb{N}_0. \tag{1.11}$$

1.5 The aim of the thesis

Results in ([78] and [80]) give sufficient conditions for a controller designed by means of the approximate model to stabilize the exact model, as well. As is emphasized by the title of [78], these results provide a framework for controller design relying on approximate discrete-time models, but they did not explain how to find controllers that satisfy the given conditions. The purpose of the present work is to construct a receding horizon controller within this framework (with a slight modification in the case $T = h$). We study the conditions under which the stabilizing receding horizon control computed for the approximate discrete-time model also stabilizes the exact discrete-time system both in the cases when the sampling period T is fixed, but the integration parameter h used in obtaining the approximate model can be chosen to be arbitrarily small and when these two parameters coincide, but can be adjusted. It should be emphasized that these conditions concern directly the data of the problem and the design parameters of the method, but not the results of the design procedure. From a practical point of view, it is important to know whether the basin of attraction is sufficiently large when some stabilizing controller is applied.

In this thesis we drive a multirate version of the receding horizon algorithm based on discrete-time approximate models of the plant, and establish sufficient conditions which guarantee that the proposed control stabilizes the original exact model in the case of "low measurement rate" in the presence of measurement and computational delays. In this approach the sampling period T is considered as fixed and the integration period h is allowed to vary.

In the second part of the work, we apply the theoretical results for two different HIV/AIDS models. The drug dose is considered as a control input and the goal is to stabilize the system around the uninfected steady state. We use both the one-step and the $\ell-$step versions of the receding horizon control method to determine the treatment schedules.

Chapter 2

Stabilization of sampled-data systems

Results on stabilizing receding horizon control of sampled-data nonlinear systems via their approximate discrete-time models are presented. The proposed receding horizon control is based on the solution of Bolza-type optimal control problems for the parametrized family of approximate discrete-time models. We investigate both situations when the sampling period T is fixed and the integration parameter h can be chosen to be arbitrarily small, and when these two parameters coincide but can be adjusted arbitrarily. Both single-rate and multirate versions of the receding horizon algorithm for the stabilization of sampled-data nonlinear systems are investigated. In the latter case "low measurement rate" is assumed, and the presence of measurement and computational delays are taken into account. Sufficient conditions are established which guarantee that the controller that renders the origin to be asymptotically stable for the approximate model also stabilizes the exact discrete-time model for sufficiently small integration and/or sampling parameters.

The results presented in this chapter were published in ([18], [20], [22], [41] and [42]).

2.1 Problem statement

Consider the nonlinear control system described by

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad x(0) = x_0 \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, with $f(0,0) = 0$, U is closed and $0 \in U$. Assumption $f(0,0) = 0$ is not very restrictive, since if $f(x_s, u_s) = 0$, one can always shift the origin of the system to (x_s, u_s).

Assumption A1 (i) The function f is continuous.

(ii) For any pair of positive numbers (Δ', Δ'') there exists an $L_f = L_f(\Delta', \Delta'')$ such that

$$\|f(x,u) - f(y,u)\| \leq L_f \|x - y\|,$$

for all $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$. □

Let $\Gamma \subset \mathbb{R}^n$ be a given compact set containing the origin and consisting of all initial states to be taken into account. The system is to be controlled digitally using piecewise constant control functions

$$u(t) = u(kT) =: u_k, \quad \text{if} \quad t \in [kT, (k+1)T), \quad k \in \mathbb{N},$$

where $T > 0$ is the sampling period. First, we assume that all state measurements are available at the sampling instants kT, $k \in \mathbb{N}$ and the sampling rates of the control function and the state measurement coincide, i.e. $y(k) = x(kT)$.

Under the conditions on f, for any $\overline{x} \in \mathcal{B}_{\Delta'}$ and $\overline{u} \in \mathcal{B}_{\Delta''}$ there exists an $\omega = \omega(\overline{x}, \overline{u}) > 0$ such that equation (2.1) with $u(t) \equiv \overline{u}$, $(t \in [0, \omega))$ and initial condition $x(0) = \overline{x}$ has a unique solution on $[0, \omega)$ denoted by $\phi^E(., \overline{x}, \overline{u})$.

The exact discrete-time model of system (2.1), can be defined as

$$x_{k+1} = F_T^E(x_k, u_k), \tag{2.2}$$

where $F_T^E(x,u) := \phi^E(T,x,u)$, if $T < \omega(x,u)$ otherwise $F_T^E(x,u)$ is defined to be an arbitrary element of $\mathbb{R}^n$ with a sufficiently large norm.

Remark 1 If Assumption A1 is valid, then F_T^E is continuous in x and u and it satisfies a local Lipschitz condition of the following type: for any pair of positive numbers (Δ', Δ'') there exist $T^* > 0$ and $L_f > 0$ such that

$$\|F_T^E(x,u) - F_T^E(y,u)\| \le e^{L_f T}\|x - y\|, \tag{2.3}$$

holds for all $u \in \mathcal{B}_{\Delta''}$, all $T \in (0, T^*]$, and all $x, y \in \mathcal{B}_{\Delta'}$. □

We emphasize that F_T^E in (2.2) is not known in most cases, therefore the controller design can be carried out by means of an approximate discrete-time model

$$x_{k+1} = F_{T,h}^A(x_k, u_k), \tag{2.4}$$

where $T \in (0, T^*]$ is again the sampling parameter, while parameter h is a modelling parameter, which is typically the step size of the underlying numerical method. We define the control sequence $\mathbf{u} = \{u_0, u_1, ...\}$ where u_i, $i = 0, 1, ...$ are piecewise constant control functions. For the solutions of (2.2) and (2.4) with the control sequence $\mathbf{u} = \{u_0, u_1, ...\}$ satisfying the initial conditions $x_0^E = x$ and $x_0^A = x$ we shall use the notation $\phi_k^E(x, \mathbf{u})$ and $\phi_k^A(x, \mathbf{u})$, respectively.

When $T = h$ we use the short-hand notation $F_T^A(x,u) := F_{T,T}^A(x,u)$. We use this notation for other functions in the sequel; that is, whenever we drop the h subscript, we refer to the situation when $h = T$.

Assumption A2 (i) $F_{T,h}^A(0,0) = 0$, $F_{T,h}^A$ is continuous in both variables uniformly in small h, and it satisfies a local Lipschitz condition: for any pair of positive numbers (Δ', Δ'') there exist $h^* > 0$ and $L_{F^A} > 0$ such that

$$\|F_{T,h}^A(x,u) - F_{T,h}^A(y,u)\| \le e^{L_{F^A} T}\|x - y\|,$$

holds for all $u \in \mathcal{B}_{\Delta''}$, all $h \in (0, h^*]$, and all $x, y \in \mathcal{B}_{\Delta'}$.

(ii) In the case of $T = h$ there exist $r_0 > 0$, $T^* > 0$ and $L_{r_0}^A > 0$ such that for any $T \in (0, T^*]$ we have

$$\left\|F_T^A(x,u) - x\right\| \le T L_{r_0}^A(\|x\| + \|u\|), \quad \|x\| + \|u\| \le r_0. \; \square \tag{2.5}$$

Remark 2 Observe that, if Assumption A1 holds true, then for many one-step numerical methods, the assertions of Assumption A2 can be proven.

Let $\widetilde{\Gamma}$ be a suitable set containing at least Γ.

The problem is to define a state-feedback controller

$$v^A_{T,h} : \widetilde{\Gamma} \to U \tag{2.6}$$

using the approximate model (2.4) which stabilizes the origin for the exact model (2.2) in an appropriate sense.

One may think that if a controller is designed for an approximate discrete-time model (obtained by a convergent numerical method) of the plant with a sufficiently small modeling parameter then the same controller will stabilize the exact discrete-time model. Note that if this was true then one could directly apply the existing theory that assumes that the exact discrete-time model is known. However, this reasoning is wrong since no matter how small the modeling parameter is, we may find a controller that stabilizes the approximate model for that modeling parameter but destabilizes the exact model for the same modeling parameter as illustrated by the following examples.

2.2 Motivating examples

In this section, we present three examples for which a family of receding horizon control law is designed to stabilize the family of approximate models, but the exact discrete-time model is destabilized by the same family of controllers. These examples are parallel to those in ([36], [78] and [80]). In fact, the systems of Example 2 and Example 3 were also investigated in [36], [78] and [80], respectively, but the controllers are derived here by the receding horizon method, while the controllers of the cited papers come from different considerations. In these examples the exact discrete-time models of the corresponding systems can be computed, but we base our control algorithm on the family of approximate discrete-time models in order to illustrate possible wrong situation that may arise in receding horizon control design based on approximate discrete-time models.

Example 1 We consider the sampled-data control of the double integrator

$$\dot{x}_1 = x_{2,} \qquad \dot{x}_2 = u,$$

for which the family of exact discrete-time models can be given as

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k) + \frac{T^2}{2}u(k), \\ x_2(k+1) &= x_2(k) + Tu(k). \end{aligned} \tag{2.7}$$

The family of Euler approximate models is

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k), \\ x_2(k+1) &= x_2(k) + Tu(k). \end{aligned} \tag{2.8}$$

Since the Euler method is convergent, the trajectory of the approximate model (2.8) converges to the trajectory of the exact model (2.7) as $T \to 0$.

The receding horizon control for (2.8) with a standard quadratic cost function

$$J_T(x_0, \mathbf{u}) = \sum_{k=0}^{N-1} (x_k' Q_T x_k + u_k' R_T u_k) + x_N' G x_N, \tag{2.9}$$

with the choice of $N = 2$, $G = 0$, $R_T = T^2$, $Q_T = \begin{pmatrix} 17/T^2 & 4/T \\ 4/T & 1 \end{pmatrix}$ can be computed:

$$v_T^A(x(k)) = -\frac{2}{T^2} x_1(k) - \frac{2.5}{T} x_2(k). \tag{2.10}$$

This gives the family of approximate closed-loop models

$$\begin{aligned} x_1(k+1) &= x_1(k) + T x_2(k), \\ x_2(k+1) &= -\frac{2}{T} x_1(k) - \frac{3}{2} x_2(k). \end{aligned}$$

The eigenvalues of the coefficient matrix are $|\lambda_1| = |\lambda_2| = \sqrt{2}/2$, thus the approximate closed-loop model is asymptotically stable.

The family of exact closed-loop models is

$$\begin{aligned} x_1(k+1) &= -\frac{T}{4} x_2(k), \\ x_2(k+1) &= -\frac{2}{T} x_1(k) - \frac{3}{2} x_2(k), \end{aligned}$$

which has a pole at -1.78078 for all $T > 0$, hence the receding horizon controller destabilizes the exact model for any sampling period.

Note that the receding horizon controller (2.10) is not uniformly bounded in T : in fact, $v_T^A(x) \to \infty$ as $T \to 0$ for any $x \neq 0$. Similar observation can also be made regarding the trajectories of the approximate model: e.g. let $x_0 = (1, 0)'$, we have $|x_2(1)| = \frac{2}{T} \to \infty$ as $T \to 0$.

Example 2 We consider the sampled-data control of the triple integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u$$

for which the family of exact discrete-time models can be given as

$$\begin{aligned} x_1(k+1) &= x_1(k) + T x_2(k) + \frac{1}{2} T^2 x_3(k) + \frac{1}{6} T^3 u(k), \\ x_2(k+1) &= x_2(k) + T x_3(k) + \frac{1}{2} T^2 u(k), \\ x_3(k+1) &= x_3(k) + T u(k). \end{aligned}$$

The family of Euler approximate models is

$$\begin{aligned} x_1(k+1) &= x_1(k) + T x_2(k), \\ x_2(k+1) &= x_2(k) + T x_3(k), \\ x_3(k+1) &= x_3(k) + T u(k). \end{aligned} \tag{2.11}$$

Let us choose the following $N = 2$, $G = 0$, $R_T = T^2$,

$$Q_T = \begin{pmatrix} a^2/T^5 & ab/(2T^4) & a/T^2 \\ ab/(2T^4) & b^2/(T^4) & b/T \\ a/T^2 & b/T & 1 \end{pmatrix} > 0, \ 0 < T < 4/5,$$

where $a = 1 - \sqrt[3]{2} + \sqrt[3]{4}$ and $b = 5 + \sqrt[3]{4}$. The receding horizon control for (2.11) with a cost function of type (2.9) can be computed:

$$v_T^A(x(k)) = -\frac{1}{2}\left(\frac{a}{T^3}x_1(k) + \frac{a+b}{T^2}x_2(k) + \frac{b+1}{T}x_3(k)\right).$$

This gives the family of approximate closed-loop models

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k), \\ x_2(k+1) &= x_2(k) + Tx_3(k), \\ x_3(k+1) &= -\frac{1}{2}\left(\frac{a}{T^2}x_1(k) + \frac{a+b}{T}x_2(k) + (b-1)x_3(k)\right). \end{aligned}$$

The eigenvalues of the coefficient matrix are $|\lambda_1| = |\lambda_2| = |\lambda_2| = 1/\sqrt[3]{2}$, thus the approximate closed-loop model is asymptotic stable. The family of exact closed-loop models is

$$\begin{aligned} x_1(k+1) &= \left(1 - \frac{a}{12}\right)x_1(k) + T\left(1 - \frac{a+b}{12}\right)x_2(k) + \frac{T^2(5-b)}{12}x_3(k), \\ x_2(k+1) &= -\frac{a}{4T}x_1(k) + \left(\frac{4-a-b}{4}\right)x_2(k) + \frac{T(3-b)}{4}x_3(k), \\ x_3(k+1) &= -\frac{1}{2}\left(\frac{a}{T^2}x_1(k) + \frac{a+b}{T}x_2(k) + (b-1)x_3(k)\right), \end{aligned}$$

which has a pole at -3.98226, hence the receding horizon controller destabilizes the exact model for all $0 < T < 4/5$.

We note that similarly to Example 1 both the controller and the trajectories of the approximate closed-loop model are not uniformly bounded in T. It may appear that these are the only reasons why instability of the exact model occurs. To show that there are other reasons we shall show the following example.

Example 3 Consider the scalar linear system

$$\dot{x} = x + u.$$

The family of exact discrete-time models is

$$x(k+1) = e^T x(k) + (e^T - 1)u(k), \tag{2.12}$$

while the family of "partial Euler" approximate discrete-time models is

$$x(k+1) = (1+T)x(k) + (e^T - 1)u(k). \tag{2.13}$$

We note that the "partial Euler" method is convergent, so the trajectory of the approximate model (2.13) converges to the trajectory of the exact model (2.12) as $T \to 0$.

The receding horizon control for (2.13) with a cost function of type (2.9) and with $N = 2$, $G = 0$, $R_T = (e^T - 1)^2$, $Q_T = T + \frac{1}{2}T^2$ can be given by

$$v_T^A(x(k)) = -\frac{(1+T)(T+\frac{1}{2}T^2)}{(e^T-1)(1+T+\frac{1}{2}T^2)}x(k)$$

which yields the stable approximate closed-loop model

$$x(k+1) = \left(1 - \frac{\frac{1}{2}T^2}{1+T+\frac{1}{2}T^2}\right)x(k),$$

and the solution of the approximate closed-loop model is given by

$$x(k) = \left(1 - \frac{1}{2}T^2\frac{1}{1+T+\frac{1}{2}T^2}\right)^k x_0.$$

However, the same family of controllers yields the unstable exact closed loop model

$$x(k+1) = \left(1 + \frac{2}{3}T^3 + O(T^4)\right)x(k).$$

In this example, both the controller and the trajectories of the approximate closed-loop model are uniformly bounded in T over any compact set, but the rate of convergence is not uniform in T. The reason for these problems in the three examples is the bad choice of cost function. Similar phenomena may arise in the case of discontinuous Lyapunov function (see [78]).

Remark 3 The interpretation of the above results is as follows. One cannot first find a sufficiently "good" approximate plant model with a sufficiently small sampling and/or integration period and then assume that the receding horizon controller for the approximate model with respect to a given cost would stabilize the exact model. Indeed, because of the fact that we are considering parametrized systems and costs, the examples illustrate that, given an arbitrarily small sampling period (and hence an arbitrarily "good" plant model), there still exists a cost function for which the receding horizon controller obtained by the approximate model would destabilize the exact model. Hence, a careful investigation of stability is needed to avoid situations like those presented in the examples.

2.3 Practical asymptotic controllability and stabilizability

Since we want to find a state-feedback controller, it seems to be reasonable to investigate when it does exist. In this section, we present a necessary and sufficient condition for the existence of such a controller.

Let $\Gamma \subset \mathbb{R}^n$ be a given compact set, containing a neighborhood of the origin and let $\Delta > 0$ be such that $\Gamma \subset \mathcal{B}_\Delta$. Consider the family of discrete-time systems

$$x_{k+1} = F_T(x_k, u_k), \tag{2.14}$$

parametrized by $T > 0$ together with a parametrized family of admissible control sequences $\mathcal{U}^h$, $h > 0$, $\mathcal{U}^h = \left\{\mathbf{u}^h = (u_0^h, u_1^h, \ldots), u_k^h \in U\right\}$, where the two parameters may be different or may coincide.

System (2.14) can be obtained by discretizing the continuous-time model (2.1). The discrete-time model F_T is called an exact discrete-time model if it is obtained as the exact solution of the initial value problem of the continuous-time model over the sampling interval and it is called an approximate discrete-time model if it is obtained via numerical approximation.

For continuous-time systems, asymptotic stability can be achieved, but this is in general not the case for sampled-data systems. We show in an example (see Appendix) that, if we have a stabilizing controller for a continuous-time system and discretize this system, then practical stability can only be achieved even if the exact discrete-time model is known.

Definition 4 System (2.14) is *practically asymptotically controllable* (PAC) from Γ to the origin with the parametrized family $\mathcal{U}^h$, if there exist a $\beta(.,.) \in \mathcal{KL}$ and a continuous, positive and nondecreasing function $\sigma(.)$ which are independent of T and h, and such that for any $r > 0$ there exists a $h^* > 0$ so that for all $x \in \Gamma$ and for all $h \in (0, h^*]$ there exists a control sequence $\mathbf{u}^h(x) \in \mathcal{U}^h$, such that $\left\| u_k^h(x) \right\| \leq \sigma(\|x\|)$, and the corresponding solution ϕ of (2.14) satisfies the inequality

$$\left\| \phi_k(x, \mathbf{u}^h(x)) \right\| \leq \max \left\{ \beta(\|x\|, kT), r \right\}, \qquad k \in \mathbb{N}. \tag{2.15}$$

Definition 5 System (2.14) is *practically asymptotically stabilizable* (PAS) in Γ about the origin with the parametrized family $\mathcal{U}^h$, if there exist a $\beta(.,.) \in \mathcal{KL}$ and a continuous, positive and nondecreasing function $\sigma(.)$ which are independent of T and h, and such that for any $r > 0$ there exists a $h^* > 0$ so that for all $h \in (0, h^*]$ there exists a feedback $\kappa^h : \Gamma \to U$ with the property $\left\| \kappa^h(x) \right\| \leq \sigma(\|x\|)$ such that for any $x \in \Gamma$ the solution ϕ^c of $x_{k+1} = F_T\left(x_k, \kappa^h(x_k)\right)$, $x_0 = x$ satisfies the inequality

$$\|\phi_k^c(x)\| \leq \max \left\{ \beta(\|x\|, kT), r \right\}, \qquad k \in \mathbb{N}. \tag{2.16}$$

Note that practical asymptotic controllability (PAC) and stabilizability (PAS) specialize to asymptotic controllability (AC) and asymptotic stabilizability (AS), respectively, in the limit, when $r = 0$.

Theorem 1 System (2.14) is practically asymptotically stabilizable in Γ about the origin if and only if it is practically asymptotically controllable from Γ to the origin.

Proof. The necessity: Let the control sequence $\mathbf{u}^h(x)$ be defined by

$$\mathbf{u}^h(x) = \left\{ \kappa^h(\phi_k^c(x)), \quad k = 0, 1, ... \right\},$$

then we obtain that system (2.14) is PAC.

The sufficiency can be shown as follows. Let $\overline{r} > 0$ be arbitrary, fixed. Let $\beta \in \mathcal{KL}$ and σ be the functions given in Definition 4 and let $\widetilde{\varphi} \in \mathcal{K}$ be such that $\sum_{k=0}^{\infty} \widetilde{\varphi}(\beta(\Delta, kT)) < \infty$. Let $\gamma_0 : [0, \Delta) \to \mathbb{R}_{\geq 0}$ be defined by

$$\gamma_0(r) = \sum_{k=0}^{\infty} \widetilde{\varphi}(\beta(r, kT)). \tag{2.17}$$

Now we show that $\gamma_0 \in \mathcal{K}$. Because of the theorem of Weierstrass, γ_0 is continuous. Let

$r_1 < r_2$, since $\beta \in \mathcal{KL}$ then $\beta(r_1, kT) < \beta(r_2, kT)$, $\forall k$,

$$\begin{aligned}
\gamma_0(r_1) &= \sum_{k=0}^{\infty} \widetilde{\varphi}(\beta(r_1, kT)) = \lim_{m\to\infty} \sum_{k=0}^{m} \widetilde{\varphi}(\beta(r_1, kT)) \\
&= \widetilde{\varphi}(\beta(r_1, 0)) + \lim_{m\to\infty} \sum_{k=1}^{m} \widetilde{\varphi}(\beta(r_1, kT)) \\
&\le \widetilde{\varphi}(\beta(r_1, 0)) + \lim_{m\to\infty} \sum_{k=1}^{m} \widetilde{\varphi}(\beta(r_2, kT)) \\
&= \widetilde{\varphi}(\beta(r_1, 0)) + \lim_{m\to\infty} \sum_{k=0}^{m} \widetilde{\varphi}(\beta(r_2, kT)) - \widetilde{\varphi}(\beta(r_2, 0)) \\
&= \widetilde{\varphi}(\beta(r_1, 0)) - \widetilde{\varphi}(\beta(r_2, 0)) + \gamma_0(r_2) < \gamma_0(r_2).
\end{aligned}$$

Since $\beta(0, kT) = 0$, $\forall k$ we have $\gamma_0(0) = 0$, thus $\gamma_0 \in \mathcal{K}$.

Let $r' = \min\left\{\gamma_0^{-1}(\widetilde{\varphi}(\overline{r})), \overline{r}\right\}$ and let $r'' \le r'$ be such that

$$\|F_T(x, 0)\| \le r', \quad \text{if} \quad \|x\| \le r''. \tag{2.18}$$

Then we take $r = \frac{1}{2} r''$ and define function $l : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ by

$$l(x) = \begin{cases} \widetilde{\varphi}(\|x\|), & \|x\| \ge r'', \\ 0, & \|x\| < r''. \end{cases}$$

Let system (2.14) be subject to cost function

$$J(x, \mathbf{u}^h) = \sum_{k=0}^{\infty} l(\phi_k(x, \mathbf{u}^h(x))).$$

For any $x \in \mathbb{R}^n$, we consider the minimization problem of this cost function with respect to the control constraint set $U(x) = U \cap \mathcal{B}_{\widetilde{\sigma}(\|x\|)}$, where $\widetilde{\sigma}(\|x\|) = \sigma(\|x\|)$ if $\|x\| \le \Delta$ and $\widetilde{\sigma}(\|x\|) = \sigma(\Delta)$ otherwise. Let

$$V(x) := \inf\left\{J(x, \mathbf{u}^h) : \mathbf{u}^h = \{u_0, u_1, ...\} \in \mathcal{U}^h,\ u_i \in U(x)\right\}.$$

Then for any $x \in \Gamma$, $V(.)$ has finite value and $V(x) \le J(x, \mathbf{u}^h(x))$, where $\mathbf{u}^h(x)$ is the control sequence given in Definition 4. Because of (2.15), there is an $N_1 \in \mathbb{N}$ such that $\left\|\phi_k(x, \mathbf{u}^h(x))\right\| \le r''$ for all $x \in \Gamma$, if $k \ge N_1$, therefore $l(\phi_k(x, \mathbf{u}^h(x))) = 0$, if $k \ge N_1$ and

$$\begin{aligned}
V(x) &\le \sum_{k=0}^{N_1} \widetilde{\varphi}(\left\|\phi_k(x, \mathbf{u}^h(x))\right\|) \le \sum_{k=0}^{N_1} \widetilde{\varphi}(\beta(\|x\|, kT)) \\
&\le \sum_{k=0}^{\infty} \widetilde{\varphi}(\beta(\|x\|, kT)) = \gamma_0(\|x\|).
\end{aligned}$$

Let us consider the set $M^* \subset \mathbb{R}^n$, for which $V(.)$ is finite and let us take its level set

$$\Gamma_\Delta = \{x \in M^* : V(x) \le \gamma_0(\Delta)\}.$$

Clearly $\Gamma \subset \Gamma_\Delta$, and for any $x \in \Gamma_\Delta$, V satisfies the dynamic programming equation

$$V(x) = \inf_{u \in U(x)} \{l(x) + V(F_T(x,u))\}.$$

It follows then, that for any $\varepsilon > 0$, and for any $x \in \Gamma_\Delta$ there exists a $u = \kappa^h_\varepsilon(x) \in U(x)$ such that

$$V(x) \geq l(x) + V(F_T(x, \kappa^h_\varepsilon(x))) - \varepsilon.$$

If $x \notin \mathcal{B}_{r''}$, then $l(x) = \widetilde{\varphi}(\|x\|)$, so if we take $\varepsilon^* = \frac{\widetilde{\varphi}(r'')}{2}$, then

$$V(F_T(x, \kappa^h_{\varepsilon^*}(x))) - V(x) \leq -l(x) + \frac{\widetilde{\varphi}(\|x\|)}{2} = -\frac{\widetilde{\varphi}(\|x\|)}{2} =: -\gamma^*(\|x\|). \tag{2.19}$$

On the other hand,

$$J(x, \mathbf{u}^h) \geq l(x) = \begin{cases} \widetilde{\varphi}(\|x\|), & \text{if} \quad x \in \Gamma_\Delta \backslash \mathcal{B}_{r''}, \\ 0, & \text{if} \quad x \in \mathcal{B}_{r''}. \end{cases}$$

Since this is valid for any $\mathbf{u}^h$ with $u^h_i \in U(x)$, we have that

$$V(x) = \inf J(x, \mathbf{u}^h) \geq \widetilde{\varphi}(\|x\|), \text{ if } \quad x \in \Gamma_\Delta \backslash \mathcal{B}_{r''}. \tag{2.20}$$

Let the required feedback be defined by

$$\kappa^h(x) = \begin{cases} \kappa^h_{\varepsilon^*}(x), & \text{if } x \in \Gamma_\Delta \backslash \mathcal{B}_{r''}, \\ 0, & \text{if } x \in \mathcal{B}_{r''}. \end{cases}$$

Then $\kappa^h(x) \in U(x)$ for any $x \in \Gamma$. Moreover, $\mathcal{B}_{r''} \subset \mathcal{B}_{r'} \subset \Omega_{\overline{r}} = \{x : V(x) \leq \widetilde{\varphi}(\overline{r})\} \subset \mathcal{B}_{\overline{r}}$. Because of (2.18) and (2.19), $\Omega_{\overline{r}}$ is positively invariant.

The remaining part of the proof is the construction of a class $\mathcal{KL}$ function β. The proof given below is similar to the proofs of Theorem 8 in [81] and Theorem 2 in [80]. Let $d = \widetilde{\varphi}(\overline{r})$. If $x \in \Gamma_\Delta \backslash\ \mathcal{B}_{r''}$, then from (2.19) we have $V\left(F_T\left(x, \kappa^h_{\varepsilon^*}(x)\right)\right) \leq V(x)$. If $x \in \mathcal{B}_{r''}$ then from (2.18) $V(F_T(x, 0)) \leq d$, therefore, for all $x \in \Gamma_\Delta$, $V(F_T(x, \kappa^h(x))) \leq \max\{V(x), d\}$. Then by induction we obtain

$$V(\phi^c_k(x)) \leq \max\{V(x), d\}.$$

If $\phi^c_k(x) \in \Gamma_\Delta \backslash\ \mathcal{B}_{r''}$, then

$$\begin{aligned} V(\phi^c_{k+1}(x)) - V(\phi^c_k(x)) &\leq -\gamma^*(\|\phi^c_k(x)\|) \\ &\leq -\gamma^*\left(\gamma_0^{-1}\left(V\left(\phi^c_k(x)\right)\right)\right) = -\alpha\left(V\left(\phi^c_k(x)\right)\right), \end{aligned}$$

where $\alpha = \gamma^* \circ \gamma_0^{-1} \in \mathcal{K}$.

In the rest of the proof we will use the notation $V(k)$ in place of $V\left(\phi^c_k(x)\right)$, then

$$V(k+1) - V(k) \leq -\alpha(V(k)). \tag{2.21}$$

We introduce a variable $t \in \mathbb{R}$ and define

$$y(t) := V(k) + (t - kT)\left(V(k+1) - V(k)\right) \qquad t \in [kT, (k+1)T), \quad k \in \mathbb{N}.$$

Then $y(t)$ is a continuous function of "time" t. Moreover, it is absolutely continuous in t (in fact, piecewise linear) and we can write for almost all t:

$$\dot{y}(t) = V(k+1) - V(k).$$

Since $V(k+1) \leq V(k)$ for all $\phi_k^c(x) \in \Gamma_\Delta \backslash \mathcal{B}_{r''}$ then, $y(t) \leq V(k)$, $t \in [kT, (k+1)T)$ and

$$\dot{y}(t) \leq -\alpha(y(t)), \quad t \in [kT, (k+1)T).$$

It follows from the arguments in ([95], Section VI) that there exists $\beta_1 \in \mathcal{KL}$ that is determined by α such that

$$y(t) \leq \beta_1(y(0), t), \quad y(0) \in \Gamma_\Delta \backslash \mathcal{B}_{r''}.$$

This implies, using $y(k) = V(k)$, with $t = kT$, $y(0) = V(0)$, that

$$V(k) \leq \beta_1(V(0), kT), \;\; k \in \mathbb{N},$$

for all $x \in \Gamma_\Delta \backslash \mathcal{B}_{r''}$.

From (2.20) it follows that if $\phi_k^c(x) \in \Gamma_\Delta \backslash \mathcal{B}_{r''}$ then

$$\begin{aligned} \widetilde{\varphi}(\|\phi_k^c(x)\|) &\leq V(\phi_k^c(x)) \\ &\leq \beta_1(V(0), kT) \leq \beta_1(\gamma_0(\|x\|), kT). \end{aligned}$$

and

$$\|\phi_k^c(x)\| \leq \widetilde{\varphi}^{-1}\left(\beta_1(\gamma_0(\|x\|)), kT\right)$$

If for some $i = 0, 1, ..., k-1$, $\phi_i^c(x) \in \mathcal{B}_{r''}$ then $\|\phi_k^c(x)\| \leq \overline{r}$ for $k = 0, 1, \ldots$. Let $\beta(s, \tau) = \widetilde{\varphi}^{-1}\left(\beta_1(\gamma_0(s), \tau)\right) \in \mathcal{KL}$, then, for each $x \in \Gamma_\Delta$ we have

$$\|\phi_k^c(x)\| \leq \max\left\{\beta(\|x\|, kT), \overline{r}\right\}. \; \square$$

Remark 4 From a practical point of view, it may be desirable to achieve stabilization over as "large" sets as possible. The realization of this aim may be limited by different factors: the trajectories of (2.1) may have finite escape times within a given sampling interval $[0, T]$, or "small" neighborhoods of the origin may be unreachable from initial points with a "large" norm because of control constraints, etc. In view of Theorem 1, it is reasonable to consider the problem of PAS for system (2.14) over a compact set Γ from which it is PAC. If this latter property is valid for each compact Γ, then the same will also be true for the stabilization. In keeping with the usage of ([44], p. 439), we shall speak in this case about semiglobal (practical) stability.

Remark 5 Theorem 1 remains valid if, similarly to [53] the properties PAC and PAS are required with vanishing controllers, i.e. the left-hand sides of (2.15) and (2.16) are substituted by $\left\|\phi_k(x, \mathbf{u}^h(x))\right\| + \left\|u_k^h(x)\right\|$ and $\|\phi_k^c(x)\| + \left\|\kappa^h(\phi_k^c(x))\right\|$, respectively.

Assumption A3 There exists a $T^* > 0$ such that the exact discrete-time model (2.2) is PAC from Γ to the origin for all $T \in (0, T^*]$. $\square$

Remark 6 Observe that Assumption A3 implies that for any $x \in \Gamma$ there exists a control function $\mathbf{u}^h(x) \in \mathcal{U}^h$ for which no finite escape time occurs.

Let $\beta(.,.)$ and $\sigma(.)$ be functions generated by Assumption A3 and let Δ_1 be such that $\Delta_1 \geq 1 + \beta(\Delta, 0)$. Moreover, for $0 < \rho \leq \Delta_1$, we introduce the notation $U_\rho = U \cap \mathcal{B}_{\sigma(\rho)}$.

The question is, how to find an approximate model, a family of controllers and a suitable Lyapunov function so that conditions (i), (ii) and (iii) of Theorem A should be satisfied. We shall show that, in the case of $T \neq h$, a suitable version of the receding horizon method gives such a controller and Lyapunov function. In the case of $T = h$, we shall prove analogous stability results, though the conditions of Theorem 2 in [80] (which is the counterpart of Theorem A) cannot be entirely satisfied.

2.4 The receding horizon control method

The aim of this section is to describe a version of the RHC method suitable in connection with the stabilization problem of exact discrete-time systems via their approximate discrete-time model. In order to define a receding horizon feedback controller, let (2.4) be subject to the cost function

$$J_{T,h}(N, x, \mathbf{u}) = \sum_{k=0}^{N-1} T l_h(x_k^A, u_k) + g(x_N^A), \tag{2.22}$$

where $0 < N \in \mathbb{N}$ and $\mathbf{u} = \{u_0, u_1, ..., u_{N-1}\}$ and $x_k^A = \phi_k^A(x, \mathbf{u})$, $k = 0, 1, ..., N$, denote the solution of (2.4), l_h and g are given functions, satisfying assumptions to be formulated later.

Consider the optimization problem

$$P_{T,h}^A(N, x): \quad \min \left\{ J_{T,h}(N, x, \mathbf{u}) : u_k \in U_{\Delta_1} \right\}. \tag{2.23}$$

If this optimization problem has a solution denoted by $\mathbf{u}^*(x) = \left\{u_0^*(x), u_1^*(x), ..., u_{N-1}^*(x)\right\}$ then its first element, i.e. $u_0^*(x)$, is applied at the state x. Since the optimal solution of $P_{T,h}^A(N, x)$ naturally depends on x, in this way a feedback has been defined on the basis of the approximate discrete-time model i.e.

$$v_{T,h}^A(x) := u_0^*(x).$$

We pause to clarify the notation adopted in the sequel. The variable t represents the real time, while $t_k = kT$, $k = 0, 1, ...$, denote the sampling instants. The vector x_k^E will denote the actual (exact discrete-time) state of the plant measured at t_k. The optimal trajectory for the approximate model is denoted by $\phi_j^A(x_k^E, \mathbf{u}^*)$, $j = 0, 1, ..., N$ where $\mathbf{u}^*$ is the optimal controller. We define the control sequence $\mathbf{u}_v$ produced by $v_{T,h}^A$ and F_T^E. The process $(\phi_i^E(x_0, \mathbf{u}_v), \mathbf{u}_v)$, $i = 0, 1, ..., k$ is the closed-loop exact discrete-time trajectory and control resulting from the RHC strategy.

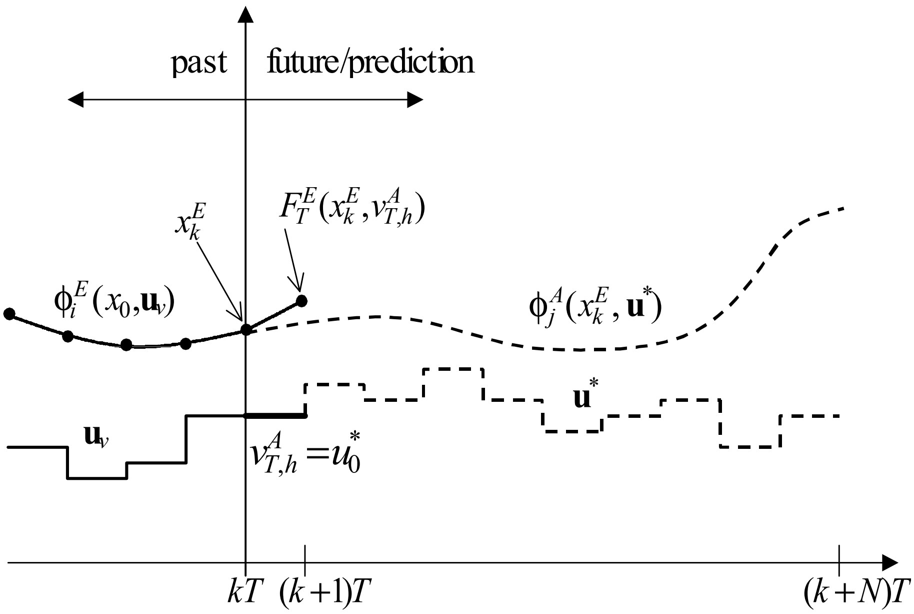

Fig. 2. The RHC strategy.

The RHC conceptual algorithm consists of performing the following steps at certain instants t_k (see Figure 2):

(1) Measure the current state of the plant x_k^E.

(2) Compute the open-loop optimal control $\mathbf{u}^*$ to the problem $P_{T,h}^A(N, x_k^E)$.

(3) The control $v_{T,h}^A(x_k^E) := u_0^*$ is applied to the plant in the interval $[kT, (k+1)T)$ (the remaining $\{u_1^*, ..., u_{N-1}^*\}$ is discarded).

(4) The procedure is repeated from (1) for the next sampling instants $t_{k+1} = (k+1)T$.

Conditions, under which $v_{T,h}^A$ asymptotically stabilizes the origin for a fixed discrete-time system of type (2.4) are well-established: here we only refer the reader to the review papers ([2], [15] and [69]) and to the references therein. In our work we establish sufficient conditions which guarantee that the receding horizon controller $v_{T,h}^A$ that renders the origin to be asymptotically stable for the approximate discrete-time model also stabilizes the exact discrete-time model for sufficiently small integration and/or sampling parameters.

The running and the terminal cost functions have to be chosen according to the following assumption.

Assumption A4 (i) g is continuous, positive definite, there exists a class-$\mathcal{K}_\infty$ function γ_1 such that $\gamma_1(\|x\|) \leq g(x)$, and, for any Δ' there exists a constant $L_g = L_g(\Delta') > 0$ such that $|g(x) - g(y)| \leq L_g\|x - y\|$ for all $x \in \mathcal{B}_{\Delta'}$.

(ii) l_h is continuous with respect to x and u, uniformly in small h, and for any $\Delta' > 0$, $\Delta'' > 0$ there exist $h^* > 0$ and $L_l = L_l(\Delta', \Delta'') > 0$ such that

$$|l_h(x, u) - l_h(y, u)| \leq L_l\|x - y\|$$

for all $h \in (0, h^*]$, $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$.

(iii) There exist a $h^* > 0$ and two class-$\mathcal{K}_\infty$ functions φ_1 and φ_2 such that the inequality

$$\varphi_1(\|x\| + \|u\|) \leq l_h(x, u) \leq \varphi_2(\|x\|) + \varphi_2(\|u\|),$$

holds for all $x \in \mathbb{R}^n$, $u \in U$ and $h \in (0, h^*]$. □

Remark 7 In the case of $T = h$, logically, the existence of a $T^* > 0$ is required in Assumption A4 in place of h^* and the conditions have to be satisfied for $T \in (0, T^*]$. The same comment should be added to the Assumption A5 below.

The terminal cost function g and/or a terminal constraint set given explicitly or implicitly play crucial role in establishing the desired stabilizing property. We shall assume that g is chosen according to the following assumption.

Assumption A5 There exist $h^* > 0$ and $\eta > 0$ such that for all $x \in \mathcal{G}_\eta = \{x : g(x) \leq \eta\}$ there exists a $\kappa(x) \in U_{\rho_0}$ (which may depend on parameters h and/or T) such that inequality

$$Tl_h(x, \kappa(x)) + g\left(F^A_{T,h}(x, \kappa(x))\right) \leq g(x) \tag{2.24}$$

holds true for all $h \in (0, h^*]$, where ρ_0 is such that $\mathcal{G}_\eta \subset \mathcal{B}_{\rho_0}$. $\square$

In what follows, we shall refer to the set $\mathcal{G}_\eta$ as the *terminal set*.

We note that there is a $\rho_1 > 0$ such that $\mathcal{B}_{\rho_1} \subset \mathcal{G}_\eta$. All over this thesis, ρ_0 and ρ_1 denote these constants. Without loss of generality we may assume that $\rho_0 \leq \Delta_1$.

Remark 8 Assumptions like A5 emerge in almost all works which define the RHC on the basis of Bolza-type optimization problems (one exception is e.g. [35]). It is not accidental, because such a condition is necessary (and sufficient) for the monotonicity of the value function with respect to the horizon length, which is important in proving that the value function is a suitable Lyapunov function (see [38]) for continuous-time problems, but the same is true for a discrete-time system, as well. (We emphasize that this doesn't involve the necessity of A5 for the stabilizing property of the RHC method (see [35]). It can be observed that a function g satisfying Assumption A5 is a local control Lyapunov function. To find a suitable g, several approaches have been proposed in the literature: in the case when the system has stabilizable linearization and a quadratic cost function is applied, one can find g in quadratic form by solving an algebraic Riccati equation (though the corresponding level set may be unacceptably small). More sophisticated methods are e.g. the quasi-infinite horizon method of [9], the method of infinite horizon costing of [14]. (More detailed analysis can be found in the review papers cited above.)

For any $x \in \mathbb{R}^n$, let

$$V_N(x) = \inf\left\{J_{T,h}(N, x, \mathbf{u}) : u_k \in U_{\Delta_1}\right\},$$

if the right-hand side is finite, and let $V_N(x) = \infty$ otherwise. (Evidently, function V_N depends also on the parameters T, h, but, for simplicity, this dependence is not reflected in the notation.)

Assume that parameters h and/or T are chosen sufficiently small so that the conditions of A2-A4 are satisfied. Then, from these Assumptions, it follows immediately that for any $x \in \Gamma$, $P^A_{T,h}(N, x)$ has a solution $\mathbf{u}^*(x)$, function $V_N(.)$ is positive definite and continuous on Γ uniformly in small h.

We shall use the optimal value function $V_N(.)$ as a Lyapunov function to establish the stability of receding horizon control of an approximate discrete-time model.

Lemma 1 Suppose that Assumptions A2, A4 and A5 hold true. Then for any $N \geq 1$, and any $\eta > 0$ the following statements are valid:

(i) For any $x_0 \in \mathcal{G}_\eta$, $V_N(x_0) \leq g(x_0)$ and $\phi^A_N(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$.

(ii) If $\phi^A_N(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$ for some $x_0 \in \mathbb{R}^n$, then

$$V_N(F^A_{T,h}(x_0, u^*_0(x_0))) - V_N(x_0) \leq -Tl_h(x_0, u^*_0(x_0)).$$

(iii) If for some $x_0 \in \mathbb{R}^n$ and for some $k \in \mathbb{N}$, $0 \leq k < N$, $\phi_k^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$, then $\phi_N^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$.

Proof. We shall prove this lemma by a technique similar to [47] and [69]. Let $x_0 \in \mathcal{G}_\eta$, and let $\boldsymbol{\kappa}(x_0) = \{\kappa_0, ..., \kappa_{N-1}\}$ where $\kappa_i = \kappa(x_i)$ and $x_{i+1} = F_{T,h}^A(x_i, \kappa_i(x_i))$, $i = 0, 1, ...,$ and κ is given by Assumption A5. By repeated use of Assumption A5 we obtain

$$g\left(\phi_N^A\left(x_0, \boldsymbol{\kappa}(x_0)\right)\right) - g(x_0) \leq -\sum_{k=0}^{N-1} Tl_h\left(\phi_k^A\left(x_0, \boldsymbol{\kappa}(x_0)\right), \kappa_k\right).$$

Therefore,

$$\begin{aligned} V_N(x_0) &\leq J_{T,h}(N, x_0, \boldsymbol{\kappa}(x_0)) \\ &= \sum_{k=0}^{N-1} Tl_h\left(\phi_k^A\left(x_0, \boldsymbol{\kappa}(x_0)\right), \kappa_k\right) + g\left(\phi_N^A\left(x_0, \boldsymbol{\kappa}(x_0)\right)\right) \leq g(x_0). \end{aligned}$$

Since $V_N(x_0) = J_{T,h}(N, x_0, \mathbf{u}^*(x_0)) \leq g(x_0)$, we get

$$g\left(\phi_N^A\left(x_0, \mathbf{u}^*(x_0)\right)\right) + \sum_{k=0}^{N-1} Tl_h\left(\phi_k^A\left(x_0, \mathbf{u}^*(x_0)\right), u_k^*(x_0)\right) \leq g(x_0),$$

$$g\left(\phi_N^A\left(x_0, \mathbf{u}^*(x_0)\right)\right) \leq g(x_0) \leq \eta.$$

This yields $\phi_N^A\left(x_0, \mathbf{u}^*(x_0)\right) \in \mathcal{G}_\eta$.

Now we prove (ii). Suppose that $x_0 \in \mathbb{R}^n$ and that the control sequence $\mathbf{u}^*(x_0)$ that solves $P_{T,h}^A(N, x_0)$ has been determined. The receding horizon control $u_0^*(x_0)$ steers the initial state x_0 to the successor state $x^+ = F_{T,h}^A(x_0, u_0^*(x_0))$. We wish to determine an admissible control sequence $\overline{\mathbf{u}}$ for x^+ and hence an upper bound for $V_N(x^+)$. Since the optimal control sequence $\left\{u_0^*, ..., u_{N-1}^*\right\}$ steers x_0 to $x_N^A = \phi_N^A(x_0, \mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$ then the control sequence $\left\{u_1^*, ..., u_{N-1}^*\right\}$ steers x^+ to x_N^A. Therefore, by using Assumption A5, we can define a control sequence $\overline{\mathbf{u}}(x^+) = \left\{u_1^*(x), ..., u_{N-1}^*(x), \kappa(x_N^A)\right\}$ which is suitable for $P_{T,h}^A(N, x^+)$. The state trajectory resulting from initial state x^+ and control sequence $\overline{\mathbf{u}}(x^+)$ is $\left\{x^+, ..., x_N^A, F_{T,h}^A\left(x_N^A, \kappa(x_N^A)\right)\right\}$. The associated cost is

$$\begin{aligned} J_{T,h}(x^+, \overline{\mathbf{u}}(x^+)) &= \sum_{k=0}^{N-1} Tl_h\left(\phi_k^A\left(x^+, \overline{\mathbf{u}}(x^+)\right), \overline{u}_k(x^+)\right) + g\left(\phi_N^A\left(x^+, \overline{\mathbf{u}}(x^+)\right)\right) \\ &= \sum_{k=0}^{N-2} Tl_h\left(\phi_k^A\left(x^+, \overline{\mathbf{u}}(x^+)\right), \overline{u}_k(x^+)\right) \\ &\quad + Tl_h\left(\phi_{N-1}^A\left(x^+, \overline{\mathbf{u}}(x^+)\right), \overline{u}_{N-1}(x^+)\right) + g\left(\phi_N^A\left(x^+, \overline{\mathbf{u}}(x^+)\right)\right) \\ &\quad + Tl_h\left(x_0, u_0^*(x_0)\right) - Tl_h\left(x_0, u_0^*(x_0)\right) + g\left(x_N^A\right) - g\left(x_N^A\right) \\ &= \sum_{k=0}^{N-1} Tl_h\left(\phi_k^A\left(x_0, \mathbf{u}^*(x_0)\right), u_k^*(x_0)\right) + g\left(x_N^A\right) \\ &\quad - Tl_h\left(x_0, u_0^*(x_0)\right) - g\left(x_N^A\right) + Tl_h\left(\phi_{N-1}^A\left(x^+, \overline{\mathbf{u}}(x^+)\right), \overline{u}_{N-1}(x^+)\right) \\ &\quad + g\left(\phi_N^A\left(x^+, \overline{\mathbf{u}}(x^+)\right)\right). \end{aligned}$$

Since $\phi_{N-1}^A(x^+,\overline{\mathbf{u}}(x^+)) = x_N^A$ and $\phi_N^A(x^+,\overline{\mathbf{u}}(x^+)) = F_{T,h}^A\left(x_N^A,\kappa(x_N^A)\right)$ then we obtain

$$\begin{aligned} J_{T,h}(x^+,\overline{\mathbf{u}}(x^+)) \quad \leq \quad & V_N(x_0) - Tl_h(x_0,u_0^*(x_0)) - g(x_N^A) + Tl_h\left(x_N^A,\kappa(x_N^A)\right) \\ & + g\left(F_{T,h}^A\left(x_N^A,\kappa(x_N^A)\right)\right). \end{aligned} \tag{2.25}$$

By using A5, the sum of the last three terms in (2.25) is less than or equal to zero. Since $V_N(x^+) \leq J_{T,h}(x^+,\overline{\mathbf{u}}(x^+))$, then we obtain

$$V_N(F_{T,h}^A(x_0,u_0^*(x_0))) - V_N(x_0) \leq -Tl_h(x_0,u_0^*(x_0)).$$

Now we prove (iii). Let $x_k^A = \phi_k^A(x_0,\mathbf{u}^*(x_0)) \in \mathcal{G}_\eta$, then by optimality principle we have

$$V_N(x_0) = \sum_{i=0}^{k-1} Tl_h\left(\phi_i^A(x_0,\mathbf{u}^*(x_0)),u_i^*(x_0)\right) + V_{N-k}(x_k^A).$$

Since $x_k^A \in \mathcal{G}_\eta$ then from (i) we obtain $V_{N-k}(x_k^A) \leq g(x_k^A)$

$$V_{N-k}(x_k^A) = \sum_{i=k}^{N-1} Tl_h\left(\phi_i^A(x_0,\mathbf{u}^*(x_0)),u_i^*(x_0)\right) + g\left(\phi_N^A(x_0,\mathbf{u}^*(x_0))\right) \leq g(x_k^A)$$

and $g\left(\phi_N^A(x_0,\mathbf{u}^*(x_0))\right) \leq g(x_k^A) \leq \eta$. □

In the remaining part of this section we shall formulate some properties of the RHC method which depend on the connection between the exact and the approximate model: as an auxiliary result we shall show a control sequence that steers the trajectory of the approximate system to the terminal set, while this trajectory remains uniformly bounded. As a consequence, we obtain that the approximate model is asymptotically controllable from Γ to the origin. Moreover, we shall derive a uniform upper bound for V_N and a criterion for the horizon length (in terms of the real continuous-time t) ensuring that the optimal trajectory ends in the terminal set. With this end in view, we need an assumption describing the "closeness" of the exact and the approximate models in the sense of Definition 2.

Assumption A6 In the case of $T \neq h$, the family $F_{T,h}^A$ is $(T,\Delta_1,\sigma(\Delta_1))$-consistent with F_T^E. In the case of $T = h$, the family $F_{T,h}^A$ is semiglobally consistent with F_T^E. □

In what follows, let h_0^* denote the minimum of the values h^* generated by A2-A6 with $\Delta' = \Delta_1$, $\Delta'' = \sigma(\Delta_1)$ (in the case of $T = h$, T_0^* is used instead of h_0^*).

Lemma 2 If Assumptions A3, A5 and A6 hold true, then there exists a $h_1^* > 0$, ($T_1^* > 0$, if $T = h$), and for any $h \in (0,h_1^*]$ and for any $x \in \Gamma$, there exists a control sequence $\widetilde{\mathbf{u}}^h(x)$ with $\widetilde{u}_k^h(x) \in U_{\Delta_1}$ such that

$$\left\|\phi_k^A(x,\widetilde{\mathbf{u}}^h(x))\right\| \leq \Delta_1, \quad k = 0,1,\dots.$$

Moreover, there is a $\mathcal{T}_1^*$ such that, if $\mathcal{T}_1^* \leq N_1 T \leq \mathcal{T}_1^* + 1$, then $\phi_k^A(x,\widetilde{\mathbf{u}}^h(x)) \in \mathcal{G}_\eta$ for all $k \geq N_1$.

Proof. For the sake of definiteness, consider the case of $T = h$ (the case of $T \neq h$ can be treated analogously). Consider an arbitrary $T \in (0,T_0^*]$ and $x \in \Gamma$ and let $r = \rho_1/2$ where $\rho_1 > 0$ is such that $\mathcal{B}_{\rho_1} \subset \mathcal{G}_\eta$ and let $\mathbf{u}^T(x)$ be the sequence given in Definition 4. Then $\left\|u_k^T(x)\right\| \leq \sigma(\|x\|) \leq \sigma(\Delta_1)$. Because of Assumption A6, there exists a T',

$(0 < T' \leq T_0^*)$ such that as long as $\left\|\phi_{k-1}^A(x, \mathbf{u}^T(x))\right\| \leq \Delta_1$, then for all $T \in (0, T']$ we have

$$\begin{aligned}
\left\|\phi_k^A(x, \mathbf{u}^T(x))\right. &- \left.\phi_k^E(x, \mathbf{u}^T(x))\right\| = \\
& \left\|F_T^A\left(\phi_{k-1}^A(x, \mathbf{u}^T(x)), u_{k-1}^T(x)\right) - F_T^E\left(\phi_{k-1}^E(x, \mathbf{u}^T(x)), u_{k-1}^T(x)\right)\right\| \\
\leq & \left\|F_T^A\left(\phi_{k-1}^A(x, \mathbf{u}^T(x)), u_{k-1}^T(x)\right) - F_T^E\left(\phi_{k-1}^A(x, \mathbf{u}^T(x)), u_{k-1}^T(x)\right)\right\| \\
& + \left\|F_T^E\left(\phi_{k-1}^A(x, \mathbf{u}^T(x)), u_{k-1}^T(x)\right) - F_T^E\left(\phi_{k-1}^E(x, \mathbf{u}^T(x)), u_{k-1}^T(x)\right)\right\| \\
\leq & \; T\gamma(T) + e^{L_f T}\left\|\phi_{k-1}^A(x, \mathbf{u}^T(x)) - \phi_{k-1}^E(x, \mathbf{u}^T(x))\right\|
\end{aligned}$$

It follows by induction that

$$\left\|\phi_k^A(x, \mathbf{u}^T(x)) - \phi_k^E(x, \mathbf{u}^T(x))\right\| \leq T\gamma(T)\frac{e^{L_f kT} - 1}{e^{L_f T} - 1}.$$

Let $\mathcal{T}_1^*$ denote such a number that $\beta(\Delta_1, kT) < \rho_1/2$, if $kT \geq \mathcal{T}_1^*$, and let N_1 be taken so that $\mathcal{T}_1^* \leq N_1 T \leq \mathcal{T}_1^* + 1$. Let $K = T'(e^{L_f(\mathcal{T}_1^*+1)} - 1)/(e^{L_f T'} - 1)$, then there exists a T_1^* such that $\gamma(T_1^*)K < \min\left\{1, \frac{\rho_1}{2}\right\}$. Therefore, $\left\|\phi_k^A(x, \mathbf{u}^T(x))\right\| \leq \Delta_1$, if $k = 0, 1, ..., N_1$ and $\left\|\phi_{N_1}^A(x, \mathbf{u}^T(x))\right\| \leq \rho_1$. Let $\widetilde{\mathbf{u}}^T(x)$ be a control sequence defined by

$$\widetilde{u}_k^T(x) = \begin{cases} u_k^T(x), & \text{if} \quad 0 \leq k \leq N_1, \\ \kappa(\widetilde{x}_k), & \text{if} \quad k \geq N_1, \end{cases} \tag{2.26}$$

where

$$\widetilde{x}_{k+1} = F_T^A\left(\widetilde{x}_k, \widetilde{u}_k\right), \quad \widetilde{x}_{N_1} = \phi_{N_1}^A(x, \mathbf{u}^T(x)), \tag{2.27}$$

$k = N_1, N_1 + 1, ...$ and κ is given by Assumption A5. By construction $\left\|\widetilde{u}_k^T(x)\right\| \leq \sigma(\Delta_1)$. It follows from Assumption A5 that $\phi_k^A(x, \widetilde{\mathbf{u}}^T(x)) \in \mathcal{G}_\eta$ for all $k \geq N_1$. □

Proposition 1 If Assumptions A3-A6 hold true, then there exists $h^* > 0$ (i.e. $T^* > 0$) such that, for any $h \in (0, h^*]$ system (2.4) is asymptotically controllable from Γ to the origin.

Proof. From Assumptions A4(i) and A5 it follows that $\phi_k^A(x, \widetilde{\mathbf{u}}^T(x)) \longrightarrow 0$ as $k \longrightarrow \infty$, where $\widetilde{\mathbf{u}}^T(x)$ is given by (2.26) and (2.27). A suitable function β can be derived by considerations like those in the proofs of Theorem 8 in [81] and Theorem 2 in [80]. □

Lemma 3 If Assumptions A1-A6 hold true, then there exists a constant $V_{\max}^A$ such that $V_N(x) \leq V_{\max}^A$ for all $x \in \Gamma$ and $h \in (0, h_1^*]$ (i.e. $T \in (0, T_1^*]$) and $N \in \mathbb{N}$, where h_1^* (i.e. T_1^*) is given by Lemma 2.

Proof. Let $\mathcal{T}_1^*$ and N_1 be defined by Lemma 2, and let $x \in \Gamma$ be arbitrary. Consider the control sequence given by (2.26) and (2.27). (For simplicity we omit the notation of parameter dependence.) If $N < N_1$, then

$$\begin{aligned}
V_N(x) \leq & \; J_{T,h}(N, x, \widetilde{\mathbf{u}}(x)) = \sum_{k=0}^{N-1} T l_h(\phi_k^A(x, \widetilde{\mathbf{u}}(x)), \widetilde{u}_k(x)) + g(\phi_N^A(x, \widetilde{\mathbf{u}}(x))) \\
\leq & \sum_{k=0}^{N-1} T\left[\varphi_2\left(\left\|\phi_k^A(x, \widetilde{\mathbf{u}}(x)\right\|\right) + \varphi_2\left(\left\|\widetilde{u}_k(x)\right\|\right)\right] + g(\phi_N^A(x, \widetilde{\mathbf{u}}(x))) \\
\leq & \; NT[\varphi_2(\Delta_1) + \varphi_2(\sigma(\Delta_1))] + \max_{\|x\| \leq \Delta_1} g(x) \\
\leq & \; (\mathcal{T}_1^* + 1)[\varphi_2(\Delta_1) + \varphi_2(\sigma(\Delta_1))] + \max_{\|x\| \leq \Delta_1} g(x) =: V^{(1)}.
\end{aligned}$$

If $N \geq N_1$, then

$$\begin{aligned} V_N(x) &\leq \sum_{k=0}^{N-1} Tl_h(\phi_k^A(x,\widetilde{\mathbf{u}}(x)),\widetilde{u}_k(x)) + g(\phi_N^A(x,\widetilde{\mathbf{u}}(x))) \\ &= \sum_{k=0}^{N_1-1} Tl_h(\phi_k^A(x,\widetilde{\mathbf{u}}(x)),\widetilde{u}_k(x)) + \sum_{k=N_1}^{N-1} Tl_h(\phi_k^A(x,\widetilde{\mathbf{u}}(x)),\widetilde{u}_k(x)) + g(\phi_N^A(x,\widetilde{\mathbf{u}}(x))). \end{aligned}$$

By repeated use of A5 we have

$$\sum_{k=N_1}^{N-1} Tl_h(\phi_k^A(x,\widetilde{\mathbf{u}}(x)),\widetilde{u}_k(x)) + g(\phi_N^A(x,\widetilde{\mathbf{u}}(x))) \leq g(\phi_{N_1}^A(x,\widetilde{\mathbf{u}}(x))).$$

Then

$$\begin{aligned} V_N(x) &\leq \sum_{k=0}^{N_1-1} Tl_h(\phi_k^A(x,\widetilde{\mathbf{u}}(x)),\widetilde{u}_k(x)) + g(\phi_{N_1}^A(x,\widetilde{\mathbf{u}}(x))) \\ &\leq (\mathcal{T}_1^*+1)[\varphi_2(\Delta_1)+\varphi_2(\sigma(\Delta_1))] + \eta =: V^{(2)}. \end{aligned}$$

Therefore $V_{\max}^A = \max\{V^{(1)}, V^{(2)}\}$ is a suitable upper bound. □

Let $\mathcal{T}_2^*$ be defined by

$$\mathcal{T}_2^* = \frac{V_{\max}^A - \eta}{c}, \tag{2.28}$$

where $c > 0$ is such a positive constant that $\varphi_1(\|x\|) \geq c$ for all $x \notin \mathcal{G}_\eta$.

Lemma 4 Suppose that Assumptions A1-A6 hold true. In the case of $T \neq h$, if $h \in (0, h_1^*]$, and $N \in \mathbb{N}$ are chosen so that

$$TN > \mathcal{T}_2^*, \tag{2.29}$$

or, in the case of $T = h$, if $T \in (0, T_1^*]$, and T and $N \in \mathbb{N}$ are chosen according to (2.29), then $\phi_N^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$ for all $x \in \mathcal{B}_\Delta$.

Proof. Assume that, in contrary, $\phi_N^A(x_0, \mathbf{u}^*(x_0)) \notin \mathcal{G}_\eta$ for some $x_0 \in \mathcal{B}_\Delta$. Then, from Lemma 1, it follows that $\phi_k^A(x_0, \mathbf{u}^*(x_0)) \notin \mathcal{G}_\eta,\ \ k = 0, 1, ..., N$. Therefore

$$\begin{aligned} V_N(x_0) &= \sum_{k=0}^{N-1} Tl_h(\phi_k^A(x_0,\mathbf{u}^*(x_0)),u_k^*(x_0)) + g(\phi_N^A(x_0,\mathbf{u}^*(x_0))) \\ &\geq \sum_{k=0}^{N-1} T\varphi_1\left(\left\|\phi_k^A(x_0,\mathbf{u}^*(x_0))\right\|\right) + \eta \geq NTc + \eta. \end{aligned}$$

Since $V_N(x_0) \leq V_{\max}^A$, this contradicts to (2.29). □

2.5 Stabilization with fixed sampling parameter T

In this section we discuss the case when the sampling parameter $T \in (0, T_0^*]$ is fixed and the discretization parameter h can be assigned arbitrarily and independently of T. On the basis of Theorem A we shall show that the family $(F_T^E, v_{T,h}^A)$ is locally practically asymptotically stable about the origin.

Theorem 2 Suppose that Assumptions A1-A6 are valid and N is chosen according to (2.28), (2.29). Then, there exist $\beta \in \mathcal{KL}$ and $D_1 \in (0, \Delta)$, and for any $\delta > 0$ there exists $h^* > 0$ such that for all $x_0 \in \mathcal{B}_{D_1}$ and $h \in (0, h^*]$, the solutions of the family $(F_T^E, v_{T,h}^A)$ satisfy the inequality

$$\left\|\phi_k^E(x_0, \mathbf{u}_v(x_0))\right\| \leq \beta(\|x_0\|, kT) + \delta, \quad k \in \mathbb{N},$$

where $\mathbf{u}_v$ is the control sequence produced by $v_{T,h}^A$ and F_T^E.

Proof. We have to show that Assumptions A1-A6 imply the conditions (i)-(iii) of Theorem A. In fact, condition (ii) is satisfied with $D = \Delta$ and $M = \sigma(\Delta_1)$ by the construction of $v_{T,h}^A$, and condition (iii) is identical with A6 if we take again $D = \Delta$ and $M = \sigma(\Delta_1)$. It remains only to prove that the family $(F_{T,h}^A, v_{T,h}^A)$ is (T, Δ)-stable with a continuous Lyapunov function. Let h_1^* be defined by Lemma 2. The lower estimation in (1.7) is given by using assumption A4(iii)

$$\begin{aligned} V_N(x_0) &= \sum_{k=0}^{N-1} Tl_h(\phi_k^A(x_0, \mathbf{u}^*(x_0)), u_k^*(x_0)) + g(\phi_N^A(x_0, \mathbf{u}^*(x_0))) \\ &\geq Tl_h(x_0, u_0^*(x_0)) \geq T\varphi_1\left(\|x_0\| + \|u_0^*(x_0)\|\right) \\ &\geq T\varphi_1\left(\|x_0\|\right) =: \sigma_1\left(\|x_0\|\right). \end{aligned}$$

To verify the upper estimation in (1.7) with a class-$\mathcal{K}_\infty$ function, let us introduce the function $\vartheta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\vartheta(s) = \max_{\|x\| \leq s} g(x) + s^2/2 \tag{2.30}$$

and let

$$\sigma_2(s) = \max\left\{\vartheta(s), \vartheta(\frac{\rho_1}{2}) + \frac{2}{\rho_1} V_{\max}^A \left(s - \frac{\rho_1}{2}\right)\right\}. \tag{2.31}$$

Obviously, $\sigma_2 \in \mathcal{K}_\infty$. According to Lemma 1 (i), $V_N(x) \leq g(x)$ if $x \in \mathcal{G}_\eta$, thus $V_N(x) \leq \vartheta(\|x\|) \leq \sigma_2(\|x\|)$. Since $\sigma_2(\|x\|) \geq V_{\max}^A$, if $\|x\| \geq \rho_1$, therefore $V_N(x) \leq \sigma_2(\|x\|)$ for all $x \in \mathcal{B}_\Delta$. Because of the choice of N, the optimal trajectories starting from any $x_0 \in \mathcal{B}_\Delta$ terminate in $\mathcal{G}_\eta$, therefore inequality (1.8) follows from Lemma 1 (ii). Since l_h, g, $F_{T,h}^A$ and $J_{T,h}$ are continuous and the infimum of continuous functions is continuous, therefore, $V_N(.)$ is continuous in $\mathcal{B}_\Delta$ uniformly in small h, it is also uniformly continuous in $\mathcal{B}_\Delta$, because $\mathcal{B}_\Delta$ is compact. Thus the statement of the theorem is an immediate consequence of Theorem A. □

Remark 9 Observe that $V_{\max}^A$ and $\mathcal{T}_2^*$ (i.e. the criterion for the choice of the horizon length) remain the same, if problem $P_{T,h}^A(N, x)$ is modified so that the minimization with respect to $\mathbf{u}$ is performed over U instead of U_{Δ_1}. Since $T\varphi_1(\|u_0^*(x)\|) \leq V_N(x) \leq V_{\max}^A$, $v_{T,h}^A$ is (T, Δ, M)-uniformly bounded with $M = M_T = \varphi_1^{-1}\left(V_{\max}^A/T\right)$. If the family $F_{T,h}^A$ is (T, Δ_1, M_T)-consistent with F_T^E, then Theorem 2 remains valid, if the receding horizon controller $v_{T,h}^A$ is based on this modified version of the problem $P_{T,h}^A(N, x)$.

2.5.1 Illustrative Example

Consider the continuous-time system (this example is taken from [70] cf. Section V)

$$\begin{aligned}\dot{x}_1 &= -x_2 + (\mu + (1-\mu)x_1)u,\\ \dot{x}_2 &= x_1 + (\mu - 4(1-\mu)x_2)u.\end{aligned}$$

Let the approximate discrete-time model be defined by the Euler method as follows: let $\overline{x} = x^A(k), \quad \overline{u} = u(k)$. With $x_0 = \overline{x}$ let

$$\begin{aligned}x_{1,i+1} &= x_{1,i} + h\left[-x_{2,i} + (\mu + (1-\mu)x_{1,i})\overline{u}\right],\\ x_{2,i+1} &= x_{2,i} + h\left[x_{1,i} + (\mu - 4(1-\mu)x_{2,i})\overline{u}\right],\end{aligned}$$

$i = 0, 1, ..., m-1$, and let $x^A(k+1) = x_m$, where $h = T/m$. The running and the terminal costs are given by $l_h(x,u) = \frac{1}{2}\|x\|^4 + u^2$, $g(x) = x'Px$, where P is given below. All computations were carried out using MATLAB. Especially, the optimal control sequence was computed by the `constr` code of the Optimization toolbox. Simulations for the continuous-time system were carried out using `ode45` program in MATLAB when $T = 0.1$, $\mu = 0.5$ and

$$P = \begin{pmatrix} 2.8438 & -0.1589 \\ -0.1589 & 4.1541 \end{pmatrix},$$

and m was chosen subsequently as $m = 1$, $m = 10$, $m = 15$. The trajectories and the norms of the state of the continuous-time system are shown in Figures 3 and 4. It can be seen that the radius of the ball around the origin to which the trajectory tends is decreasing as m is increasing (or h is decreasing), though the convergence of this radius to zero (which is proven theoretically) can't be shown "experimentally" because for "sufficiently large" m, the computational errors will dominate. The control sequence $\mathbf{u}_v$ produced by the receding horizon controller $v^A_{T,h}$ is shown in Figure 5.

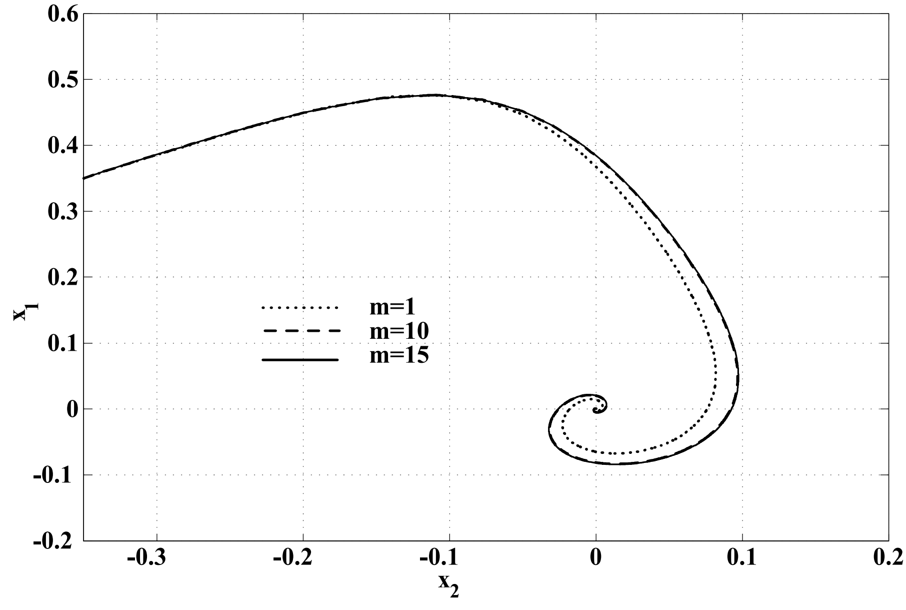

Fig. 3. Trajectories of continuous-time system for different values of parameter h.

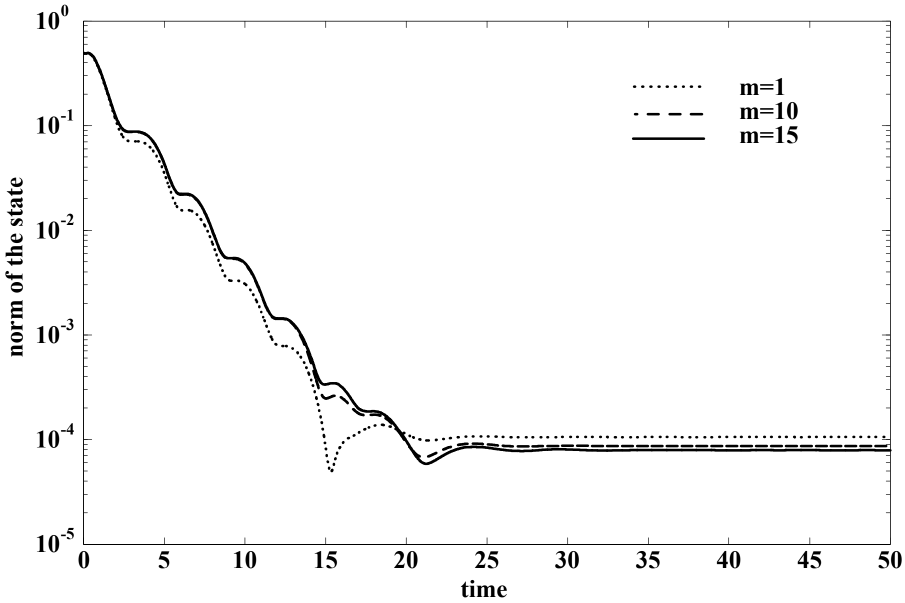

Fig. 4. Norm of trajectories of continuous-time system for different values of parameter h.

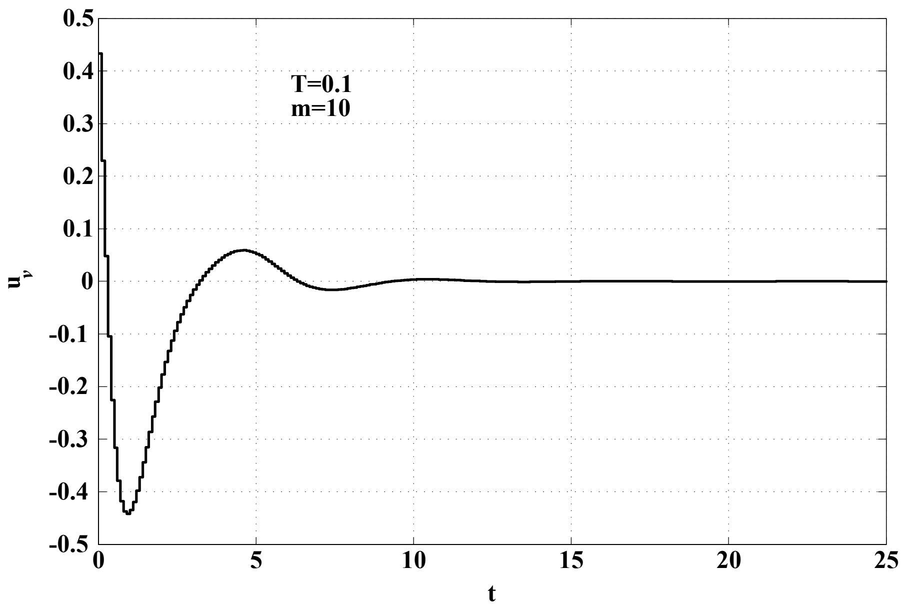

Fig. 5. Receding horizon controller when $T = 0.1$ and $m = 10$.

2.6 Stabilization with varying sampling rate $T = h$

The case of $T = h$ is of special interest because several approximations of this type (such as the Euler approximation) preserve the structure and types of nonlinearities of the continuous-time system and, hence may be preferable to the designer.

In this section we state conditions that guarantee that the receding horizon controller defined for the approximate discrete-time model with $T = h$ results in a sampled-data system which is practically asymptotically stable about the origin with the prescribed basin of attraction Γ. The main difficulty in this case is to derive a class-$\mathcal{K}_\infty$ lower bound for V_N which is independent of T. In [36], the lower bound for V_N is not proven under checkable conditions, instead, they are simply assumed. In fact, we can only determine a uniform lower bound outside an arbitrary small ball around the origin. Basically, this is why the results of ([80] Theorem 2) cannot directly be applied. In order to derive the necessary estimation, we need the following assumption on the relation between f and l_T.

Assumption A7 There exist two positive constants c_1, c_2 such that $\|f(x,u)\| \leq c_1 + c_2 l_T(x,u)$, for all $x \in \mathbb{R}^n$ and $u \in U$. □

Let $\Gamma_{\max} = \left\{x \in \mathbb{R}^n : V_N(x) \leq V^A_{\max}\right\}$. Clearly, $\Gamma \subset \Gamma_{\max}$.

We summarize the basic properties of V_N in the following theorem.

Theorem 3 Suppose that Assumptions A1-A7 are valid. Then there exist such positive numbers $\mathcal{T}^*$ and T^* that for any $T \in (0, T^*]$ and $N \in \mathbb{N}$ with $\mathcal{T}^* \leq NT \leq \mathcal{T}^* + 1$

1.) there exists a function $\psi_2 \in \mathcal{K}_\infty$ such that

$$V_N(x) \leq \psi_2(\|x\|)$$

for any $x \in \Gamma_{\max}$;

2.) for any $\overline{r} > 0$ there exists a function $\psi_1^{\overline{r}} \in \mathcal{K}_\infty$ such that

$$\psi_1^{\overline{r}}(\|x\|) \leq V_N(x), \tag{2.32}$$

for any $x \in \Gamma_{\max} \backslash \mathcal{B}_{\overline{r}}$. Moreover $V_N(0) = 0$ and $V_N(x) > 0$ for any $x \in \mathcal{B}_{\overline{r}} \backslash \{0\}$;

3.) for any $x \in \Gamma_{\max}$,

$$V_N(F^A_T(x, v^A_T(x))) - V_N(x) \leq -T l_T(x, v^A_T(x)); \tag{2.33}$$

4.) $V_N(.)$ is locally Lipschitz continuous in $\Gamma_{\max}$ uniformly in small T.

Proof. The assertions of the theorem follow from Lemmas 1-4 and Lemmas 5-7 below by taking $T^* = T^*_3$ and $\mathcal{T}^* = \mathcal{T}^*_2$. □

In what follows, we shall assume that

$$\mathcal{T}^*_2 \leq NT \leq \mathcal{T}^*_2 + 1. \tag{2.34}$$

Lemma 5 Suppose that Assumptions A1, A4, A6 and A7 hold true. For any $\delta > 0$ let

$$K_\delta = 2\left(\frac{c_1}{\varphi_1(\delta)} + c_2\right) + 1,$$

and let $R_\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as

$$R_\delta(s) = \min\left\{\gamma_1\left(\frac{s}{2}\right), \frac{s}{4K_\delta}\right\}. \tag{2.35}$$

Then there exists a $T_2^* > 0$ such that

$$\left\|F_T^A(x,u) - x\right\| \leq K_\delta T l_T(x,u)$$

for all $x \in \mathbb{R}^n$ and $u \in U_{\Delta_1}$ with $\delta \leq \|x\| + \|u\|$ and $\|x\| \leq \Delta^\delta$, where $\Delta^\delta = R_\delta^{-1}\left(V_{\max}^A\right)$.

Proof. Let $\delta > 0$ be arbitrary but fixed and let $M_1 > 0$.

If $\|x\| + \|u\| \geq \delta$, then from Assumption A4 we obtain $l_T(x,u) \geq \varphi_1(\delta)$ and

$$\|f(x,u)\| \leq \left(\frac{c_1}{\varphi_1(\delta)} + c_2\right) l_T(x,u) = K_1 l_T(x,u).$$

On the other hand, for any Δ^δ there exists $M_1 > 0$ such that for all $(x,u) \in \mathcal{B}_{\Delta^\delta} \times \mathcal{B}_{\sigma(\Delta_1)}$ we have $\|f(x,u)\| \leq M_1$.

Let $x_0 \in \Gamma$ (we may assume that $\Delta^\delta \gg \Delta$), we obtain

$$\|x(t) - x_0\| \leq \int_0^t \|f(x(s), u(s))\|\, ds \leq M_1 t,$$

as long as $\|x(s)\| \leq \Delta^\delta$. Let T_1' be such that $T_1' \leq \frac{1}{M_1}$, then $\|x(t)\| \leq \Delta + 1$.

Let L_f be the Lipschitz constant of f on $\mathcal{B}_{\Delta^\delta+1} \times \mathcal{B}_{\sigma(\Delta_1)}$. Then for any $0 < T \leq T_1'$, we have

$$\begin{aligned}\|x(T) - x_0\| &\leq \int_0^T \left(\|f(x(s),\overline{u}) - f(x_0,\overline{u})\| + \|f(x_0,\overline{u})\|\right) ds \\ &\leq TK_1 l_T(x_0,\overline{u}) + L_f \int_0^T \|x(s) - x_0)\|\, ds,\end{aligned}$$

thus by Gronwall lemma we obtain

$$\|x(T) - x_0\| \leq e^{TL_f} K_1 T l_T(x_0,\overline{u}).$$

If $0 < T \leq T_1'' := \frac{\ln 2}{L_f}$, then

$$\left\|F_T^E(x_0,\overline{u}) - x_0\right\| = \|x(T) - x_0\| \leq 2K_1 T l_T(x_0,\overline{u}).$$

From Assumption A6 with $\Delta' = \Delta^\delta + 1$, $\Delta'' = \sigma(\Delta_1)$ we have

$$\begin{aligned}\left\|F_T^A(x,u) - x\right\| &\leq \left\|F_T^A(x,u) - F_T^E(x,u)\right\| + \left\|F_T^E(x,u) - x\right\| \\ &\leq T\gamma(T) + 2K_1 T l_T(x,u).\end{aligned}$$

Let T_1''' be such that $T_1''' = \gamma^{-1}(\varphi_1(\delta))$, and $T_2^* = \min\{T_0^*, T_1', T_1'', T_1'''\}$ then for any $0 < T \leq T_2^*$ we obtain

$$\begin{aligned}\left\|F_T^A(x,u) - x\right\| &\leq T\varphi_1(\delta) + 2K_1 T l_T(x,u). \\ &\leq T l_T(x,u) + 2K_1 T l_T(x,u) = K_\delta T l_T(x,u). \; \square\end{aligned}$$

Up to this point Δ^δ can be given arbitrarily, but $\Delta^\delta > \Delta + 1$.

Lemma 6 Suppose that Assumptions A1-A7 hold true. Then there exist a $T_3^* > 0$ and a class-$\mathcal{K}_\infty$ function ψ_2 such that

$$V_N(x) \leq \psi_2(\|x\|) \tag{2.36}$$

for all $x \in \Gamma_{\max}$ and all $T \in (0, T_3^*]$. Moreover, for any $\overline{r} > 0$ there exist a class-$\mathcal{K}_\infty$ function $\psi_1^{\overline{r}}$ such that

$$V_N(x) \geq \psi_1^{\overline{r}}(\|x\|) \tag{2.37}$$

for all $x \in \Gamma_{\max} \backslash \mathcal{B}_{\overline{r}}$ and all $T \in (0, T_3^*]$.

Proof. Let NT satisfy (2.34). We observe that function $\psi_2 = \sigma_2$, where σ_2 is given by (2.30) and (2.31), is a suitable upper bound.

To find a lower estimation, let r_0, $\overline{T}$ be defined as in assumption A2 (ii). Let us choose a positive δ so that

$$0 < \delta \leq \min \left\{ r_0, \overline{r} / \left(4L_{r_0}^A (\mathcal{T}_2^* + 1) \right) \right\} \tag{2.38}$$

and let $T_3^* = \min \left\{ \overline{T}, T_1^*, T_2^* \right\}$ where T_2^* is generated by Lemma 5. Let $x_0 \in \mathcal{B}_\Delta$ be arbitrary with $\|x_0\| \geq \overline{r}$ and let the corresponding optimal trajectory be $\xi_k^* = \phi_k^A(x_0, \mathbf{u}^*(x_0))$. Then there are two possibilities: a.) If $\|\xi_N^* - x_0\| \leq \frac{1}{2} \|x_0\|$, then $\|\xi_N^*\| \geq \frac{1}{2} \|x_0\|$, therefore

$$\begin{aligned} V_N(x_0) &= \sum_{k=0}^{N-1} T l_T(\xi_k^*, u_k^*(x_0)) + g(\xi_N^*) \\ &\geq g(\xi_N^*) \geq \gamma_1(\xi_N^*) \geq \gamma_1(\frac{1}{2} \|x_0\|). \end{aligned}$$

b.) If $\|\xi_N^* - x_0\| > \frac{1}{2} \|x_0\|$, then we shall introduce the set of integers

$$\iota_1 := \{k : 0 \leq k \leq N-1, \quad \|\xi_k^*\| + \|u_k^*(x_0)\| > \delta\},$$

$$\iota_2 = \{0, 1, ..., N-1\} \backslash \iota_1.$$

With this definition we have that

$$\begin{aligned} \frac{1}{2} \|x_0\| &\leq \|\xi_N^* - x_0\| \leq \left\|\xi_N^* - \xi_{N-1}^*\right\| + \left\|\xi_{N-1}^* - \xi_{N-2}^*\right\| + ... + \|\xi_1^* - x_0\| \\ &= \sum_{k=0}^{N-1} \left\| F_T^A(\xi_k^*, u_k^*(x_0)) - \xi_k^* \right\| \\ &\leq \sum_{k \in \iota_1} T K_\delta l_T(\xi_k^*, u_k^*(x_0)) + \sum_{k \in \iota_2} T L_{r_0}^A (\|\xi_k^*\| + \|u_k^*(x_0)\|) \\ &\leq K_\delta V_N(x_0) + \delta N T L_{r_0}^A \leq K_\delta V_N(x_0) + \delta(\mathcal{T}_2^* + 1) L_{r_0}^A. \end{aligned}$$

By the choice of δ, $\delta(\mathcal{T}_2^* + 1) L_{r_0}^A \leq \frac{1}{4} \|x_0\|$, thus $V_N(x_0) \geq \frac{1}{4K_\delta} \|x_0\|$. Let $\psi_1^{\overline{r}} : [0, \infty) \to \mathbb{R}_{\geq 0}$ be defined as

$$\psi_1^{\overline{r}}(s) = \min \left\{ \gamma_1(\frac{s}{2}), \frac{s}{4K_\delta} \right\}.$$

Then $\psi_1^{\overline{r}} \in \mathcal{K}_\infty$ and (2.37) is valid. □

Corollary 1 Under the assumptions of Lemma 6, there exists a $\Delta(r_0, \rho_1)$ such that

$$\Gamma_{\max} = \left\{ x \in \mathbb{R}^n : V_N(x) \leq V_{\max}^A \right\} \subset \mathcal{B}_{\Delta(r_0, \rho_1)}.$$

Proof. The proof is given by taking $\delta_0 = \min\left\{r_0, \rho_1/(4L^A_{r_0}(\mathcal{T}^*_2+1))\right\}$, and $\Delta(r_0,\rho_1) = R^{-1}_{\delta_0}(V^A_{\max})$, where R_{δ_0} is defined by (2.35). □

Lemma 7 Suppose that Assumptions A1-A7 hold true. Then $V_N(.)$ is locally Lipschitz continuous in $\Gamma_{\max}$, uniformly in small T, i.e. there exist $L_V > 0$ and $\delta_V > 0$ such that for all $T \in (0, T^*_3]$ and $N \in \mathbb{N}$ with $TN \le (\mathcal{T}^*_2 + 1)$, inequality

$$|V_N(x) - V_N(y)| \le L_V\|x-y\| \tag{2.39}$$

holds true for all $x, y \in \Gamma_{\max}$ with $\|x-y\| \le \delta_V$.

Proof. We have seen that $\Gamma_{\max} \subset \mathcal{B}_{\Delta(r_0,\rho_1)}$. Let L_{FA}, L_g and L_l denote the Lipschitz constants of functions F^A_T, g and l_T, respectively, corresponding to $\Delta' = 2\Delta(r_0,\rho_1)$ and $\Delta'' = \sigma(\Delta_1)$. Let $y \in \Gamma_{\max}$ and let $\mathbf{u}^*(y)$ denote the corresponding optimal control sequence. Let $L_V = [(\mathcal{T}^*_2+1)L_l + L_g]\, e^{(\mathcal{T}^*_2+1)L_{FA}}$ and $\delta_V = \Delta(r_0,\rho_1)e^{-(\mathcal{T}^*_2+1)L_{FA}}$. Consider an $x \in \Gamma_{\max}$ with $\|x-y\| \le \delta_V$. Assume that for some $k \in \mathbb{N}$, $k \ge 1$ we know that

$$\left\|\phi^A_{k-1}(x,\mathbf{u}^*(y)) - \phi^A_{k-1}(y,\mathbf{u}^*(y))\right\| \le e^{(k-1)TL_{FA}}\|x-y\|.$$

(According to Assumption A2 this is true for $k=1$). From Theorem 3 it follows that $\phi^A_{k-1}(y,\mathbf{u}^*(y)) \in \Gamma_{\max} \subset \mathcal{B}_{\Delta(r_0,\rho_1)}$ for all $k = 1,2,...,N+1$. Because of the choice of δ_V it follows that $\phi^A_{k-1}(x,\mathbf{u}^*(y)) \in \mathcal{B}_{2\Delta(r_0,\rho_1)}$. Then

$$\begin{aligned}\left\|\phi^A_k(x,\mathbf{u}^*(y)) - \phi^A_k(y,\mathbf{u}^*(y))\right\| &\le e^{TL_{FA}}\left\|\phi^A_{k-1}(x,\mathbf{u}^*(y)) - \phi^A_{k-1}(y,\mathbf{u}^*(y))\right\| \\ &\le e^{kTL_{FA}}\|x-y\|.\end{aligned}$$

According to the choice of δ_V we have that $\phi^A_k(x,\mathbf{u}^*(y)) \in \mathcal{B}_{2\Delta(r_0,\rho_1)}$, since $\phi^A_k(y,\mathbf{u}^*(y)) \in \Gamma_{\max}$. Thus

$$\left\|\phi^A_k(x,\mathbf{u}^*(y)) - \phi^A_k(y,\mathbf{u}^*(y))\right\| \le e^{kTL_{FA}}\|x-y\|.$$

is valid for all $k = 0,1,...N$. Therefore

$$\begin{aligned}V_N(x) - V_N(y) &\le \sum_{k=0}^{N-1} T\left[l_T\left(\phi^A_k(x,\mathbf{u}^*(y)), u^*_k(y)\right) - l_T\left(\phi^A_k(y,\mathbf{u}^*(y)), u^*_k(y)\right)\right] \\ &\quad + g(\phi^A_N(x,\mathbf{u}^*(y))) - g(\phi^A_N(y,\mathbf{u}^*(y))) \\ &\le \left[TL_l\sum_{k=0}^{N-1} e^{kTL_{FA}} + L_g e^{NTL_{FA}}\right]\|x-y\| \le L_V\|x-y\|.\end{aligned}$$

Changing the role of x and y completes the proof. □

Theorem 4 Suppose that Assumptions A1-A7 hold true. Then there exist such positive numbers $\mathcal{T}^*$ and T^* that for any $T \in (0, T^*]$ and $N \in \mathbb{N}$ with $\mathcal{T}^* \le NT \le \mathcal{T}^*+1$, the exact discrete-time model with the receding horizon controller

$$x_{k+1} = F^E_T(x_k, v^A_T(x_k)) \tag{2.40}$$

is practically asymptotically stable about the origin with basin of attraction containing $\Gamma_{\max}$.

Proof. First we need the following claim.

Claim For any $d > 0$ with $d < V^A_{\max}$, there exists $T^* > 0$ such that, for all $T \in (0, T^*]$ and $x \in \Gamma_{\max}$, $\max\left\{V_N\left(F^E_T(x, v^A_T(x))\right), V_N(x)\right\} \geq d$, implies

$$V_N\left(F^E_T(x, v^A_T(x))\right) - V_N(x) \leq -\frac{1}{2}T\varphi_1(\|x\|).$$

Proof of the Claim. Let T^*_a, $\mathcal{T}^*_a$ and ψ_2 be generated by Theorem 3. By the assumptions of Theorem 4 and by the choice of T^*_a and $\mathcal{T}^*_a$, the $(\Delta_1, \sigma(\Delta_1))$-consistency is valid for $0 < T \leq T^*_a$ with a function γ, and (2.39) holds true with L_V and δ_V given in Lemma 7.

Suppose $V_N\left(F^E_T(x, v^A_T(x))\right) \geq d/2$. Because of (2.36) this implies

$$\left\|F^E_T(x, v^A_T(x))\right\| \geq \psi_2^{-1}\left(\frac{d}{2}\right). \tag{2.41}$$

From the choice of T^*_a, we have

$$\begin{aligned}\left|V_N(F^E_T(x, v^A_T(x))) - V_N(F^A_T(x, v^A_T(x)))\right| &\leq L_V\left\|F^E_T(x, v^A_T(x)) - F^A_T(x, v^A_T(x))\right\| \\ &\leq L_V T\gamma(T). \end{aligned}\tag{2.42}$$

Let T^*_b be such that $L_V T\gamma(T) \leq d/4$ for all $T \in (0, T^*_b]$, then we obtain

$$V_N(F^A_T(x, v^A_T(x))) \geq V_N(F^E_T(x, v^A_T(x))) - L_V T\gamma(T) \geq \frac{d}{4}. \tag{2.43}$$

From (2.33) and (2.36) we obtain

$$\psi_2(\|x\|) \geq V_N(x) \geq V_N(F^A_T(x, v^A_T(x))) \geq \frac{d}{4},$$

$$\|x\| \geq \psi_2^{-1}\left(\frac{d}{4}\right) = r_1. \tag{2.44}$$

Let T^*_c be such that $L_V\gamma(T) \leq \min\{\delta_V, \varphi_1(r_1)/2\}$ for all $T \in (0, T^*_c]$. Then we take $T^* = \min\{T^*_a, T^*_b, T^*_c\}$. From (2.41)-(2.44) we deduce that for any $T \in (0, T^*]$ and $V_N\left(F^E_T(x, v^A_T(x))\right) \geq d/2$ this implies

$$\begin{aligned}V_N(F^E_T(x, v^A_T(x))) - V_N(x) &= V_N(F^A_T(x, v^A_T(x))) - V_N(x) + \\ &\quad V_N(F^E_T(x, v^A_T(x))) - V_N(F^A_T(x, v^A_T(x))) \\ &\leq -T\varphi_1(\|x\|) + L_V\left\|F^E_T(x, v^A_T(x)) - F^A_T(x, v^A_T(x))\right\| \\ &\leq -T\varphi_1(\|x\|) + L_V T\gamma(T) \\ &\leq T\left(\frac{\varphi_1(r_1)}{2} - \varphi_1(\|x\|)\right) \leq -\frac{T}{2}\varphi_1(\|x\|).\end{aligned}$$

Next suppose that $V_N(F^E_T(x, v^A_T(x))) \leq d/2$ and $V_N(x) \geq d$.

We know also from (2.33) that $V_N(x) \geq T\varphi_1(\|x\|)$. So, it follows that

$$\begin{aligned}V_N(F^E_T(x, v^A_T(x))) - V_N(x) &\leq \frac{1}{2}\left(d - V_N(x) - V_N(x)\right) \\ &\leq -\frac{T}{2}l_T(x, v^A_T(x)) \leq -\frac{T}{2}\varphi_1(\|x\|).\end{aligned}$$

This completes the proof of the claim.

It follows from the claim that $\Gamma_{\max}$ is positively invariant for (2.40). On the other hand, if for some x both $V_N(x) \le d$ and $V_N(F_T^E(x, v_T^A(x))) \le d/2$ hold true, then it follows from the consideration above that for the trajectory of

$$\xi_{k+1} = F_T^E\left(\xi_k, v_T^A\left(\xi_k\right)\right), \;\; \xi_0 = x, \quad k = 0, 1, ...$$

we have that $V_N(\xi_k) \le d/2$, if $k = 1, 2, ...$.

To complete the proof let $\overline{r} > 0$ be an arbitrary number and let $d = \frac{1}{2}\psi_1^{\overline{r}}(\overline{r})$. Let $\mathbf{u}_v$ be the control sequence produced by v_T^A and F_T^E. From the above considerations, there exists $\beta_1 \in \mathcal{KL}$ (see the Appendix) such that

$$V_N(\phi_k^E(x, \mathbf{u}_v(x))) \le \max\{\beta_1(V_N(x), kT), d\}, \;\; k \ge 0$$

for all $x \in \Gamma_{\max}$.

From (2.32) it follows that if $\phi_k^E(x, \mathbf{u}_v(x)) \in \Gamma_{\max} \backslash \mathcal{B}_{\overline{r}}$

$$\begin{aligned} \psi_1^{\overline{r}}(\left\|\phi_k^E(x, \mathbf{u}_v(x))\right\|) &\le V_N(\phi_k^E(x, \mathbf{u}_v(x))) \\ &\le \beta_1(V_N(x), kT) \le \beta_1(\psi_2(\|x\|), kT) \end{aligned}$$

and

$$\left\|\phi_k^E(x, \mathbf{u}_v(x)))\right\| \le (\psi_1^{\overline{r}})^{-1} \circ \beta_1(\psi_2(\|x\|), kT) = \beta(\|x\|, kT)$$

where, $\beta(s, \tau) = (\psi_1^{\overline{r}})^{-1} \circ (\beta_1(\psi_2(s), \tau)) \in \mathcal{KL}$. On the other hand from the choice of $\overline{r}$ if the trajectory enters the ball $\mathcal{B}_{\overline{r}}$ it will remain there. Therefore

$$\left\|\phi_k^E(x, \mathbf{u}_v(x))\right\| \le \max\left\{\beta(\|x\|, kT), \overline{r}\right\}. \;\square$$

2.6.1 Illustrative Example

Consider the continuous-time system (this example is taken from [9])

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.5(1 + x_1)u, \\ \dot{x}_2 &= x_1 + 0.5(1 - 4x_2)u. \end{aligned}$$

This system is a modification of the system used in subsection 2.5.1 in that it is unstable.

Let the approximate discrete-time model be defined by Euler method as follows:

$$\begin{aligned} x_1(k+1) &= x_1(k) + T\left[x_2(k) + 0.5(1 + x_1(k))u(k)\right], \\ x_2(k+1) &= x_2(k) + T\left[x_1(k) + 0.5(1 - 4x_2(k))u(k)\right]. \end{aligned}$$

The running and the terminal costs are given by $l_h(x, u) = \frac{1}{2}\|x\|^4 + u^2$, $g(x) = x' P x$, where P is given below. All computations were carried out by MATLAB. Especially, the optimal control sequence was computed by the `constr` code of the Optimization toolbox. Simulations for the continuous-time system were carried out using `ode45` program in MATLAB when

$$P = \begin{pmatrix} 2.7778 & 2.2223 \\ 2.2223 & 2.7778 \end{pmatrix}.$$

The trajectories and the norms of the state of the continuous-time system are shown in Figures 6-8. The control sequence $\mathbf{u}_v$ produced by the receding horizon controller v_T^A is shown in Figure 9.

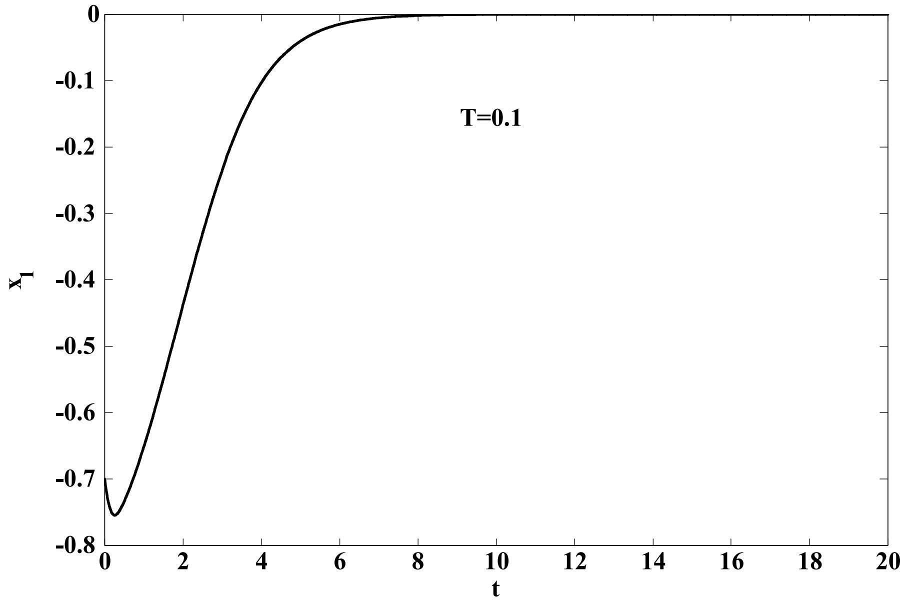

Fig. 6. The evolution of x_1 when $T = 0.1$.

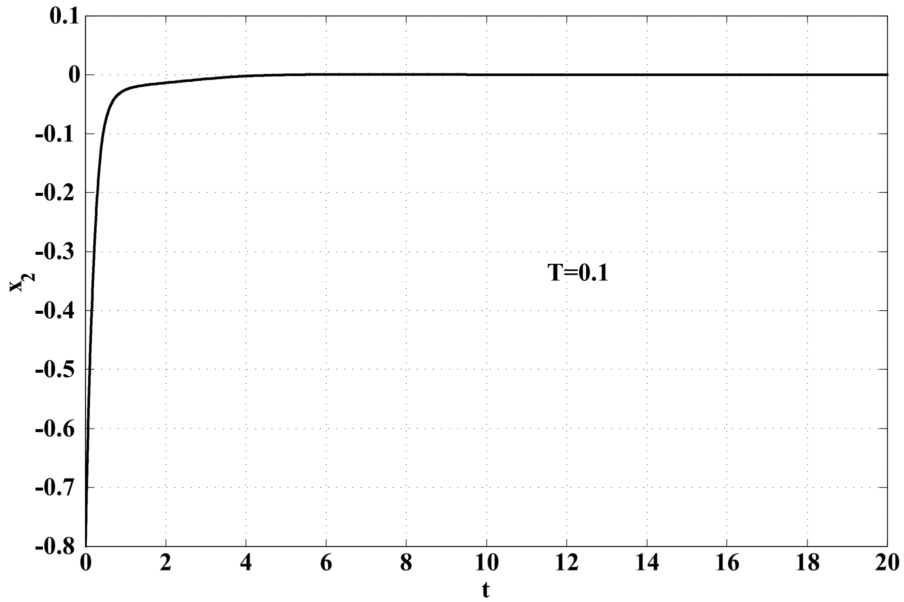

Fig. 7. The evolution of x_2 when $T = 0.1$.

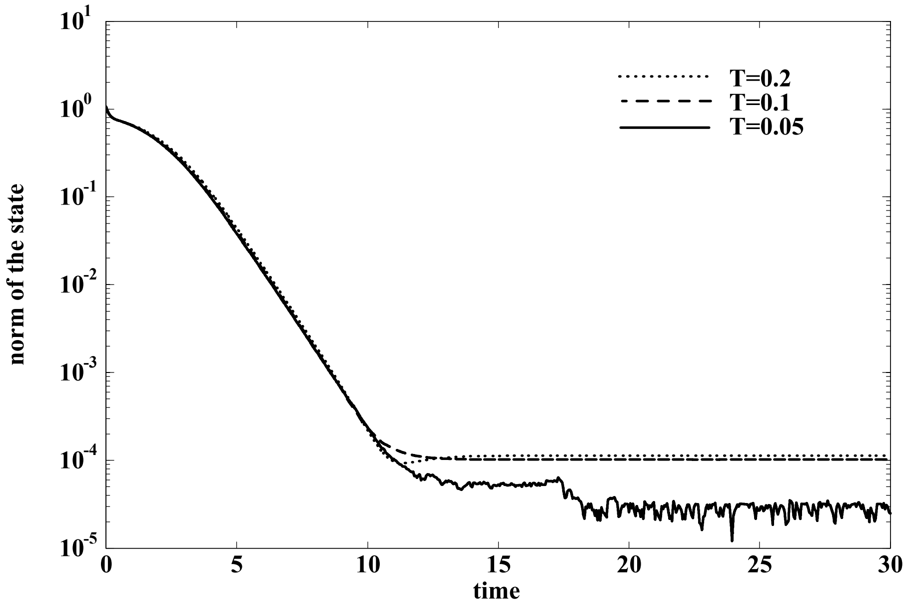

Fig. 8. Norm of trajectories for different values of parameter T.

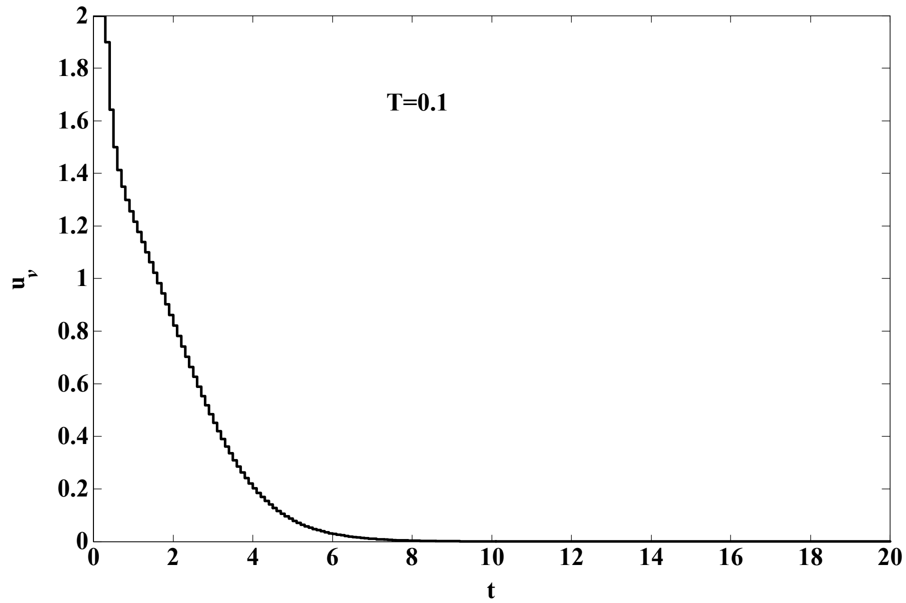

Fig. 9. Receding horizon controller when $T = 0.1$.

2.7 The Cost-to-go

The receding horizon controller redefines the performance criterion at every sampling-time instant, thus it is not optimal with respect to the given criterion. Nevertheless, a function $l_T^E(x,u)$ can be accepted as a measure of running cost, and seek a bound for the "cost-to-go" for the real process. If $l_T^E(x,u) \geq \varphi_1(\|x\|)$ with some positive definite φ_1, then this bound cannot be finite. A reasonable cost function can be defined as

$$J_\delta(x_{k_0}, k_0) = \sum_{k=k_0}^{\infty} T l_{T,\delta}^E(\widetilde{x}_k, \widetilde{u}_k) + C_\delta,$$

where $(\widetilde{x}_k, \widetilde{u}_k)$ is defined by the recursion $\widetilde{x}_{k+1} = F_T^E(\widetilde{x}_k, \widetilde{u}_k)$, $\widetilde{x}_{k_0} = x_{k_0}$, $\widetilde{u}_k = v_T^A(\widetilde{x}_k)$, $k = k_0, k_0+1, ...,$

$$l_{T,\delta}^E(x,u) = \begin{cases} l_T(x,u), & \|x\| \geq \delta, \\ 0, & \|x\| < \delta, \end{cases}$$

and term C_δ gives the cost of keeping the state inside the ball $\mathcal{B}_\delta$.

Corollary 2 Under the conditions of Theorem 4, the above-defined cost-to-go $J_{\overline{r}}$ has the following bound:

$$J_{\overline{r}}(x_{k_0}, k_0) \leq 2V_N(x_{k_0}) + C_{\overline{r}}. \tag{2.45}$$

Proof. Let d be given in the proof of Theorem 4. In the proof of Theorem 4, we have seen that

$$V_N(F_T^E(\widetilde{x}_k, \widetilde{u}_k)) - V_N(\widetilde{x}_k) \leq -\frac{T}{2} l_T(\widetilde{x}_k, \widetilde{u}_k)$$

as long as $\max\left\{V_N(F_T^E(\widetilde{x}_k, \widetilde{u}_k)), V_N(\widetilde{x}_k)\right\} \geq d$.

Summing up this inequality from k_0 to M and rearranging it, we have that

$$\frac{1}{2}\sum_{k=k_0}^{M} T l_T(\widetilde{x}_k, \widetilde{u}_k) \leq V_N(x_{k_0}) - V_N(F_T^E(\widetilde{x}_M, \widetilde{u}_M)) \leq V_N(x_{k_0}).$$

Let M be the first time instant $k = M$ such that $\max\left\{V_N(F_T^E(\widetilde{x}_k, \widetilde{u}_k)), V_N(\widetilde{x}_k)\right\} < d$ (then the same holds true for $k \geq M$, as well). It has been proven that $\widetilde{x}_k \in \mathcal{B}_{\overline{r}}$ holds true for all $k \geq M$. By the definition of $l_{T,\delta}^E$, it follows that

$$\sum_{k=k_0}^{\infty} T l_{T,\overline{r}}^E(\widetilde{x}_k, \widetilde{u}_k) \leq \sum_{k=k_0}^{M} T l_T(\widetilde{x}_k, \widetilde{u}_k) \leq 2V_N(x_{k_0}).$$

Thus estimation (2.45) follows by adding the fixed cost $C_{\overline{r}}$ to both sides of the above inequality. □

2.8 Multirate sampled-data systems with delays

A preliminary version of the results presented in this section is published in [18].

In this section we address the problem of state-feedback stabilization of (2.2) under a "low measurement rate" in the presence of measurement and computational delays. Let $\Gamma_0 \subset \Gamma$ be a given compact set containing the origin and consisting of all initial states to be taken into account. The system is to be controlled digitally using piecewise constant control functions $u(t) = u(iT) =: u_i$, if $t \in [iT, (i+1)T)$, $i \in \mathbb{N}$, where $T > 0$ is the control sampling period. Here we consider fixed sampling period T and the integration period h used in obtaining the approximate discrete-time plant model which is allowed to vary.

We shall assume that state measurements can be performed at the time instants jT^m, $j = 0, 1, \ldots$:

$$y_j := x(jT^m), \;\; j = 0, 1, \ldots,$$

where T^m is the measurement sampling period. In this case, different measurement and control sampling rates are used.

The result of the measurement y_j becomes available for the computation of the controller at $jT^m + \tau_1 (> jT^m)$, while the computation requires $\tau_2 > 0$ length of time i.e. the (re)computed controller is available at $T_j^* := jT^m + \tau_1 + \tau_2$, $j = 0, 1, \ldots$. We assume that $\tau_1 = \ell_1 T$, $\tau_2 = \ell_2 T$ and $T^m = \ell T$ for some integers $\ell_1 \geq 0$, $\ell_2 \geq 0$ and $\ell \geq \ell_1 + \ell_2 =: \overline{\ell}$.

Because of the measurement and computational delays, on the time interval $[0, \tau_1 + \tau_2)$ a precomputed control function $\mathbf{u}^c$ can only be used. It is reasonable to assume that initial states can be kept within the PAC domain of the exact system with such a precomputed controller. More precisely:

Assumption A8 (feasibility of the initial phase) There exists a $\Delta_0 > 0$ and a control sequence $\mathbf{u}^c = \{u_0^c, \ldots, u_{\overline{\ell}-1}^c\}$ with $u_i^c \in U$ can be given so that $\phi_k^E(x, \mathbf{u}^c) \in \Gamma \cap \mathcal{B}_{\Delta_0}$, $\phi_k^A(x, \mathbf{u}^c) \in \Gamma \cap \mathcal{B}_{\Delta_0}$, $k = 0, 1, \ldots, \overline{\ell}$ for all $x \in \Gamma_0$ and $h \in (0, h_0^*]$.

Furthermore, a "new" controller computed according to the measurement $y_j = x(jT^m)$ will only be available from $jT^m + \overline{\ell}T$, while in the time interval $[jT^m, jT^m + \overline{\ell}T)$, the "old" controller has to be applied. Since the corresponding exact trajectory is unknown, only an approximation ζ_j^A to the exact state $x\left(jT^m + \overline{\ell}T\right)$ can be used which can be defined as follows. Assume that a control sequence $\left\{u_0\left(\zeta_{j-1}^A\right), \ldots, u_{\ell-1}\left(\zeta_{j-1}^A\right)\right\}$ has been defined for $j \geq 1$. Let

$$\mathbf{v}^p\left(\zeta_{j-1}^A\right) = \left\{u_{\ell-\overline{\ell}}\left(\zeta_{j-1}^A\right), \ldots, u_{\ell-1}\left(\zeta_{j-1}^A\right)\right\},$$

and define ζ_j^A by

$$\zeta_j^A = \mathcal{F}_{\overline{\ell}}^A\left(y_j, \mathbf{v}^p\left(\zeta_{j-1}^A\right)\right), \quad \zeta_0^A = \phi_{\overline{\ell}}^A(x, \mathbf{u}^c),$$

where $\mathcal{F}_{\overline{\ell}}^A\left(y, \{u_0, \ldots, u_{\overline{\ell}-1}\}\right) = F_{T,h}^A\left(\ldots F_{T,h}^A\left(F_{T,h}^A\left(y, u_0\right), u_1\right) \ldots, u_{\overline{\ell}-1}\right)$.

In the stability analysis of the exact discrete-time model in the case of multirate sampling with delays outlined above, the following ℓ-step exact discrete-time model plays an important role: let $\mathbf{v}^{(j)} = \left\{u_0^{(j)}, \ldots, u_{\ell-1}^{(j)}\right\}$ and let

$$\xi_{j+1}^E = \mathcal{F}_\ell^E(\xi_j^E, \mathbf{v}^{(j)}), \qquad \xi_0^E = \phi_{\overline{\ell}}^E(x, \mathbf{u}^c),$$

where $\mathcal{F}_\ell^E(\xi_j^E, \mathbf{v}) = \phi_\ell^E(\xi_j^E, \mathbf{v})$.

Our aim is to solve the following problem: for given T, T^m, τ_1 and τ_2 find a control strategy

$$\mathbf{v}_{\ell,h} : \widetilde{\Gamma} \to \underbrace{U \times U \times ... \times U}_{\ell \text{ times}},$$

$$\mathbf{v}_{\ell,h}(x) = \{u_0(x), ..., u_{\ell-1}(x)\},$$

using the approximate model (2.4) which stabilizes the origin for the exact system (2.2) in an appropriate sense, where $\widetilde{\Gamma}$ is a suitable set containing at least $\Gamma \cap \mathcal{B}_{\Delta_0}$.

Next, we shall show that if we don't take into account the occurring delays, then instability of the closed-loop may occur.

2.8.1 Motivating examples

In this subsection, we present two examples for which a family of control laws which stabilizes the family of exact discrete-time models when no delays are presented, may destabilize the same family of exact models when the occurring delays are not taken into account.

Example 1

Consider the scalar linear system

$$\dot{x} = x + u.$$

The family of the exact discrete-time models is

$$x(k+1) = e^T x(k) + (e^T - 1)u(k). \tag{2.46}$$

The exact system can be stabilized by $u(k) = -Kx(k)$ with

$$1 \leq K \leq K^*(T) = \frac{e^T + 1}{e^T - 1}$$

for all $T > 0$.

We assume that the controller comes with a one-sampling delay i.e. $u(k) = -Kx(k-1)$. The closed-loop exact discrete-time system with delay becomes:

$$x(k+1) = e^T x(k) - (e^T - 1)Kx(k-1).$$

Let $z(k) = x(k-1)$ then the closed-loop exact system is given by

$$\begin{pmatrix} x(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} e^T & -(e^T-1)K \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ z(k) \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are

$$\lambda_1 = \frac{1}{2}\left(e^T + \sqrt{e^{2T} - 4K(e^T - 1)}\right),$$

$$\lambda_2 = \frac{1}{2}\left(e^T - \sqrt{e^{2T} - 4K(e^T - 1)}\right),$$

which are inside the unit circle for all $T < \ln(2)$ and

$$1 \leq K \leq K_d^*(T) = \frac{1}{e^T - 1}.$$

It is clear that $K_d^*(T) < K^*(T)$, for all $T > 0$. Therefore if we choose $T > \ln(2)$ or $K_d^*(T) < K \leq K^*(T)$, then the controller which stabilize the exact closed-loop system without delay will destabilize the delayed one.

In the next example we construct the state-feedback by RHC method based on the approximate discrete-time model. We show that if one does not take into account the occurring delay, then the designed receding horizon controller will destabilize the closed-loop system.

Example 2 We consider the sampled-data control of the double integrator

$$\dot{x}_1 = x_{2,} \qquad \dot{x}_2 = u,$$

for which the family of exact discrete-time models can be given as

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k) + \frac{T^2}{2}u(k), \\ x_2(k+1) &= x_2(k) + Tu(k). \end{aligned}$$

The family of Euler approximate models is

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k), \\ x_2(k+1) &= x_2(k) + Tu(k). \end{aligned} \tag{2.47}$$

The receding horizon controller for (2.47) with a cost function of type (2.9) and with the choice of $N = 2$, $G = 0$, $R_T = T^2$, $Q_T = \begin{pmatrix} 6/T^2 & 2.4/T \\ 2.4/T & 1 \end{pmatrix}$ can be computed:

$$v_T^A(x(k)) = -\frac{1.2}{T^2}x_1(k) - \frac{1.7}{T}x_2(k).$$

This gives the family of approximate closed-loop models

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k), \\ x_2(k+1) &= -\frac{1.2}{T}x_1(k) - 0.7x_2(k). \end{aligned}$$

The eigenvalues of the coefficient matrix are $|\lambda_1| = |\lambda_2| = \sqrt{2}/2$, thus the approximate closed-loop model is asymptotically stable for all $T > 0$.

The family of exact closed-loop models is

$$\begin{aligned} x_1(k+1) &= 0.4x_1(k) - 0.15Tx_2(k), \\ x_2(k+1) &= -\frac{1.2}{T}x_1(k) - 0.7x_2(k). \end{aligned}$$

The eigenvalues of the coefficient matrix are $\{-0.5,\ 0.2\}$, thus the exact closed-loop model is asymptotically stable for all $T > 0$.

We assume that the controller comes with a one-sampling delay i.e. $u(k) = v_T^A(x(k-1))$, thus the closed-loop delayed approximate and exact discrete-time models are given respectively by:

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k), \\ x_2(k+1) &= x_2(k) - TK_1x_1(k-1) - TK_2x_2(k-1). \end{aligned}$$

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k) - \frac{T^2}{2}K_1x_1(k-1) - \frac{T^2}{2}K_2x_2(k-1), \\ x_2(k+1) &= x_2(k) - TK_1x_1(k-1) - TK_2x_2(k-1). \end{aligned}$$

where $K_1 = \frac{1.2}{T^2}$, $K_2 = \frac{1.7}{T}$.

Let $z_1(k) = x_1(k-1)$ and $z_2(k) = x_2(k-1)$ then

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{pmatrix}^A = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & -TK_1 & -TK_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ z_1(k) \\ z_2(k) \end{pmatrix}^A,$$

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ z_1(k+1) \\ z_2(k+1) \end{pmatrix}^E = \begin{pmatrix} 1 & T & -\frac{T^2}{2}K_1 & -\frac{T^2}{2}K_2 \\ 0 & 1 & -TK_1 & -TK_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ z_1(k) \\ z_2(k) \end{pmatrix}^E.$$

The eigenvalues of the coefficient matrix for the approximate and exact models are $\{1.52139, 1.52139, 0.216017, 0\}$ and $\{1.6249, 1.6249, 0.4166, 0\}$, respectively. Therefore, the closed-loop for both the approximate and exact discrete-time models with delay are unstable for all $T > 0$.

In order to find a suitable controller $\mathbf{v}$, we shall apply a multistep version of the receding horizon method. To do so, we shall consider the following optimization problem: for any $N > \ell$ let

$$P^A_{T,h}(N, x): \quad \min\{J_{T,h}(N, x, \mathbf{u}) : u_k \in U\}. \tag{2.48}$$

If this optimization problem has a solution denoted by $\mathbf{u}^* = \{u_0^*, ..., u_{N-1}^*\}$, then the first ℓ elements of $\mathbf{u}^*$ are applied at the state x i.e.,

$$\mathbf{v}_{\ell,h}(x) = \{u_0^*(x), ..., u_{\ell-1}^*(x)\}.$$

For any $N > \ell$ and $x \in \mathbb{R}^n$, let

$$V_N(x) = \inf\{J_{T,h}(N, x, \mathbf{u}) : u_k \in U\}. \tag{2.49}$$

In this section we use the same optimization problem given in the previous sections, therefore the properties of the value function remain valid. The only difference is given in the following lemma. We shall use the same notation as before.

Lemma 8 If Assumptions A1-A6 hold true, then

(i) There exists a constant $M > 0$ which is independent of N and h such that

$$\|u_k^*(x)\| \leq M, \quad k = 0, 1, ..., N-1 \tag{2.50}$$

for all $x \in \Gamma_{\max}$ and $h \in (0, h_1^*]$, where h_1^* is given in Lemma 2.

(ii) If $h \in (0, h_1^*]$, and $N \in \mathbb{N}$ is chosen according to (2.29) then for any $x \in \Gamma_{\max}$, $\phi_N^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta$, and

$$V_N\left(\phi_k^A(x, \mathbf{u}^*(x))\right) - V_N(x) \leq -T\varphi_1(\|x\|). \tag{2.51}$$

for all $k = 1, ..., \ell$. Moreover, there exists $\widetilde{\Delta} > 0$ such that

$$\left\|\phi_k^A(x, \mathbf{u}^*(x))\right\| \leq \widetilde{\Delta}, \;\; k = 0, 1, ..., N. \tag{2.52}$$

Proof. The proof of (2.51) is similar to Lemma 1. To show (2.50) and (2.52), we observe that

$$V_N(x) = J_{T,h}(N, x, \mathbf{u}^*) \geq T\varphi_1\left(\|u_k^*(x)\| + \left\|\phi_k^A(x, \mathbf{u}^*(x))\right\|\right). \tag{2.53}$$

$k = 0, 1, ..., N-1$, therefore,

$$\|u_k^*(x))\| \leq \varphi_1^{-1}\left(V_{\max}^A/T\right) = M, \quad k = 0, 1, ..., N-1.$$

Since for any $x \in \Gamma_{\max}$, $\phi_N^A(x, \mathbf{u}^*(x)) \in \mathcal{G}_\eta \subset \Gamma_{\max}$ then from (2.53) we obtain

$$\left\|\phi_k^A(x, \mathbf{u}^*(x))\right\| \leq \varphi_1^{-1}\left(V_{\max}^A/T\right) = \widetilde{\Delta}, \quad k = 0, 1, ..., N. \; \square$$

2.8.2 Multistep receding horizon control in the presence of delays

In this subsection we outline an approach to the problem how the occurring measurement and computational delays can be taken into account in the stabilization of multirate sampled-data system by a receding horizon controller.

Suppose that a precomputed control sequence $\mathbf{u}^c$ satisfying Assumption A8 is given. Then the following Algorithm can be proposed.

Algorithm Let N be chosen according to (2.29), let $j = 0$, $T_{-1}^* = 0$ and let $\mathbf{u}^{(0)} = \mathbf{u}^{(p,0)} = \mathbf{u}^c = \{u_0^c, ..., u_{\ell-1}^c\}$. Measure the initial state $y(0) = x_0$.

Step j. (i) Apply the controller $\mathbf{u}^{(j)}$ to the exact system over the time interval $[T_{j-1}^*, T_j^*]$.

ii) Predict the state of the system by using the approximate model and let $\zeta_j^A = \phi_\ell^A(y(j), \mathbf{u}^{(p,j)})$.

iii) Find the solution $\mathbf{u}^* = \{u_0^*, , ..., u_{N-1}^*\}$ to the problem $P_{T,h}^A(N, \zeta_j^A)$, let $\mathbf{u}^{(j+1)} = \{u_0^*, , ..., u_{\ell-1}^*\}$ and $\mathbf{u}^{(p,j+1)} = \{u_{\ell-\bar{\ell}}^*, , ..., u_{\ell-1}^*\}$.

(iv) $j = j + 1$.

A schematic illustration of the Algorithm is sketched in Figure 10.

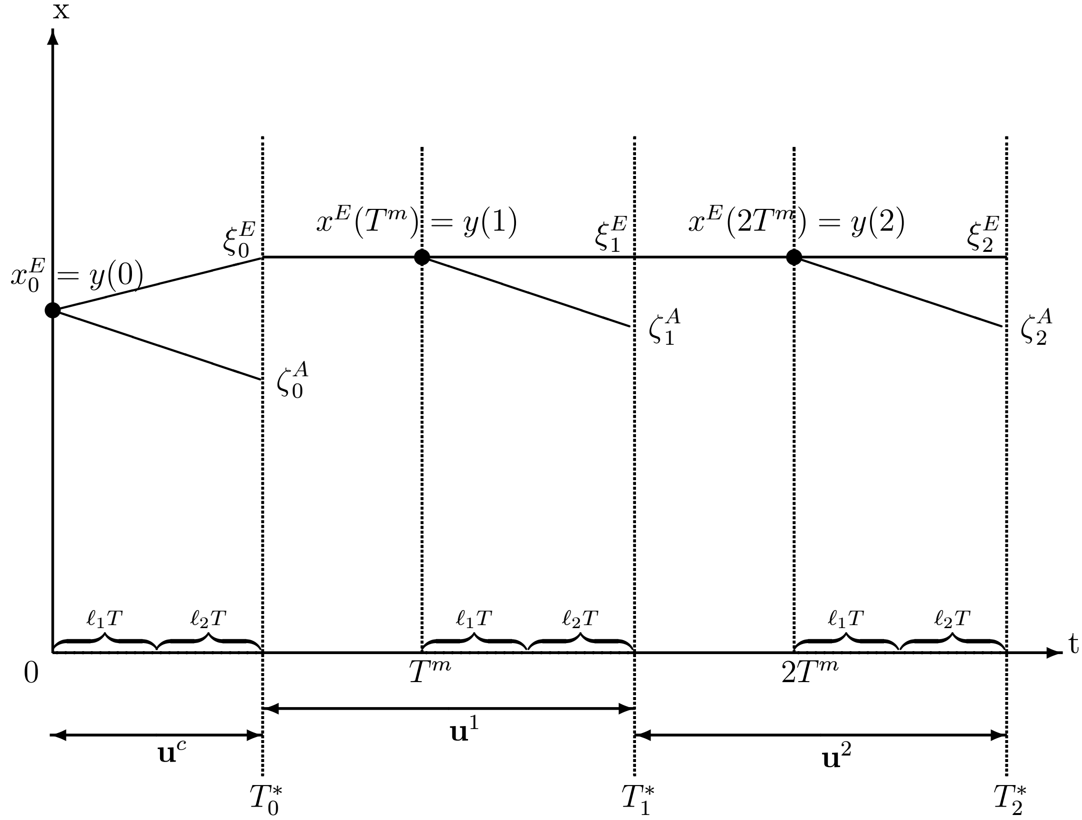

Fig. 10. Sketch of the Algorithm.

Lemma 9 Let $d > 0$ and $k \in \{1, 2, ..., \ell\}$ be arbitrary. Suppose that Assumptions A1-A6 and A8 are valid, N is chosen according to (2.29), and the following condition is satisfied:

(C) $\xi_{j-1}^E \in \Gamma_{\max}$, $\zeta_{j-1}^A \in \Gamma_{\max}$, and there exists a $\varepsilon_1(h) \in \mathcal{K}$ such that $\left\|\xi_{j-1}^E - \zeta_{j-1}^A\right\| \leq \varepsilon_1(h)$, if $h \in (0, h_1^*]$ ($j \in \mathbb{N}$, $j \geq 1$) with some $0 < h_1^* \leq h_0^*$.

Then there exists a $\overline{h}^* > 0$ (independent of k) such that for any $h \in (0, \overline{h}^*]$, inequality

$$\max\left\{V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right), V_N(\xi_{j-1}^E)\right\} \geq d \tag{2.54}$$

implies that

$$V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) - V_N(\xi_{j-1}^E) \leq -\frac{T}{2}\varphi_1(\frac{1}{2}\left\|\xi_{j-1}^E\right\|),$$

where $\mathbf{u}^{(j)}$ is the optimal solution of problem $P_{T,h}^A(N, \zeta_{j-1}^A)$.

Proof. Let Assumptions A2-A6 are satisfied with $\Delta' = \widetilde{\Delta} + 2$ and $\Delta'' = M$. Let $\varepsilon_2(h)$ be defined as

$$\varepsilon_2(h) = T\gamma(h)\frac{e^{L_f \ell T} - 1}{e^{L_f T} - 1} + e^{L_f \ell T}\varepsilon_1(h).$$

Using condition (C) we can show by induction with respect to $k = 1, ..., \ell$ that

$$\begin{aligned}\left\|\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right) - \phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right\| &\leq T\gamma(h)\frac{e^{L_f kT} - 1}{e^{L_f T} - 1} + e^{L_f kT}\left\|\xi_{j-1}^E - \zeta_{j-1}^A\right\| \\ &\leq T\gamma(h)\frac{e^{L_f \ell T} - 1}{e^{L_f T} - 1} + e^{L_f \ell T}\varepsilon_1(h) =: \varepsilon_2(h).\end{aligned}$$

Let h_2^* be such that $h_2^* = \min\left\{\varepsilon_1^{-1}\left(\frac{1}{2}e^{-L_f\ell T}\right), h_1^*\right\}$ and let $h_3^* \leq h_2^*$ be such that

$$T\gamma(h_3^*)\frac{e^{L_f\ell T}-1}{e^{L_fT}-1} \leq \frac{1}{2},$$

then for any $h \in (0, h_3^*]$ we obtain

$$\left\|\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right) - \phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right\| \leq 1.$$

Since $\zeta_{j-1}^A \in \Gamma_{\max}$ it follows from (2.52) that $\left\|\phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right\| \leq \widetilde{\Delta}$ and

$$\left\|\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right)\right\| \leq \widetilde{\Delta} + 1. \tag{2.55}$$

Suppose that $V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) \geq d/2$. Since $V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) \leq \sigma_2\left(\left\|\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right\|\right)$ then

$$\left\|\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right\| \geq \sigma_2^{-1}\left(\frac{d}{2}\right). \tag{2.56}$$

If $0 < h \leq h_4^* = \varepsilon_2^{-1}\left(\frac{1}{2}\sigma_2^{-1}\left(\frac{d}{2}\right)\right)$ then

$$\begin{aligned}\left\|\phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right\| &\geq \left\|\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right)\right\| - \left\|\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right) - \phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right\| \\ &\geq \sigma_2^{-1}\left(\frac{d}{2}\right) - \varepsilon_2(h) \geq \frac{1}{2}\sigma_2^{-1}\left(\frac{d}{2}\right).\end{aligned} \tag{2.57}$$

On the other hand from inequality (2.51) it follows that

$$V_N\left(\zeta_{j-1}^A\right) \geq V_N\left(\phi_k^A(\zeta_{j-1}^A, \mathbf{u}^{(j)})\right) \geq \sigma_1\left(\frac{1}{2}\sigma_2^{-1}\left(\frac{d}{2}\right)\right),$$

$$\left\|\zeta_{j-1}^A\right\| \geq \sigma_2^{-1}\left(\sigma_1\left(\frac{1}{2}\sigma_2^{-1}\left(\frac{d}{2}\right)\right)\right) =: \delta_1, \tag{2.58}$$

and let h_5^* be such that $h_5^* = \varepsilon_1^{-1}(\delta_1)$ then for any $h \in (0, h_5^*]$ we have

$$\begin{aligned}\left\|\xi_{j-1}^E\right\| &\leq \left\|\zeta_{j-1}^A\right\| + \left\|\xi_{j-1}^E - \zeta_{j-1}^A\right\| \\ &\leq \left\|\zeta_{j-1}^A\right\| + \varepsilon_1(h) \\ &\leq \left\|\zeta_{j-1}^A\right\| + \delta_1 \leq 2\left\|\zeta_{j-1}^A\right\|.\end{aligned} \tag{2.59}$$

Let h_6^* be such that $L_V\left(\varepsilon_1(h_6^*) + \varepsilon_2(h_6^*)\right) \leq \frac{T}{2}\varphi_1\left(\delta_1\right)$ and let $\overline{h}^* = \min\left\{h_0^*, ..., h_6^*\right\}$. From (2.55)-(2.59) and the choice of $\overline{h}^*$ we deduce that $V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) \geq d/2$ implies

$$\begin{aligned}V_N\left(\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right)\right) &- V_N\left(\xi_{j-1}^E\right) = V_N\left(\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right)\right) - V_N\left(\phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right) \\ &\quad + V_N\left(\phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right) - V_N\left(\zeta_{j-1}^A\right) + V_N\left(\zeta_{j-1}^A\right) - V_N\left(\xi_{j-1}^E\right) \\ &\leq L_V\left\|\phi_k^E\left(\xi_{j-1}^E, \mathbf{u}^{(j)}\right) - \phi_k^A\left(\zeta_{j-1}^A, \mathbf{u}^{(j)}\right)\right\| - T\varphi_1\left(\|\zeta_{j-1}^A\|\right) \\ &\quad + L_V\left\|\xi_{j-1}^E - \zeta_{j-1}^A\right\| \\ &\leq L_V\left(\varepsilon_1(h) + \varepsilon_2(h)\right) - T\varphi_1\left(\|\zeta_{j-1}^A\|\right) \\ &= -\frac{T}{2}\varphi_1\left(\|\zeta_{j-1}^A\|\right) + L_V\left(\varepsilon_1(h) + \varepsilon_2(h)\right) - \frac{T}{2}\varphi_1\left(\|\zeta_{j-1}^A\|\right) \\ &\leq -\frac{T}{2}\varphi_1\left(\frac{1}{2}\|\xi_{j-1}^E\|\right) + L_V\left(\varepsilon_1(h) + \varepsilon_2(h)\right) - \frac{T}{2}\varphi_1\left(\delta_1\right) \\ &\leq -\frac{T}{2}\varphi_1\left(\frac{1}{2}\|\xi_{j-1}^E\|\right).\end{aligned}$$

Next suppose that $V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) < d/2$, but $V_N\left(\xi_{j-1}^E\right) > d$. We know from (2.51) that $V_N\left(\xi_{j-1}^E\right) \geq T\varphi_1\left(\left\|\xi_{j-1}^E\right\|\right)$

$$\begin{aligned} V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) - V_N\left(\xi_{j-1}^E\right) &\leq \frac{d}{2} - \frac{1}{2}V_N\left(\xi_{j-1}^E\right) - \frac{1}{2}V_N\left(\xi_{j-1}^E\right) \\ &\leq -\frac{T}{2}\varphi_1\left(\left\|\xi_{j-1}^E\right\|\right) \leq -\frac{T}{2}\varphi_1\left(\frac{1}{2}\left\|\xi_{j-1}^E\right\|\right). \ \square \end{aligned}$$

Corollary 3 Under the conditions of Lemma 9 inequality

$$\max\left\{V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right), V_N(\xi_{j-1}^E)\right\} \geq d$$

implies that $\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)}) \in \Gamma_{\max}$.

Theorem 5 Suppose that Assumptions A1-A6 and A8 hold true. Then there exist a $\mathcal{T}^* > 0$ and a $\beta \in \mathcal{KL}$, and for any $r_0 > 0$ there exists a $h^* > 0$ such that for any fixed $N \in \mathbb{N}$ with $NT \geq \mathcal{T}^*$, $h \in (0, h^*]$ and $x_0 \in \Gamma_0$, the trajectory of the ℓ-step exact discrete-time system

$$\xi_{j+1}^E = \mathcal{F}_\ell^E(\xi_j^E, \mathbf{v}_{\ell,h}(\zeta_j^A)), \qquad \xi_0^E = \phi_{\overline{\ell}}^E(x_0, \mathbf{u}^c) \tag{2.60}$$

with the ℓ-step receding horizon controller $\mathbf{v}_{\ell,h}$ obtained by the prediction

$$\zeta_{j+1}^A = \mathcal{F}_{\overline{\ell}}^A\left(y_{j+1}, \mathbf{v}^p\left(\zeta_j^A\right)\right), \qquad \zeta_0^A = \phi_{\overline{\ell}}^A(x_0, \mathbf{u}^c) \tag{2.61}$$

satisfies that $\xi_j^E \in \Gamma_{\max}$ and

$$\left\|\xi_j^E\right\| \leq \max\left\{\beta\left(\left\|\xi_0^E\right\|, jT^m\right), r_0\right\}$$

for all $j \geq 0$. Moreover, $\zeta_j^A \in \Gamma_{\max}$, as well, and

$$\left\|\zeta_j^A\right\| \leq \max\left\{\beta\left(\left\|\zeta_0^A\right\|, jT^m\right) + \delta_1, r_0\right\}$$

where δ_1 can be made arbitrarily small by suitable choice of h.

Proof. To prove the theorem, we have to show that for any $j = 1, 2, ...$, lemma 9 is applicable. As in the proof of lemma 9, we take $\Delta' = \widetilde{\Delta} + 2$ and $\Delta'' = M$ in Assumptions A2-A6. Because of Assumption A8 and the choice of $\mathbf{u}^{(p,0)}$ as $\mathbf{u}^{(p,0)} = \mathbf{u}^c$, for $j = 1$ we have

$$\left\|\phi_k^E\left(y_{j-1}, \mathbf{u}^{(p,j-1)}\right) - \phi_k^A\left(y_{j-1}, \mathbf{u}^{(p,j-1)}\right)\right\| \leq T\gamma(h)\frac{e^{L_f kT} - 1}{e^{L_f T} - 1}, \tag{2.62}$$

if $k = 1, ..., \overline{\ell}$ and $h \in (0, h_1^*]$.

Let us define ε_1 by

$$\varepsilon_1(h) = T\gamma(h)\frac{e^{L_f \overline{\ell} T} - 1}{e^{L_f T} - 1}.$$

Then, from (2.62) it follows that ξ_0^E and ζ_0^A satisfy condition (C) of lemma 9.

Let $r_0 > 0$ be arbitrary, let $d = \sigma_1\left(\frac{1}{2}\sigma_2^{-1}\left(\sigma_1\left(r_0\right)\right)\right)$ and let $\delta = \sigma_2^{-1}(d)$. Suppose that for some $j \geq 1$ condition (C) is satisfied.

If $V_N\left(\xi_{j-1}^E\right) \geq d$, then $\left\|\xi_{j-1}^E\right\| \geq \sigma_2^{-1}(d) = \delta$, and for $k = 1, ..., \ell$

$$V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) - V_N\left(\xi_{j-1}^E\right) \leq -\frac{T}{2}\varphi_1\left(\frac{1}{2}\left\|\xi_{j-1}^E\right\|\right) \leq -\frac{T}{2}\varphi_1\left(\frac{\delta}{2}\right)$$

holds true. Thus $\xi_j^E \in \Gamma_{\max}$ and

$$V_N\left(\xi_j^E\right) - V_N\left(\xi_{j-1}^E\right) \leq -\frac{T}{2}\varphi_1\left(\frac{1}{2}\left\|\xi_{j-1}^E\right\|\right) \leq -\frac{T}{2}\varphi_1\left(\frac{\delta}{2}\right), \tag{2.63}$$

furthermore, $y_j \in \Gamma_{\max}$ as well.

Now we show that $\zeta_j^A \in \Gamma_{\max}$. From (2.55) we have $\left\|\phi_k^E\left(y_j, \mathbf{u}^{(p,j)}\right)\right\| \leq \widetilde{\Delta} + 1$ and by applying (2.62) for $k = 1, ..., \overline{\ell}$ with $j+1$ instead of j, we get

$$\left\|\phi_k^E\left(y_j, \mathbf{u}^{(p,j)}\right) - \phi_k^A\left(y_j, \mathbf{u}^{(p,j)}\right)\right\| \leq T\gamma(h)\frac{e^{L_f\overline{\ell}T} - 1}{e^{L_fT} - 1} = \varepsilon_1(h).$$

From the choice of h_2^* inequality $\left\|\phi_k^E\left(y_j, \mathbf{u}^{(p,j)}\right)\right\| \leq \widetilde{\Delta} + 3/2$ holds true, as well.

Let us choose $h' = \varepsilon_1^{-1}\left(\frac{T\varphi_1(\delta/2)}{2L_V}\right)$. Then

$$\begin{aligned} V_N\left(\zeta_j^A\right) &= V_N\left(\zeta_j^A\right) - V_N\left(\xi_j^E\right) + V_N\left(\xi_j^E\right) \\ &\leq L_V\left\|\zeta_j^A - \xi_j^E\right\| + V_N\left(\xi_{j-1}^E\right) - \frac{T}{2}\varphi_1\left(\frac{\delta}{2}\right) \\ &\leq V_N\left(\xi_{j-1}^E\right) + L_V\varepsilon_1(h) - \frac{T}{2}\varphi_1\left(\frac{\delta}{2}\right) \leq V_N\left(\xi_{j-1}^E\right) \leq V_{\max}^A, \end{aligned}$$

therefore $\zeta_j^A \in \Gamma_{\max}$. Thus condition (C) is valid for $j+1$ as long as $V_N\left(\xi_{j-1}^E\right) \geq d$ holds true. From (2.63) it follows that after a finite number of steps $V_N\left(\xi_{j-1}^E\right) < d$ will occur. Then, by lemma 9, we know that $V_N\left(\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right) < d$ must also be valid for $k = 1, ..., \overline{\ell}$. Define the level set $\mathcal{V}_q = \{x : V_N(x) \leq q\}$, and let $d_1 = \sigma_1(r_0)$, $r_1 = \sigma_2^{-1}(\sigma_1(r_0))$, $r_2 = r_1/2$. Obviously $\mathcal{V}_d \subset \mathcal{B}_{r_2} \subset \mathcal{B}_{r_1} \subset \mathcal{V}_{d_1} \subset \mathcal{B}_{r_0}$. Then

$$\left\|\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})\right\| \leq \sigma_1^{-1}(d) = \frac{1}{2}\sigma_2^{-1}(\sigma_1(r_0)) = \frac{r_1}{2}$$

especially $\left\|\xi_{j-1}^E\right\| \leq \sigma_1^{-1}(d) = \frac{r_1}{2}$ and $\left\|\xi_j^E\right\| \leq \frac{r_1}{2}$. On the other hand, choose $h'' = \varepsilon_1^{-1}\left(\frac{1}{2}\sigma_2^{-1}(\sigma_1(r_0))\right)$ and take $h^* = \min\left\{\overline{h}^*, h', h''\right\}$then

$$\left\|\zeta_{j-1}^A - \xi_{j-1}^E\right\| \leq \varepsilon_1(h) = \frac{r_1}{2} \quad \text{and} \quad \left\|\zeta_{j-1}^A\right\| \leq r_1.$$

Furthermore, since $y_j = \phi_{\ell-\overline{\ell}}^E(\xi_{j-1}^E, \mathbf{u}^{(j)})$ therefore $\|y_j\| \leq \frac{r_1}{2}$. Since, $\left\|\xi_j^E\right\| \leq \frac{r_1}{2}$ and from the choice of h'' it follows that

$$\left\|\zeta_j^A - \xi_j^E\right\| \leq \varepsilon_1(h) = \frac{r_1}{2} \quad \text{and} \quad \left\|\zeta_j^A\right\| \leq r_1.$$

Since

$$V_N\left(\phi_k^A(\zeta_j^A, \mathbf{u}^{(j+1)})\right) \leq V_N\left(\zeta_j^A\right) \leq \sigma_2(r_1) < \sigma_2(r_0)$$

then

$$\begin{aligned} \left\|\phi_k^A(\zeta_j^A, \mathbf{u}^{(j+1)})\right\| &\leq r_0, \\ \left\|\phi_k^E(\xi_j^E, \mathbf{u}^{(j+1)})\right\| &\leq r_1/2. \end{aligned}$$

Thus the ball $\mathcal{B}_{r_0}$ is positively invariant with respect to the exact and the approximate trajectories that appear in connection with the proposed algorithm. The existence of the function $\beta \in \mathcal{KL}$ is shown in Appendix. □

Remark 10 From Theorem 5 and lemma 9 it follows that $\phi_k^E(\xi_{j-1}^E, \mathbf{u}^{(j)})$ converges to the ball $\mathcal{B}_{r_0}$ as $j \to \infty$ for all k.

Remark 11 We note that the statement of Theorem 5 is similar to the practical asymptotic stability of the closed-loop system (2.60)-(2.61) with respect to the initial state ξ_0^E, ζ_0^A. This is not true for the original initial state x_0, because - due to the initial phase - the ball $\mathcal{B}_{r_0}$ is not invariant over the time interval $[0, \overline{\ell}T)$.

2.9 Illustrative Example

Consider the continuous-time system (this example is taken from [9])

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.5(1 + x_1)u, \\ \dot{x}_2 &= x_1 + 0.5(1 - 4x_2)u. \end{aligned}$$

Let the approximate discrete-time model be defined by the Euler method as follows: let $z_0 = x_k^A$, $\overline{u} = u_k$, $h = T/m$ and let

$$\begin{aligned} z_{1,i+1} &= z_{1,i} + h\left[z_{2,i} + 0.5(1 + z_{1,i})\overline{u}\right], \\ z_{2,i+1} &= z_{2,i} + h\left[z_{1,i} + 0.5(1 - 4z_{2,i})\overline{u}\right]. \end{aligned}$$

$i = 0, 1, ..., m-1$. Take $x_{k+1}^A = z_m$. The running and the terminal costs are given by $l_h(x, u) = \frac{1}{2}\|x\|^4 + u^2$, $g(x) = 2.7778x_1^2 + 2.2223x_2^2$. All computations were carried out by MATLAB. Especially, the optimal control sequence was computed by the `constr` code of the Optimization toolbox. Simulations for the continuous-time system were carried out using `ode45` program in MATLAB when $T = 0.05$, $m = 10$, $\ell_1 = \ell_2 = 1$, $\ell = 3$.

The trajectories of the continuous-time system are shown in Figures 11-12. In these figures, three cases are shown; 1) The ideal instantaneous ℓ-step receding horizon controller is applied under the condition that no delays are presented (ideal RHC); 2) The ideal instantaneous receding horizon controller is applied without taking into account the occurring delays (RHC delay neglected); 3) The receding horizon controller obtained by the proposed Algorithm applied to the system when the delays are present (RHC delay considered).

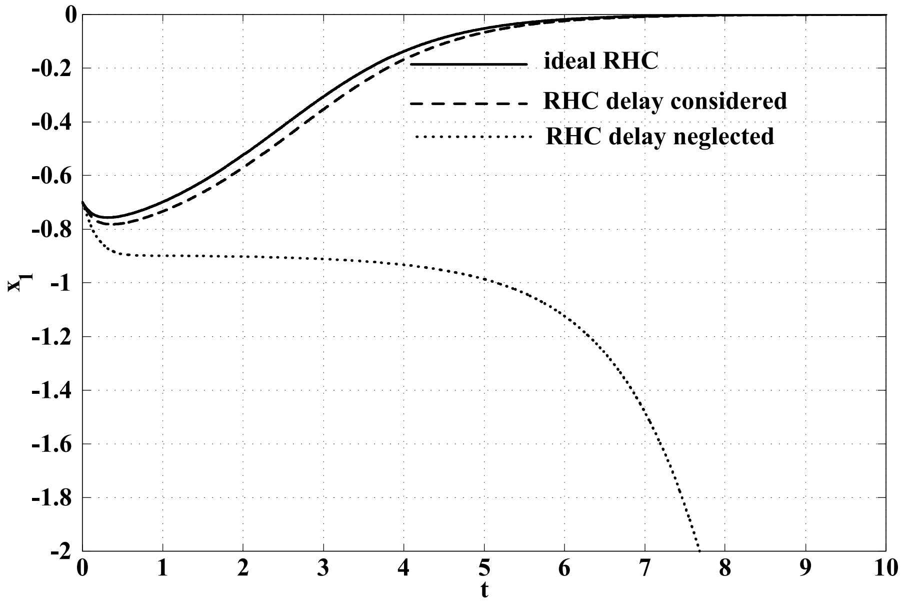

Fig. 11. The evolution of x_1 with different controllers.

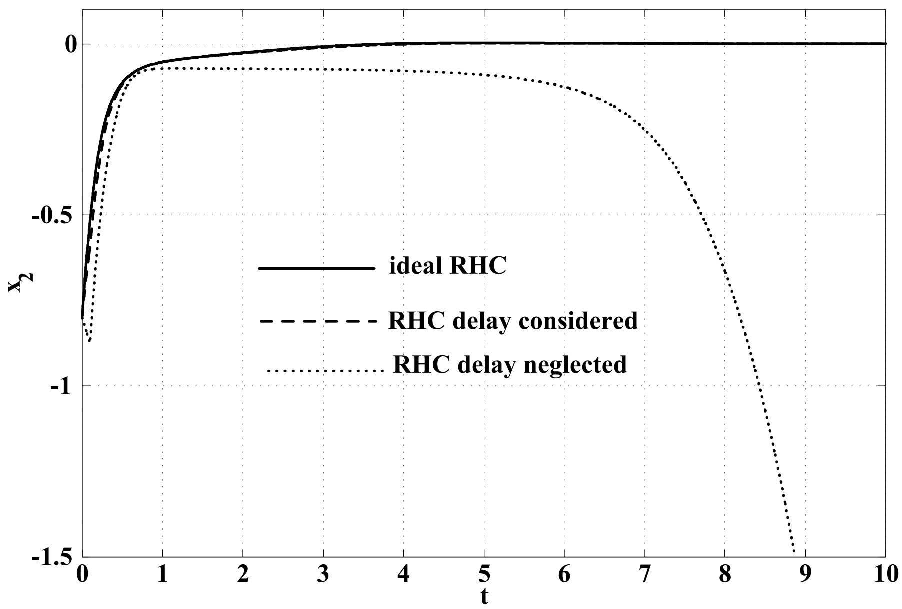

Fig. 12. The evolution of x_2 with different controllers.

Chapter 3

HIV/AIDS Model

Using recently-developed models of the interaction of the HIV virus and the immune system of the human body, we developed both one step and $\ell-$step receding horizon control (RHC) based methods for determining treatment schedules. Two kinds of four-dimensional models are considered. Reverse transcriptase inhibitors (RTI) and protease inhibitors (PI) are used. The drug dose is considered as a control input and the goal is to stabilize the system around the uninfected steady state. Simulation results are discussed.

3.1 Introduction

Over the last decade much collaborative effort involving biologists and mathematicians has been devoted towards designing mathematical models of HIV pathogenesis and antiretroviral therapies (see e.g. [45], [62], [82], [83], [87], [88], [99]). Anti-retroviral drug therapy has successfully been used to significantly suppress viral replication and to delay disease progression in many patients. These drugs act by two mechanisms: reverse transcriptase inhibitors (RTI) and protease inhibitors (PI). Reverse transcriptase inhibitors prevent the infection of new host immune cells by blocking reverse transcription of the HIV RNA into host-cell DNA. Protease inhibitors prevent already infected host cells from producing infectious virus particles. Recently, highly active antiretroviral therapies (HAART) containing a combination of RTI and PI drugs, can rapidly suppress HIV in plasma below detectable levels.

Very recent papers ([94], [100]) used a receding horizon control (RHC) method for determining treatment schedules. In these papers, the effect of the discretization of the continuous-time model on the stability analysis is completely ignored. Also, the sampling is completely ignored at the controller design step.

The aim of this chapter is to apply the theoretical results obtained in Chapter 2 to two kinds of HIV/AIDS models. Since the sampling period T can't be adjusted arbitrarily in the treatment schedule, therefore we shall only deal with the case $T \neq h$. In the first model we apply a one-step RHC method assuming the measurements are available at the sampling instants kT, $k \in \mathbb{N}$.

Sometimes, one-step RHC method is not suitable for small sampling period due to the difficulty of blood measurements. One way to overcome this problem is to increase the sampling period (see [94] and [100]). The sampling period was chosen to equal seven days i.e ($T = 7$). However, the authors did not consider the intersampling behavior. To avoid this problem we use a version of multiple step RHC, which requires the availability of blood measurements at the measurement sampling instants i.e. (kT^m, $k \in \mathbb{N}$).

3.1.1 The basic model of virus dynamics

The basic model of virus dynamics has three variables: the concentrations of uninfected CD4$^+$ T cells, x, infected CD4$^+$ T cells, y, and free infectious virus particles, v, respectively. Here, concentration refers to the population number per unit volume, mm^{-3}. The interaction between these cells and virus particles is given by the following nonlinear differential equations:

$$\dot{x} = \varrho(x, y) - kxv, \quad (3.1)$$
$$\dot{y} = kxv - \delta y, \quad (3.2)$$
$$\dot{v} = n\delta y - cv, \quad (3.3)$$

where the functional form ϱ is defined differently by different authors:
1. Perelson, Kirschner and Boer [87]:

$$\varrho(x, y) = s - \mu x + px\left(1 - \frac{x+y}{T_{\max}}\right),$$

2. Perelson and Nelson [88]:

$$\varrho(x, y) = s - \mu x + px\left(1 - \frac{x}{T_{\max}}\right).$$

We note that all of the mentioned parameters are positive. Here, s represents the rate at which new T-cells are generated from sources within the body, such as the thymus; p represents the growth rate of T-cells, which is presented as a logistic-type term; $T_{\max}$ is the T-cell population density at which proliferation shuts off; μ and δ are the (average per capita) rate of death for uninfected and for infected cells, respectively; k is the infection rate; c is the clearance rate of free virions; n is the number of virions produced from infected cell during its life-time.

Before infection the system has only one steady state $E^0(x^0, 0, 0)$ which we call uninfected steady state, where x^0 is the positive solution of $\varrho(x^0, 0) = 0$.

The basic reproductive ratio of model (3.1)-(3.3) is given by

$$R_0 = \frac{nkx^0}{c},$$

which is defined as the number of newly infected cells that arise from any one infected cell when almost all cells are uninfected (see [45] and [62]). It is shown in [62] that if $R_0 < 1$ then E^0 is asymptotically stable. Therefore at the beginning of the infection each virus-infected cell produces on average less than one newly infected cell. Thus, the infection cannot spread and the system returns to the steady state E^0. If $R_0 > 1$, then E^0 is unstable and initially each virus-infected cell produces on the average more than one newly-infected cell. In the real situation and when there is no treatment, E^0 is usually unstable for most AIDS patients. Our aim is to determine a control strategy which stabilizes this unstable equilibrium.

Here we will present two HIV/AIDS models. The first one takes into account the latently infected cells (such cells contain the virus but are not producing it) and the actively infected cells (such cells are producing the virus). The second model considers both infectious and non-infectious virions.

3.2 HIV/AIDS model 1

The results presented in this section are published in [23].

3.2.1 The model

We shall study the model proposed by [87] incorporating the effect of an antiviral therapy.

$$\dot{x} = s - \mu x + px\left(1 - \frac{x+y+y^*}{T_{\max}}\right) - e^{-\alpha_1 m(t)}kxv, \quad (3.4)$$

$$\dot{y} = e^{-\alpha_1 m(t)}kxv - \mu y - k_1 y, \quad (3.5)$$

$$\dot{y}^* = k_1 y - \delta y^*, \quad (3.6)$$

$$\dot{v} = n\delta y^* - kxv - cv, \quad (3.7)$$

where y and y^* are the concentrations of latently infected $CD4^+$ T-cells and actively infected $CD4^+$ T-cells, respectively. In this model only RTI drugs are considered. The effect of treatment is represented by a chemotherapy function $e^{-\alpha_1 m(t)}$ where $m(t)$ is the drug dose at time t and α_1 is the efficiency of drugs for reverse transcriptase inhibitors (see [6] and [25]). We shall define the control input as:

$$u(t) = \alpha_1 m(t).$$

Equation (3.5) and (3.6) model the latently and actively infected $CD4^+$ T populations, respectively. At rate k_1, the latently infected cells become actively infected. The loss of free virus due to infection of a cell is represented by the term $-kxv$ in (3.7).

It is reasonable to assume that the death by viral cytopathicity occurs faster than death by natural means (see [45] and [87]), i.e.

$$\delta > \mu. \quad (3.8)$$

We note that the system (3.4)-(3.7) is biologically acceptable because

$$\begin{aligned} \dot{x}\,|_{(x=0)} &= s \geq 0, \\ \dot{y}\,|_{(y=0)} &= e^{-u(t)}kxv \geq 0, \quad (v, x \geq 0), \\ \dot{y}^*\,|_{(y^*=0)} &= k_1 y \geq 0, \quad (y \geq 0), \\ \dot{v}\,|_{(v=0)} &= n\delta y^* \geq 0, \quad (y^* \geq 0). \end{aligned}$$

This means that the non-negative orthant $\mathbb{R}^4_+$ is positively invariant, namely, if a trajectory starts in the non-negative orthant, it remains there.

One of the properties of the logistic equation for T-cells is that if $0 \leq x(0) < T_{\max}$, then $x(t) < T_{\max}$ for all t (see [24]). If we consider the case $y = y^* = v = 0$, the equation (3.4) is given in the following form:

$$\dot{x} = s - \mu x + px\left(1 - \frac{x}{T_{\max}}\right).$$

The property mentioned above remains true if

$$s - \mu T_{\max} < 0. \tag{3.9}$$

Since HIV infection only reduces the T-cell population, this property has to remain true for the total T-cell population $T_{tot} = x + y + y^*$, namely $\dot{T}_{tot} \mid_{(T_{tot}=T_{\max})} < 0$ if (3.8) and (3.9) hold. (This corresponds to [45] and [87].)

Thus, in the case of HIV infection, $T_{tot}(t)$, and hence $x(t)$, $y(t)$ and $y^*(t)$ are all bounded by $T_{\max}$. On the other hand,

$$\dot{v} \le n\delta y^* - cv \le n\delta T_{\max} - cv,$$

therefore $v(t) \le \frac{n\delta T_{\max}}{c} =: M^*$ for all $t > 0$, if this inequality holds true for $t = 0$ and $y^*(0) \le T_{\max}$.

We introduce the notation for the set of biologically relevant states

$$\Omega = \{(x, y, y^*, v) : 0 \le x, y, y^* \le T_{\max},\ 0 \le v \le M^*\}.$$

3.2.2 Steady state analysis

Let us compute the equilibrium points of system (3.4)-(3.7) under constant controller i.e. for $u(t) = \widehat{u}$, $t \ge 0$. We get that, if

$$n \le n_{th}(\widehat{u}) = \frac{\mu + k_1}{e^{-\widehat{u}} k_1},$$

then there is only one biologically acceptable equilibrium point, namely the uninfected steady state $E^0 = (x = x^0,\ y = 0,\ y^* = 0,\ v = 0)$,

$$x^0 = \frac{1}{2q_2}\left(q_1 + \sqrt{q_1^2 + 4sq_2}\right) \tag{3.10}$$

with $q_1 = p - \mu$, $q_2 = p/T_{\max}$, which is identical with the steady state of the untreated case i.e when $\widehat{u} = 0$. We note that if $n \le n_{th}(\widehat{u})$ there is another equilibrium point, but it has no biological meaning because its first coordinate is negative ($x < 0$).

If n satisfies inequality

$$n > \widehat{n}_{crit}(\widehat{u}) = \frac{(k_1 + \mu)(c + kx^0)}{x^0 k k_1 e^{-\widehat{u}}} = n_{th}(\widehat{u})\left(1 + \frac{c}{kx^0}\right), \tag{3.11}$$

then this negative equilibrium point becomes positive with the following coordinates: $E^+ = (\overline{x}_0,\ \overline{y}_0,\ \overline{y}_0^*,\ \overline{v}_0)$, where

$$\begin{aligned}
\overline{x}_0 &= \frac{c}{n\frac{e^{-\widehat{u}} k k_1}{k_1 + \mu} - k} = \frac{c}{k\left(n/n_{th}(\widehat{u}) - 1\right)} = \frac{c}{\alpha}, \\
\overline{y}_0 &= \frac{e^{-\widehat{u}} k \overline{v}_0 \overline{x}_0}{k_1 + \mu} = \frac{k \overline{v}_0 \overline{x}_0}{k_1 n_{th}(\widehat{u})}, \\
\overline{y}_0^* &= \frac{e^{-\widehat{u}} k k_1 \overline{v}_0 \overline{x}_0}{\delta\left(k_1 + \mu\right)} = \frac{k \overline{v}_0 \overline{x}_0}{\delta n_{th}(\widehat{u})}, \\
\overline{v}_0 &= \frac{s\alpha^2 + q_1 \alpha c - q_2 c^2}{k c e^{-\widehat{u}}\left(\beta c + \alpha\right)},
\end{aligned}$$

where

$$\alpha = k\left(\frac{n}{n_{th}(\widehat{u})} - 1\right), \quad \beta = \frac{q_2}{k_1+\mu}\left(1+\frac{k_1}{\delta}\right).$$

If $n_{th}(\widehat{u}) < n < \widehat{n}_{crit}(\widehat{u})$, then $\overline{x}_0$ is positive but it is easy to see that $\overline{v}_0, \overline{y}_0$ and $\overline{y}_0^*$ are negative.

If $n = \widehat{n}_{crit}(\widehat{u})$, then $\overline{v}_0 = 0$, because $s\alpha^2 + q_1\alpha c - q_2c^2 = 0$. In fact,

$$\alpha\mid_{(n=\widehat{n}_{crit}(\widehat{u}))} = k\left(\frac{\widehat{n}_{crit}(\widehat{u})}{n_{th}(\widehat{u})} - 1\right) = \frac{c}{x^0},$$

and

$$s\frac{c^2}{\left(x^0\right)^2} + q_1\frac{c^2}{x^0} - q_2c^2 = 0 \ \text{ iff } \ q_2\left(x^0\right)^2 - q_1x^0 - s = 0,$$

which can be verified by substituting x^0 from (3.10).

In sum we get

$$\begin{aligned} \overline{v}_0 &< 0, \quad \text{if} \quad n < \widehat{n}_{crit}(\widehat{u}), \\ \overline{v}_0 &= 0, \quad \text{if} \quad n = \widehat{n}_{crit}(\widehat{u}), \\ \overline{v}_0 &> 0, \quad \text{if} \quad n > \widehat{n}_{crit}(\widehat{u}). \end{aligned}$$

It means $\overline{v}_0$ is an increasing function of n and $\overline{x}_0$ is a decreasing function of n, when $n > n_{th}(\widehat{u})$. In case of $n = \widehat{n}_{crit}(\widehat{u})$, $\overline{x}_0 = x^0$ and $\overline{v}_0 = 0$. ($\overline{y}_0$ and $\overline{y}_0^*$ are negative if $n_{th}(\widehat{u}) < n < \widehat{n}_{crit}(\widehat{u})$.)

3.2.3 Stability of uninfected steady state

In order to consider the local stability behavior of the equilibrium point let us linearize the system (3.4)-(3.7) in case of constant control i.e. $u(t) = \widehat{u}$, around E^0. The coefficient matrix is:

$$A = \begin{bmatrix} a_{11} & -q_2x^0 & -q_2x^0 & -e^{-\widehat{u}}kx^0 \\ 0 & -(k_1+\mu) & 0 & e^{-\widehat{u}}kx^0 \\ 0 & k_1 & -\delta & 0 \\ 0 & 0 & n\delta & -kx^0 - c \end{bmatrix} \tag{3.12}$$

where $a_{11} = -\mu + p - \frac{2px^0}{T_{\max}}$. Obviously, $a_{11} < 0$ iff $\frac{T_{\max}}{2}\left(1 - \frac{\mu}{p}\right) < x^0$ and this is true according to (3.10). (We note that if $\widehat{u} \to \infty$ then A is signstable (see [48]), thus E^0 is asymptotically stable for any given parameters. Of course this has no practical relevance, because we cannot increase the drugs infinitely.)

Let us consider the characteristic polynomial

$$\mathcal{D}(\lambda) = \lambda^4 - r_1\lambda^3 + r_2\lambda^2 - r_3\lambda + r_4$$

of the matrix A, where $r_1 = Trace(A)$ and $r_2 = \sum_{i=1}^{3} a_{ii}\sum_{i<j} a_{jj}$, $r_2 > 0$. After straightforward calculations we get that $r_4 > 0$ iff $n < \widehat{n}_{crit}(\widehat{u})$. In this case $r_3 < 0$ too, moreover the Routh-Hurwitz criteria holds.

The following theorem summarizes our results.

Theorem 6 *If $n < \widehat{n}_{crit}(\widehat{u})$, then the uninfected equilibrium point $E^0 = (x^0, 0, 0, 0)$ is locally asymptotically stable.*

We note that the steady state and stability analysis are similar to those in [87] when there is no treatment i.e. $\widehat{u} = 0$.

Theorem 7 If $n < n_{th}(\widehat{u})$, *then the uninfected equilibrium point $E^0 = (x^0, 0, 0, 0)$ is asymptotically stable in the whole set of biologically relevant states Ω.*

Proof. The proof is based on LaSalle's invariance principle (see e.g. Theorem 2.2 in [92] or Theorem 5.4.1 in [73]). Let us define a function $H : \mathbb{R}^4_+ \longrightarrow \mathbb{R}$ (analogously to function L in [87]) by

$$H(x, y, y^*, v) = y + ne^{-\widehat{u}}y^* + e^{-\widehat{u}}v.$$

The derivative of H along the solution of (3.4)-(3.7) with $u(t) \equiv \widehat{u}$ is

$$\dot{H}_{(3.4)-(3.7)}(x, y, y^*, v) = \left[nk_1 e^{-\widehat{u}} - k_1 - \mu\right] y - ce^{-\widehat{u}}v.$$

Hence $\dot{H}_{(3.4)-(3.7)}(x, y, y^*, v) \leq 0$ in $\mathbb{R}^4_+$, if $n < n_{th}(\widehat{u})$. From the invariance principle it follows that the trajectories of (3.4)-(3.7) with $u(t) \equiv \widehat{u}$ converge to the largest invariant set D_0 in the subset where $\dot{H}_{(3.4)-(3.7)}(x, y, y^*, v) = 0$. This set is nothing else as

$$D_0 = \{(x, y, y^*, v) : 0 \leq x \leq T_{\max}, \quad y = y^* = v = 0\}.$$

To finish the proof we have to show that the trajectories do not have any accumulation point different from E^0. Assume that, in contrary, $\overline{E} \in D_0$ is an accumulation point and $\overline{E} \neq E^0$. Since $n_{th}(\widehat{u}) < \widehat{n}_{crit}(\widehat{u})$, for $n < n_{th}(\widehat{u})$, E^0 is locally asymptotically stable, thus $\overline{E}$ is outside a ball $\mathcal{B}_\rho(E^0)$ with radius ρ around E^0. Since the right hand side of (3.4) is strictly positive and strictly negative along the x-axis for $x < x^0$ and $x > x^0$, respectively, therefore for continuity reasons there exist two positive numbers $\widehat{\delta}$ and c_1 such that $\dot{x}(t) \geq c_1$ (i.e. $\dot{x}(t) \leq -c_1$) if $0 \leq x(t) \leq x^0 - \frac{\rho}{2}$ (i.e. $x^0 + \frac{\rho}{2} \leq x(t) \leq T_{\max}$) and $y(t) < \widehat{\delta}$, $y^*(t) < \widehat{\delta}$ and $v(t) < \widehat{\delta}$. It follows then that $\overline{E}$ is a limit point and $\dot{x}(t)$ should tend to zero as $t \to \infty$, which is a contradiction. □

For a given value of n there exists a non-negative value of the drug dose u, namely

$$u_{th}(n) = \max\left\{0, \ln\left(\frac{nk_1}{k_1 + \mu}\right)\right\}$$

such that for $\widehat{u} > u_{th}(n)$ inequality $n \leq n_{th}(\widehat{u}) \leq \widehat{n}_{crit}(\widehat{u})$ holds true.

Thus the following corollary can be formulated.

Corollary 4 The uninfected equilibrium point E^0 is asymptotically stable in Ω if $u(t) \equiv \overline{u} > u_{th}$, and the control sequence $\mathbf{u} = \{\overline{u}, \overline{u}, \dots\}$ satisfies Assumption A3.

This implies that for any initial points belonging to Ω there exists a control function $\mathbf{u}$ for which no finite escape time occurs.

Remark 12 If the initial points are close enough to E^0 then one can apply a control sequence $\mathbf{u} = \{\widehat{u}, \widehat{u}, \dots\}$ with $\widehat{u} > u_c$, where

$$u_c = \max\left\{0, \ln\left(\frac{nkk_1x^0}{(k_1 + \mu)(c + kx^0)}\right)\right\}$$

to stabilize the system around E^0.

3.2.4 Infected steady state

We shall examine the stability behavior of the infected steady state E^+ only for the given parameters in Table 1. We found that if $0 \leq u(t) < u_c = 0.518117$, then the infected steady state E^+ is stable. If $u(t) > u_c$, the virus will eventually be eradicated and the infected steady state does not lie in $\mathbb{R}^4_+$.

In Figures 13 and 14, the steady state values of x and v are plotted versus u. Stable steady states are indicated by dark solid lines, unstable steady states by light lines.

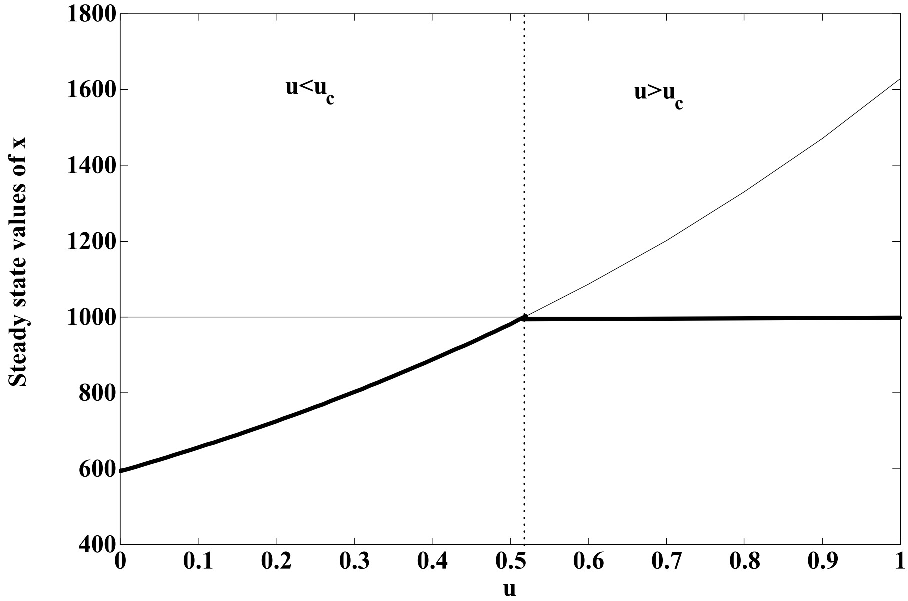

Fig. 13. Transcritical bifurcation. The steady state values of x are plotted versus $\widehat{u}$.

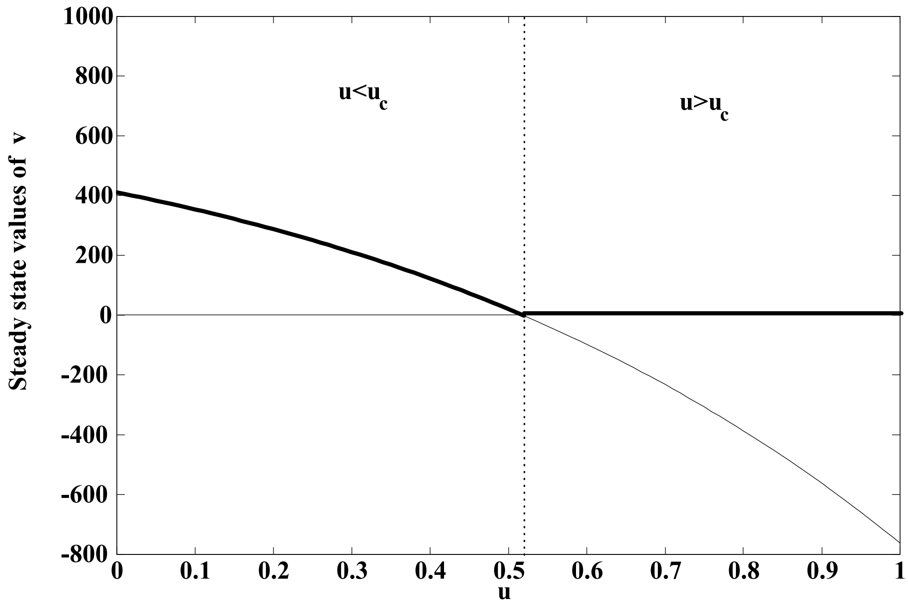

Fig. 14. Transcritical bifurcation. The steady state values of v are plotted versus $\widehat{u}$.

3.2.5 Application of RHC method

Our aim is to stabilize the system near E^0 using the receding horizon control method proposed in Chapter 2. We transform the point $(x^0, 0, 0, 0)$ into the origin with $\widetilde{x} = x - x^0$, $\widetilde{y} = y$, $\widetilde{y}^* = y^*$, $\widetilde{v} = v$, and the controller u into $\widetilde{u} = u - u_c$, then (3.4)-(3.7) become

$$\begin{aligned}
\dot{\widetilde{x}} &= s - \mu(\widetilde{x} + x^0) + p(\widetilde{x} + x^0)\left(1 - \frac{\widetilde{x} + x^0 + \widetilde{y} + \widetilde{y}^*}{T_{\max}}\right) \\
&\quad - e^{-(\widetilde{u}(t)+u_c)} k(\widetilde{x} + x^0)\widetilde{v}, &(3.13)\\
\dot{\widetilde{y}} &= e^{-(\widetilde{u}(t)+u_c)} k(\widetilde{x} + x^0)\widetilde{v} - \mu\widetilde{y} - k_1\widetilde{y}, &(3.14)\\
\dot{\widetilde{y}}^* &= k_1\widetilde{y} - \delta\widetilde{y}^*, &(3.15)\\
\dot{\widetilde{v}} &= n\delta\widetilde{y}^* - k(\widetilde{x} + x^0)\widetilde{v} - c\widetilde{v}. &(3.16)
\end{aligned}$$

Let $f(\widetilde{X}, \widetilde{u})$ be the right hand side of (3.13)-(3.16) where $\widetilde{X} = (\widetilde{x}, \widetilde{y}, \widetilde{y}^*, \widetilde{v})'$. We can see that f is continuous and Lipshitz continuous and $f(0, 0) = 0$, thus Assumption A1 is satisfied. Let $F^A_{T,h}$ be constructed using multiple steps of a one-step second order Rung-Kutta scheme, then Assumptions A2 and A6 are satisfied. From Corollary 4 it follows that Assumption A3 holds true, as well.

The sampling period was chosen to be $T = 1$. To verify Assumptions A4 and A5, let A_C be the coefficient matrix of the linearized system (3.13)-(3.16) in case of constant control i.e. $\widetilde{u}(t) = \widetilde{u} > \widetilde{u}_{th}$, where $\widetilde{u}_{th} = u_{th} - u_c$. The discrete-time model for the linearized continuous-time model is given by:

$$\widetilde{X}(k + 1) = e^{A_C T}\widetilde{X}(k). \tag{3.17}$$

The running cost and the terminal cost can be chosen as:

$$\begin{aligned}
l_h(\widetilde{X}, \widetilde{u}) &= 0.1\widetilde{X}'Q\widetilde{X} + 0.001\widetilde{u}^2, &(3.18)\\
g(\widetilde{X}) &= \widetilde{X}'P\widetilde{X}, &(3.19)
\end{aligned}$$

where P is a positive definite diagonal matrix given by

$$P = diag(100, 100, 100, 0.002)$$

and Q is a positive definite symmetric matrix

$$Q = \begin{bmatrix} 5.8235 & 1.8933 & 2.1203 & 0.0022 \\ 1.8933 & 4.4559 & -0.7156 & -0.0022 \\ 2.1203 & -0.7156 & 18.1818 & -0.0177 \\ 0.0022 & -0.0022 & -0.0177 & 0.0020 \end{bmatrix}.$$

This P and Q satisfy the Lyapunov equation for the discrete-time system (3.17)

$$Q = -(A'_T P A_T - P), \qquad A_T = e^{A_C T}.$$

From (3.18)-(3.19), Assumption A4 is satisfied. It has been verified numerically by solving a constrained minimization problem with several starting points that Assumption A5 is satisfied over the whole set Ω.

All computations were carried out by MATLAB. Especially, the optimal control sequence was computed by the `constr` code of the Optimization toolbox. Simulations for the continuous-time system were carried out using `ode45` program in MATLAB.

Remark 13 We note that, when applying the receding horizon algorithm, the cost function $J_{T,h}$ is redefined at each sampling instant, thus the applied control doesn't minimize it over any interval. The optimization of this cost is not the aim of the computations but it serves only as an aid for finding the desired stabilizing controller. Therefore the biological content doesn't play any role in its choice. Nevertheless, it is important to investigate the performance of the proposed control function by means of a biologically substantial cost function. Of course, the proposed controller may only be suboptimal with respect to the latter criterion. Caetano and Yoneyama [6] proposed a running and terminal costs which have a biological meaning as follows:

$$l^r(X,u) = \omega\left(1 - e^{-\varepsilon\left(\frac{u}{\alpha_1}\right)^2}\right) + \frac{\gamma_1}{x^2} + \gamma_2 v^2, \tag{3.20}$$

$$g^r(X) = \frac{\gamma_1}{x^2} + \gamma_2 v^2, \tag{3.21}$$

with $\gamma_1 = \gamma_2 = 25 \times 10^4$, $\omega = 1$, $\varepsilon = 10^{-6}$. The function g^r represents the target of maximizing uninfected CD4$^+$ T-cell and minimizing the viral load. The coefficient ω is the weight that reflects the dose-related side effects of the drug $m(t)$. The two last terms in (3.20) are included to force x (uninfected CD4$^+$ T-cell) to increase and v (viral load) to decrease with treatment. Here, ε is the sensibility of the patient with respect to reverse transcriptase inhibitors. We note that the value of l^r and g^r is important from a practical point of view, but can't be used to construct the receding horizon control because they don't satisfy Assumption A4. Therefore we use l_h and g to stabilize the system. Consider l^r and g^r as the costs of the real system. We define the following cumulative performance index

$$J^r(0, M, X_0, \mathbf{u}_v) = \sum_{k=0}^{M-1} T l^r(X_k^E, u_k) + g^r(X_M^E), \quad M = 1, ..., M^*$$

where $X_{k+1}^E = F_T^E\left(X_k^E, u_k\right)$, $u_k = v_{T,h}^A(X_k^E)$ and $\mathbf{u}_v$ is the control sequence produced by F_T^E and $v_{T,h}^A$. We shall compute the cumulative performance index $J^r(0, M, X_0, \mathbf{u}_v)$ as a function of the elapsed time M along the receding horizon trajectory at the sampling instants $t_M = MT$.

3.2.6 Numerical results

Definitions and numerical information for the parameters presented in Table 1 are obtained from [87].

Parameters and constants	Values
$\mu =$ death rate of uninfected and latently infected CD4$^+$ T-cells population	0.02 day^{-1}
$\delta =$ death rate of actively infected CD4$^+$ T-cells	0.24 day^{-1}
$c =$ death rate of free virus	2.4 day^{-1}
$k =$ rate CD4$^+$ T-cells becomes infected	2.4$\times$10^{-5} mm^3 day^{-1}
$k_1 =$ rate y cells convert to actively infected	3$\times$10^{-3} day^{-1}
$p =$ rate of growth for the CD4$^+$ T-cell population	0.03 day^{-1}
$n =$ number of free viruses produced by y^* cells	1300
$T_{\max} =$ maximum CD4$^+$ T-cell population level	1500 mm^{-3}
$s =$ source term for uninfected CD4$^+$ T.	10 mm^{-3} day^{-1}

Table 1.

For the parameters given in Table 1, we can see that $n > n_{crit} \mid_{(\widehat{u}=0)} = 774$, so E^0 is unstable for untreated case i.e. $\widehat{u} = 0$.

We assume that the infection occurs with a certain amount of virus particles $v = 0.001$. Thus the initial conditions are $x(0) = x^0$, $y(0) = y^*(0) = 0$ and $v(0) = 0.001$. Figures 15-18 show the case when there is no treatment ($u = 0$). It can be seen that the concentration of uninfected CD4$^+$ T cells is decaying while the concentrations of latently and actively infected cells and free viruses are increasing. Also, we note that the trajectory tends to the stable infected steady state $E^+ = (593.242,\ 254.01,\ 3.17513,\ 410.332)$. In Figures 19-24 the time measured after 1100 days from the onset of infection i.e. the treatment is initiated at the state $(25.706,\ 19.5871,\ 0.2326,\ 29.7937)$.

In Figure 19, the concentration of uninfected CD4$^+$ T cell is seen to increase and tends to the normal value x^0 when the receding horizon control is used. Figures 20-22 show that the concentrations of latently and actively infected cells and free viruses are decreasing under the application of the receding horizon control.

In this model, the efficiency of drugs for reverse transcriptase inhibitors is given by $\alpha_1 = 0.005$ (see [6]). The drug doses obtained by the RHC method as a function of the time is shown in Figure 23. It is observed that the system can be kept at the stationary point $(x^0, 0, 0, 0)$ by small drug doses i.e. $m(t) \geq m_c = \frac{u_c}{\alpha_1}$. The cumulative performance index $J^r(0, M, X_0, \mathbf{u}_v)$ with $M^* = 600$ is shown in Figure 24.

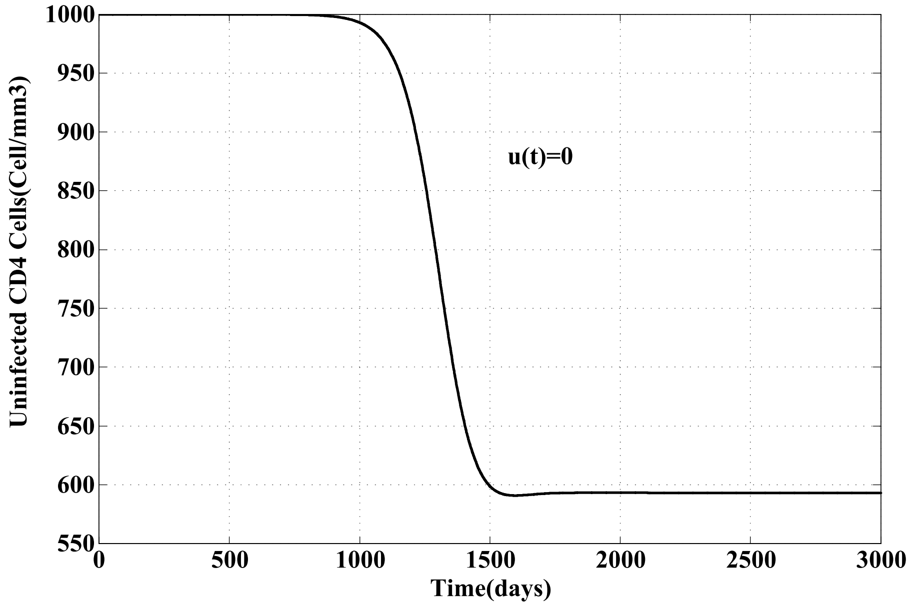

Fig. 15. The evolution of CD4 T cells for the untreated case.

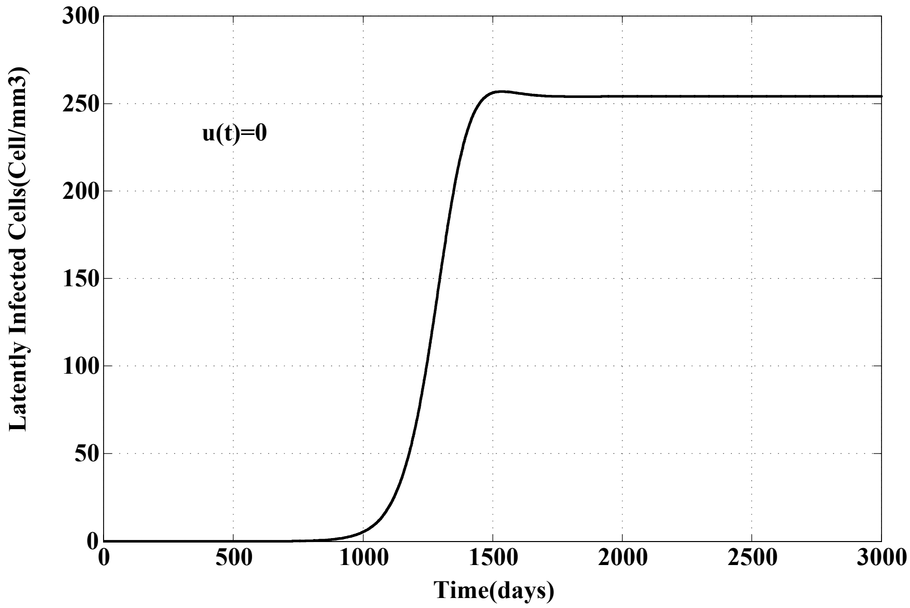

Fig. 16. The evolution of latently infected cells for the untreated case.

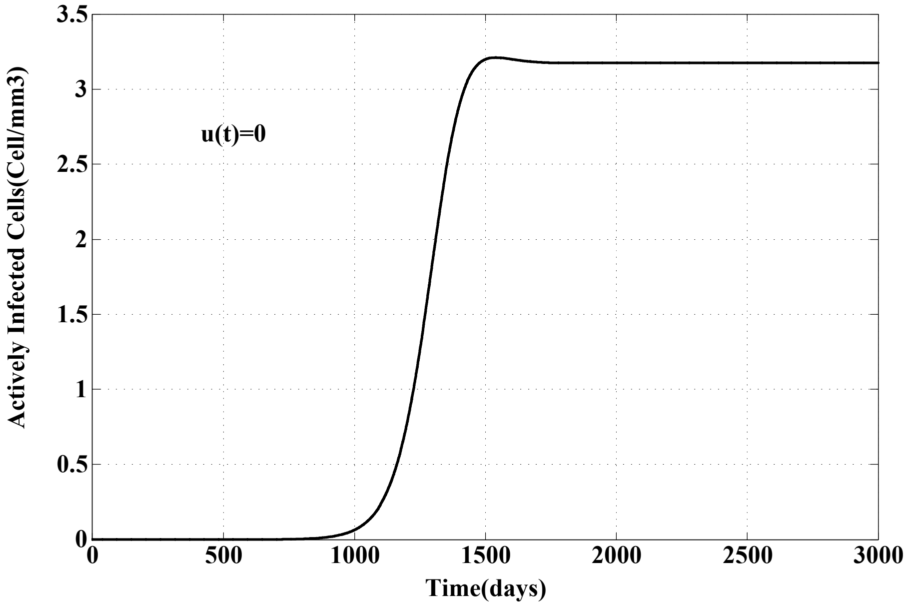

Fig. 17. The evolution of actively infected cells for the untreated case.

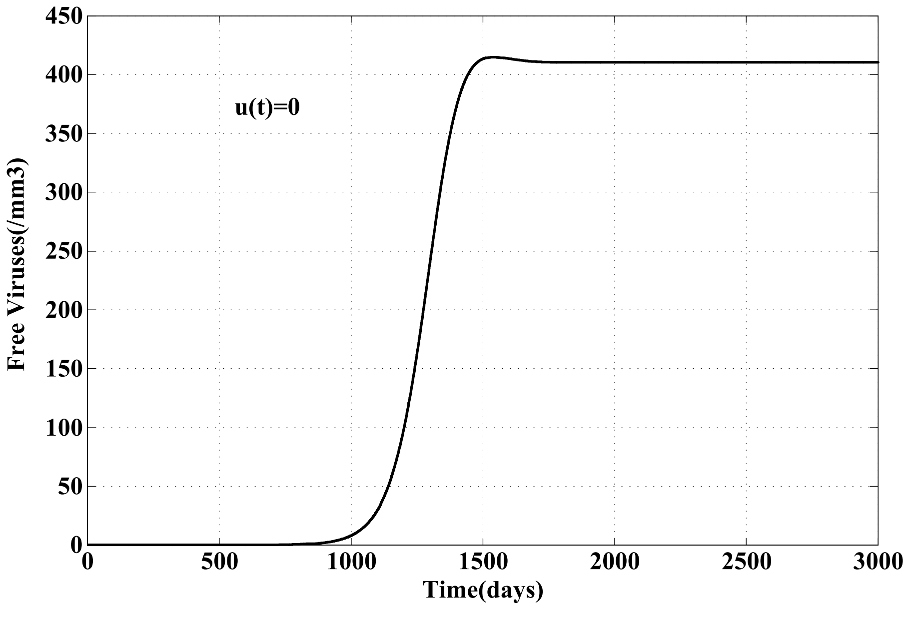

Fig. 18. The evolution of free viruses for the untreated case.

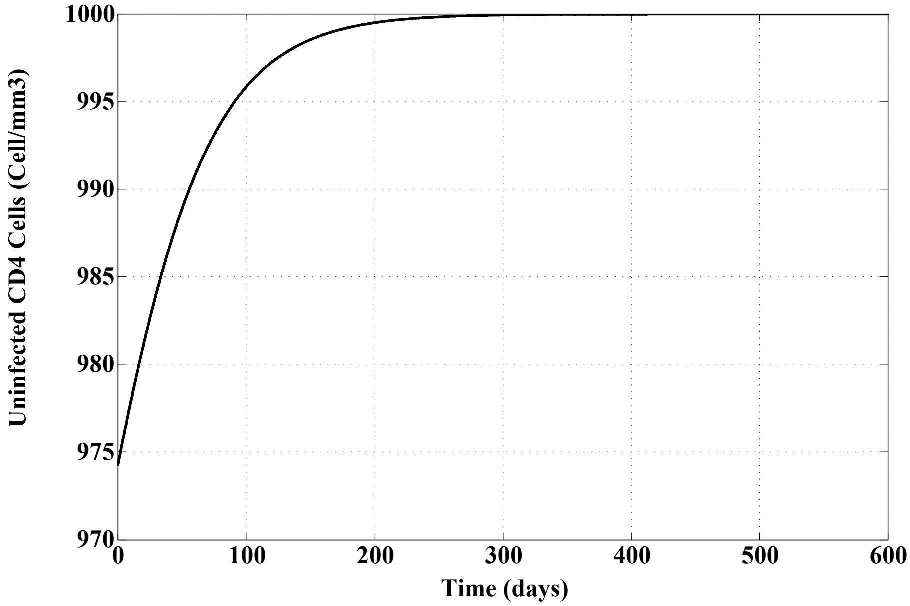

Fig. 19. The evolution of CD4 T cells under RHC drug doses.

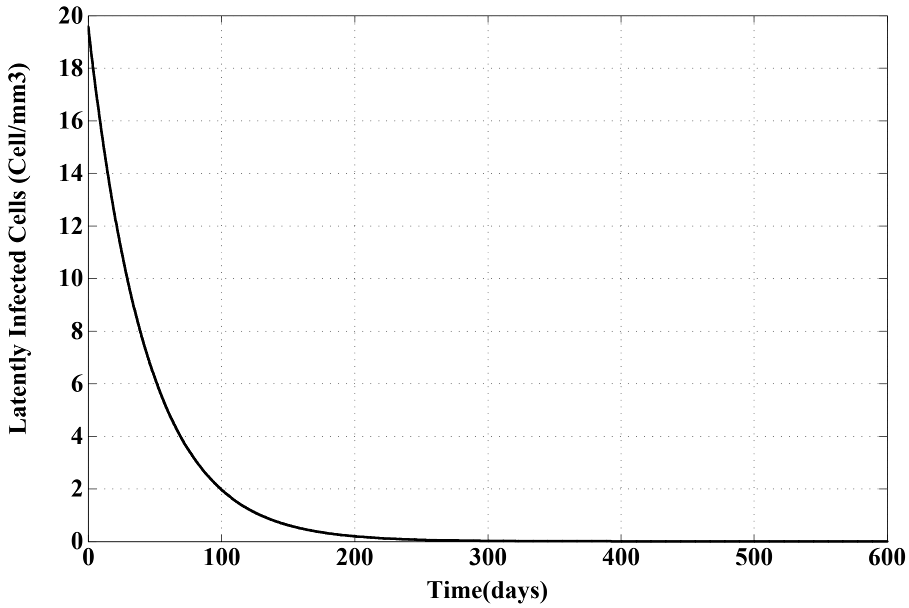

Fig. 20. The evolution of latently infected cells under RHC drug doses.

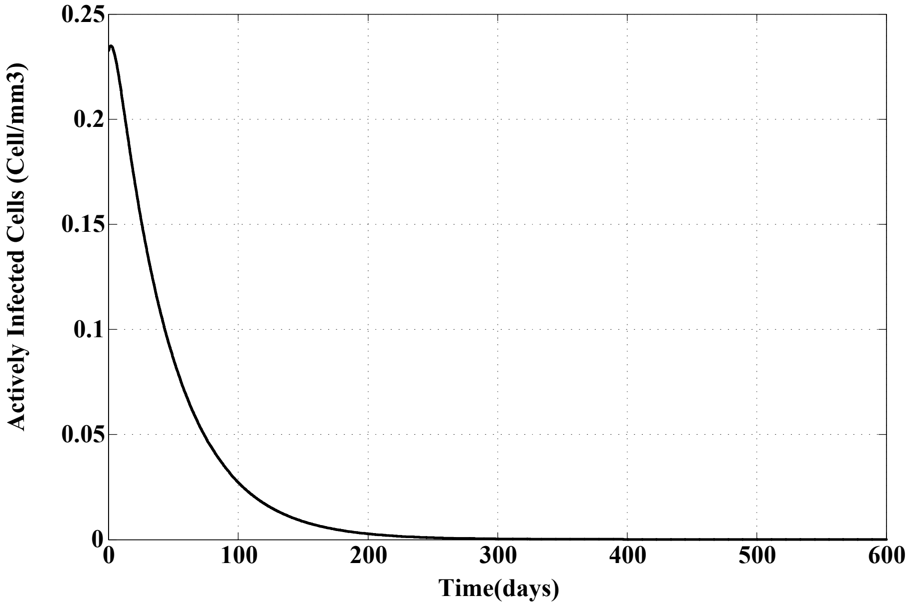

Fig. 21. The evolution of actively infected cells under RHC drug doses.

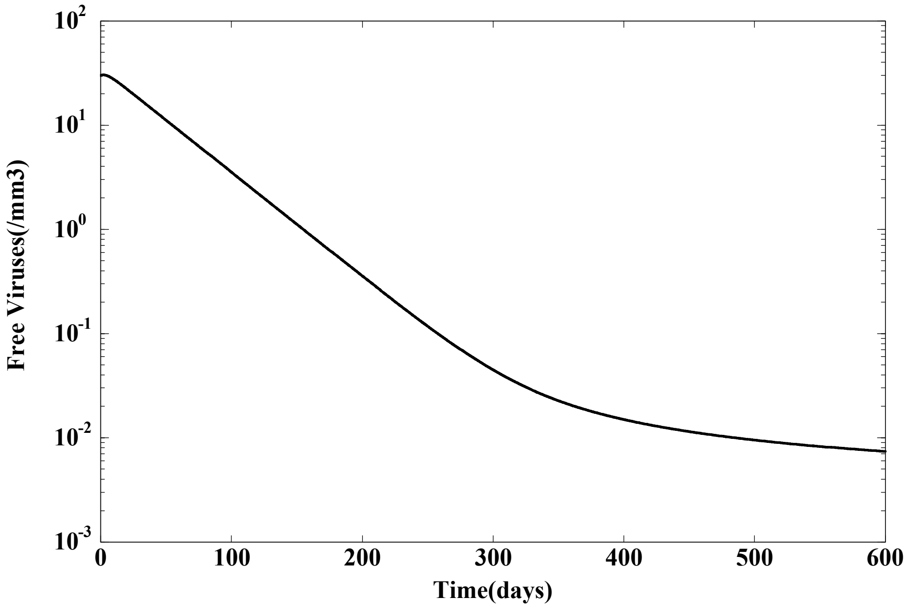

Fig. 22. The evolution of free viruses under RHC drug doses.

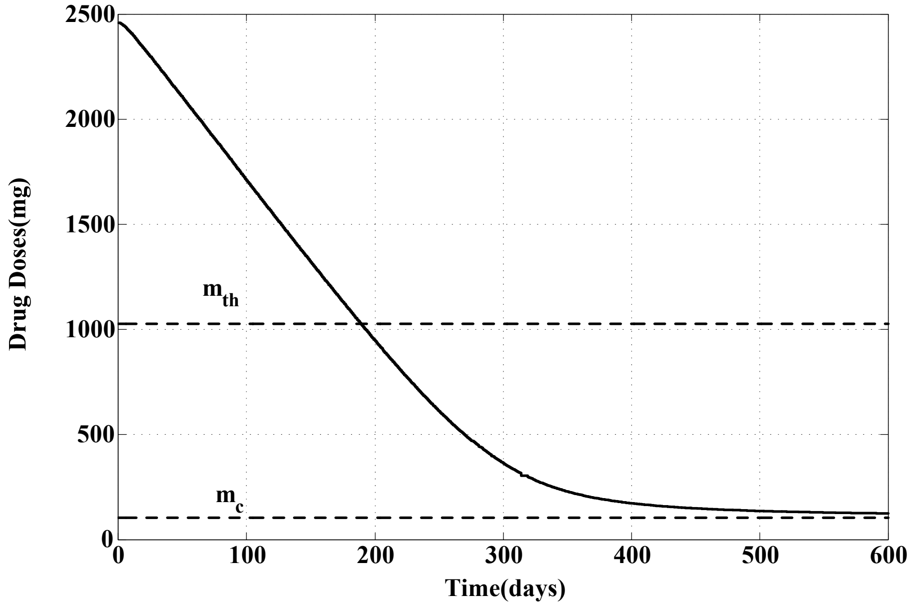

Fig. 23. Reverse transcriptase inhibitor under RHC.

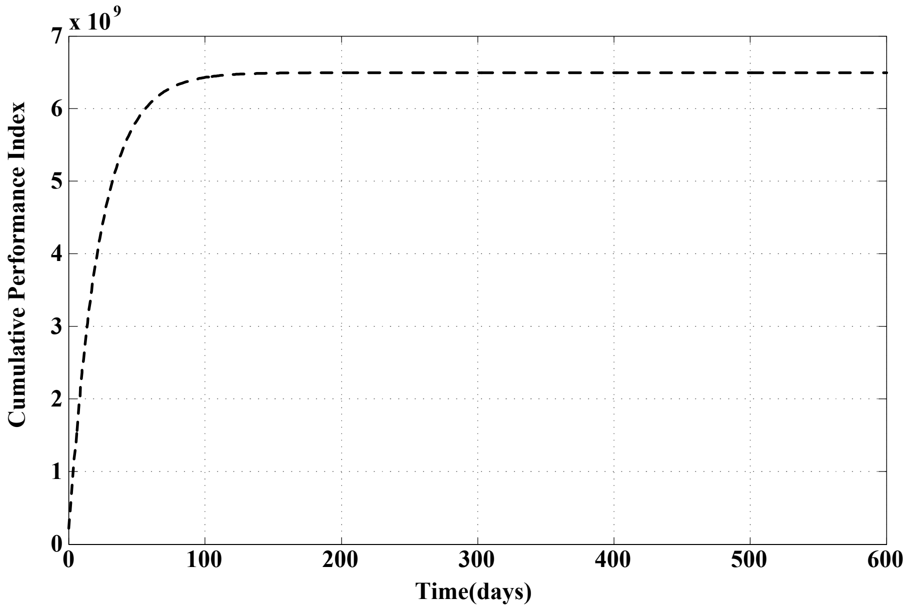

Fig. 24. Cumulative performance index under RHC drug doses.

3.3 HIV/AIDS model 2

The results presented in this section are published in [21].

3.3.1 The model

To develop a method for the scheduling of antiviral therapy which consists of RTI and PI drugs, we consider a model described in ([83] and [88]) and exhaustively investigated by ([45] and [62]). This model can be written as

$$\dot{x} = s - \mu x + px\left(1 - \frac{x}{T_{\max}}\right) - e^{-u_1(t)}kxv_I, \tag{3.22}$$

$$\dot{y} = e^{-u_1(t)}kxv_I - \delta y, \tag{3.23}$$

$$\dot{v}_I = (1-\eta_0)e^{-u_2(t)}n\delta y - cv_I, \tag{3.24}$$

$$\dot{v}_{NI} = \left\{1 - (1-\eta_0)e^{-u_2(t)}\right\} n\delta y - cv_{NI}, \tag{3.25}$$

where, v_I and v_{NI} are the concentrations of infectious and non-infectious virions, respectively. Here the control input is $u_1(t) = \alpha_1 m_1(t)$ and $u_2(t) = \alpha_2 m_2(t)$ where $m_1(t)$ and $m_2(t)$ are the drug doses at time t of RTI and PI, respectively, and α_1 and α_2 are the efficiency of drugs for RTI and PI, respectively; η_0 is the proportion of non-infectious virus in the total virus pool before the intervention of antiviral drugs. The model (3.22)-(3.25) leads to the model proposed by [45] if we replace $e^{-u_1(t)}$ and $e^{-u_2(t)}$ with $(1-\gamma(t))$ and $(1-\eta(t))$, respectively, where, $\gamma(t)$ and $\eta(t)$ are the drug efficacies of RTI and PI, respectively.

We note that the system (3.22)-(3.25) is biologically acceptable because

$$\begin{aligned}
\dot{x}\,|_{(x=0)} &= s \geq 0,\\
\dot{y}\,|_{(y=0)} &= e^{-u_1(t)}kv_I x \geq 0, \quad (v_I, x \geq 0),\\
\dot{v}_I\,|_{(v_I=0)} &= (1-\eta_0)e^{-u_2(t)}n\delta y \geq 0, \quad (y \geq 0),\\
\dot{v}_{NI}\,|_{(v_{NI}=0)} &= \left\{1-(1-\eta_0)e^{-u_2(t)}\right\} n\delta y \geq 0, \quad (y \geq 0).
\end{aligned}$$

It has been shown in [62] that all solutions of (3.22)-(3.24) are ultimately bounded (which also involves the ultimate boundedness of the solutions of (3.22)-(3.25)). Here we need a slightly stronger statement.

Proposition 2 If $s - \mu T_{\max} < 0$ then there exists such a positive number M^* that the compact set

$$\Omega = \left\{(x, y, v_I, v_{NI}) : 0 \leq x, y \leq T_{\max},\ 0 \leq v_I, v_{NI} \leq M^*\right\},$$

is positively invariant.

Proof. Assume that $(x, y, v_I, v_{NI})' \in \mathbb{R}^4_+$ and let $\varrho(x) = s - \mu x + px\left(1 - \frac{x}{T_{\max}}\right)$. We can see that for x^0 defined by (3.10), inequalities

$$\begin{aligned}
\varrho(x) &> 0 \quad \text{if} \quad 0 \leq x < x^0,\\
\varrho(x) &< 0 \quad \text{if} \quad x > x^0,\\
\varrho(x^0) &= 0, \quad \varrho'(x^0) < 0,
\end{aligned}$$

hold true. Since $\dot{x} \leq \varrho(x)$ and $\dot{x} + \dot{y} \leq \varrho(x)$ in $\mathbb{R}^4_+$, and $x^0 < T_{\max}$ if $s - \mu T_{\max} < 0$, therefore $x(t) \leq T_{\max}$, $y(t) \leq T_{\max}$ for all $t \geq 0$, if $x(0) \leq T_{\max}$ and $y(0) \leq T_{\max}$. On the other hand, $\dot{v}_I + \dot{v}_{NI} = n\delta y - c(v_I + v_{NI})$, therefore $v_I(t) + v_{NI}(t) \leq \frac{n\delta T_{\max}}{c} =: M^*$ for all $t > 0$, if this inequality holds true for $t = 0$ and $y(0) \leq T_{\max}$.

Remark 14 Note that Ω contains all of the biologically relevant states. Moreover, there is no finite escape time.

Let us compute the equilibrium points of system (3.22)-(3.25) under constant controller i.e. for $u_i(t) = \widehat{u}_i$, $i = 1, 2$, $t \geq 0$. There are two possible steady states, which we call uninfected and infected. The uninfected steady state is $E^0 = (x^0, 0, 0, 0)$ and the infected steady state is $E^+ = (\overline{x},\ \overline{y},\ \overline{v}_I,\ \overline{v}_{NI})$ where

$$\begin{aligned} \overline{x} &= \frac{ce^{\widehat{u}_1+\widehat{u}_2}}{(1-\eta_0)kn}, \quad \overline{y} = \frac{ce^{\widehat{u}_2}}{(1-\eta_0)\delta n}\overline{v}_I, \\ \overline{v}_I &= \frac{e^{\widehat{u}_1}}{k}\left[\frac{s}{\overline{x}} + q_1 - q_2\overline{x}\right], \quad \overline{v}_{NI} = \frac{e^{\widehat{u}_2} - 1 + \eta_0}{1-\eta_0}\overline{v}_I. \end{aligned}$$

The crucial quantity is the basic reproduction ratio, R_0^c, given by

$$R_0^c(\widehat{u}_1, \widehat{u}_2) = \frac{nkx^0(1-\eta_0)e^{-\widehat{u}_1-\widehat{u}_2}}{c},$$

(see [45] and [62]).

It is easy to see that

$$\begin{aligned} \overline{x} &> x^0, \quad \overline{v}_I < 0, \quad \text{if} \quad R_0^c(\widehat{u}_1, \widehat{u}_2) < 1, \\ \overline{x} &= x^0, \quad \overline{v}_I = 0, \quad \text{if} \quad R_0^c(\widehat{u}_1, \widehat{u}_2) = 1, \\ \overline{x} &< x^0, \quad \overline{v}_I > 0, \quad \text{if} \quad R_0^c(\widehat{u}_1, \widehat{u}_2) > 1. \end{aligned}$$

Therefore, if $R_0^c(\widehat{u}_1, \widehat{u}_2) < 1$, then E^0 is the only biologically acceptable equilibrium point.

It has been proven in [62] that if $R_0^c(\widehat{u}_1, \widehat{u}_2) < 1$, then E^0 is asymptotically stable. If $R_0^c(\widehat{u}_1, \widehat{u}_2) > 1$, then E^0 becomes unstable and the infected steady state E^+arises.

Our aim is to determine a control strategy which steers the system near the uninfected steady state.

Proposition 3 Any initial point is asymptotically controllable from the set Ω to E^0 with piecewise constant controllers.

Proof. Let $u_1(t) = \widehat{u}_1$ and $u_2(t) = \widehat{u}_2$ with $\widehat{u}_1 + \widehat{u}_2 > u_c$, where

$$u_c = \left\{0, \ln\left(\frac{nkx^0(1-\eta_0)}{c}\right)\right\},$$

then $R_0^c(\widehat{u}_1 + \widehat{u}_2) < 1$, therefore the corresponding trajectory will tend to E^0 as $t \to \infty$.□

This implies that for any initial points which belonging to Ω, no finite escape time occurs.

This proposition shows that the stabilization of E^0 can be achieved by piecewise constant controllers, but the speed of convergence may be unacceptably low, if the controllers are applied with $\widehat{u}_1 + \widehat{u}_2 > u_c$, $\widehat{u}_1 + \widehat{u}_2 \sim u_c$.

3.3.2 Application of ℓ-step RHC method

In this subsection we apply the ℓ-step receding horizon control method proposed in Chapter 2. The computations were carried out when the state is measured at the instants $T_j^m = j\ell T$, $j = 0, 1, \dots$. Simulations for the continuous-time system were carried out when ℓ and T were chosen to be $\ell = 4$ and $T = 1$. We transform the point $(x^0, 0, 0, 0)$ into the origin with $\widetilde{x} = x - x^0$, $\widetilde{y} = y$, $\widetilde{v}_I = v_I$, $\widetilde{v}_{NI} = v_{NI}$ and the control input $\widetilde{u}_1 = u_1 - u_c^{(1)}$, $\widetilde{u}_2 = u_2 - u_c^{(2)}$ with $u_c^{(1)} + u_c^{(2)} = u_c$. We can see that Assumption A1 is satisfied. Let $F_{T,h}^A$ be constructed using multiple steps of a one-step second-order Rung-Kutta scheme, then Assumptions A2 and A6 are also satisfied. From Proposition 3 it follows that Assumption A3 holds true, as well.

To verify Assumptions A4 and A5, let A_C be the coefficient matrix of the linearized system (3.22)-(3.25) in case of constant control i.e. $\widetilde{u}_1(t) = \widetilde{u}_1 > 0$, and $\widetilde{u}_2(t) = \widetilde{u}_2 > 0$.

Similarly to the previous section we choose the running and the terminal costs as

$$l_h(\widetilde{X}, \widetilde{u}) = 0.1\widetilde{X}'Q\widetilde{X} + 5\widetilde{u}_1^2 + 2\widetilde{u}_2^2, \tag{3.26}$$
$$g(\widetilde{X}) = \widetilde{X}'P\widetilde{X}, \tag{3.27}$$

where $\widetilde{X} = (\widetilde{x}, \widetilde{y}, \widetilde{v}_I, \widetilde{v}_{NI})'$, P is a positive definite diagonal matrix given by

$$P = diag(1,\ 20,\ 3,\ 0.0002)$$

and Q is a positive definite symmetric matrix

$$Q = \begin{bmatrix} 0.1943 & 0.1140 & 0.1056 & 0 \\ 0.1140 & 6.1242 & -1.7775 & -0.0012 \\ 0.1056 & -1.7775 & 2.7558 & -0.0001 \\ 0 & -0.0012 & -0.0001 & 0.0002 \end{bmatrix}.$$

This P and Q satisfy the Lyapunov equation for the discrete-time model of the linearized system (3.22)-(3.25)

$$Q = -(A_T' P A_T - P), \qquad A_T = e^{A_C T}.$$

From (3.26)-(3.27), Assumption A4 is satisfied. It has been verified numerically by solving a constrained minimization problem with several starting points that Assumption A5 is satisfied over the whole set Ω.

3.3.3 Numerical results

We conduct simulation studies using the parameter values taken from [45]. These values are listed in Table 2.

Parameters	Description	Values
s	Rate at which new T-cells are generated	36 $\text{day}^{-1}\text{mm}^{-3}$
p	Rate of growth for the T-cell population	0.108 day^{-1}
$T_{\max}$	Maximum T-cell population level	1500 mm^{-3}
μ	Death rate of uninfected cells	0.072 day^{-1}
δ	Death rate of infected cells	0.5 day^{-1}
k	Rate at which T-cells become infected by virus	0.001 $\text{mm}^3\text{day}^{-1}$
η_0	Proportion of non-infectious virions	0.99
n	Number of new virions produced per infected cell	1000
c	Clearance rate of free virions	3.0 day^{-1}

Table 2

For the parameters given in Table 2, we can see that $R_0^c(\widehat{u}_1, \widehat{u}_2) \mid_{(\widehat{u}_1=\widehat{u}_2=0)} > 1$ so E^0 is unstable when there is no treatment.

We assume that the infection started near the uninfected steady state with the values $x = x^0$, $y = v_{NI} = 0$ and $v_I = 0.0001$. Figures 25 and 26, show the case when there is no treatment ($u_1 = u_2 = 0$). It can be seen that the concentration of uninfected CD4$^+$ T cells is decaying, and the concentrations of infected cells, infectious and non-infectious virions are increasing. We assume that the system is in the infected steady state before initiating the treatment i.e at $E^+ \mid_{(\widehat{u}_1=\widehat{u}_2=0)} = (300,\ 80.64,\ 134.4,\ 13305.6)$.

Figure 27 shows that when the RHC strategy is applied, the concentrations of the uninfected and infected T cells tend rapidly to the normal value x^0 and zero, respectively. In Figure 28, we can see that the concentrations of infectious and non-infectious virions are decaying when the RHC strategy is applied.

The efficiencies of drugs for RTI and PI are given by $\alpha_1 = \alpha_2 = 0.005$ (see [6]). The drug doses m_1 and m_2 as functions of the time under ℓ-step receding horizon controller are shown in Figure 29. It is observed that the system can be kept near the stationary point $E^0 = (x^0, 0, 0, 0)$ by small drug doses.

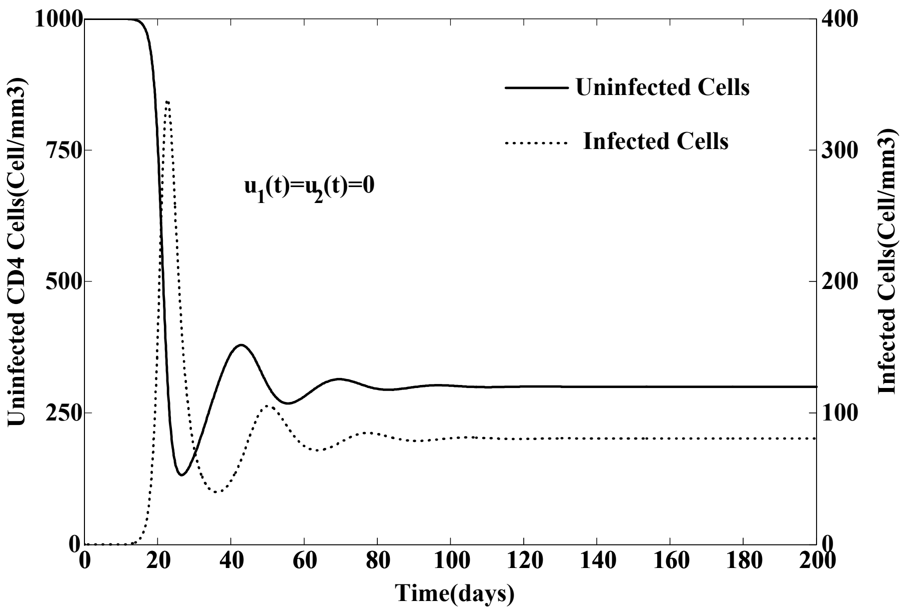

Fig. 25. The evolution of uninfected and infected cells for the untreated case.

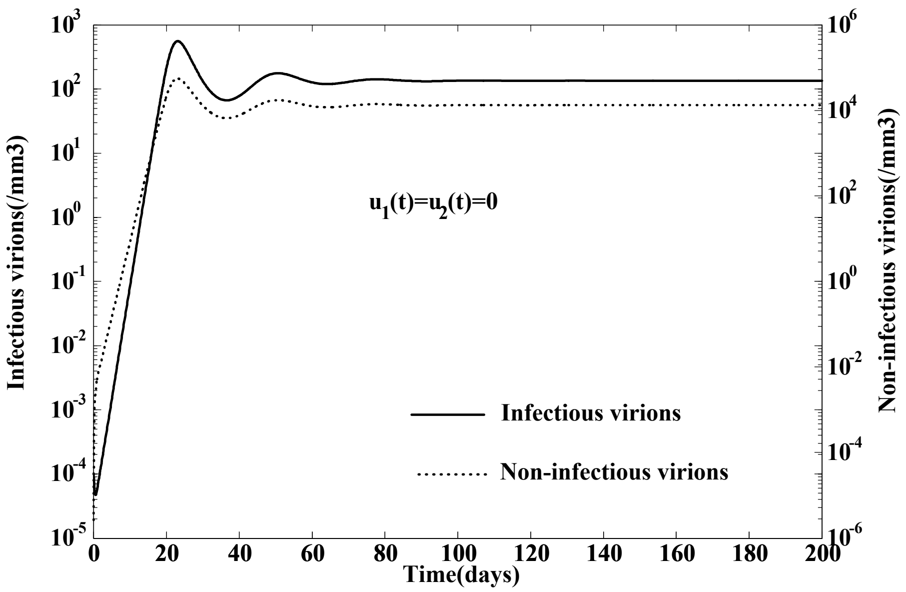

Fig. 26. The evolution of infectious and non-infectious virions for the untreated case.

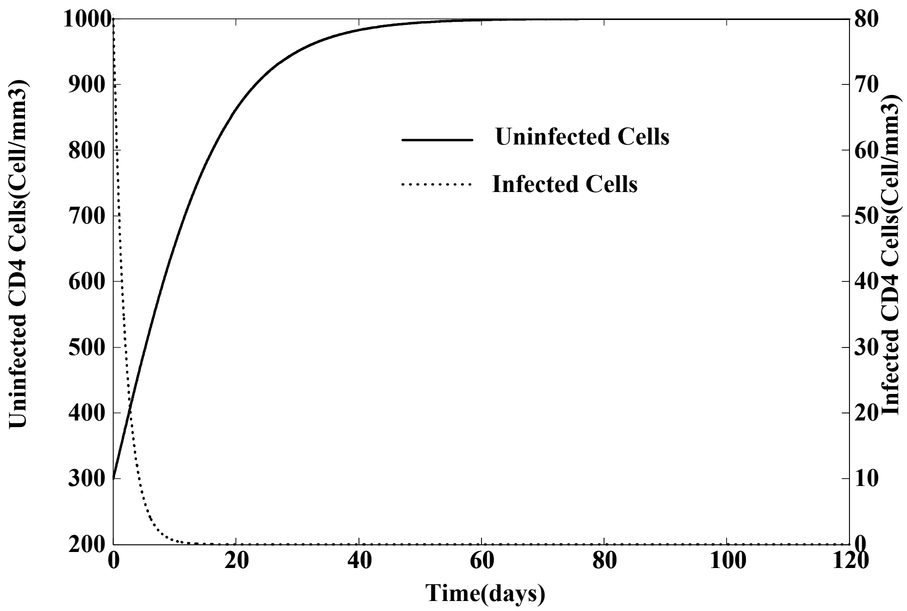

Fig. 27. The evolution of uninfected and infected cells under RHC.

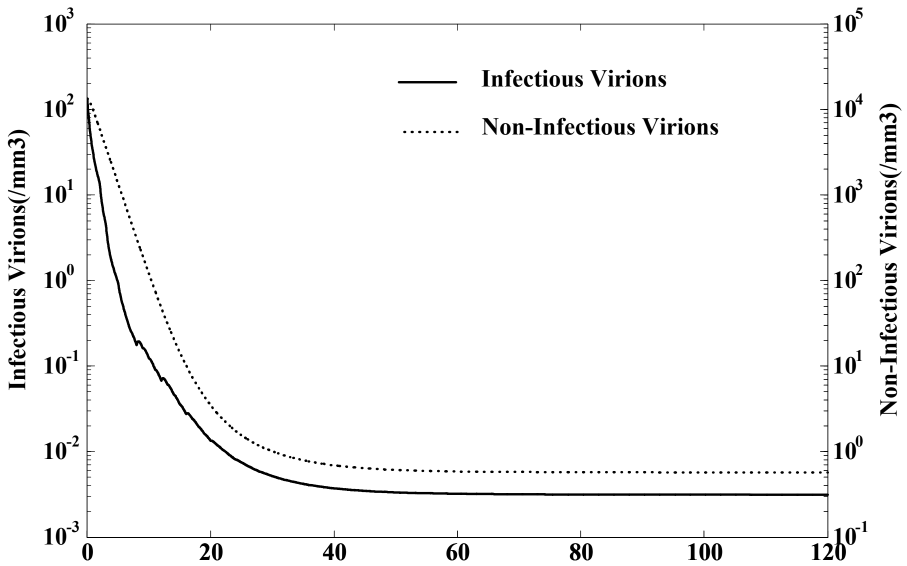

Fig. 28. The evolution of infectious and non-infectious virions under RHC.

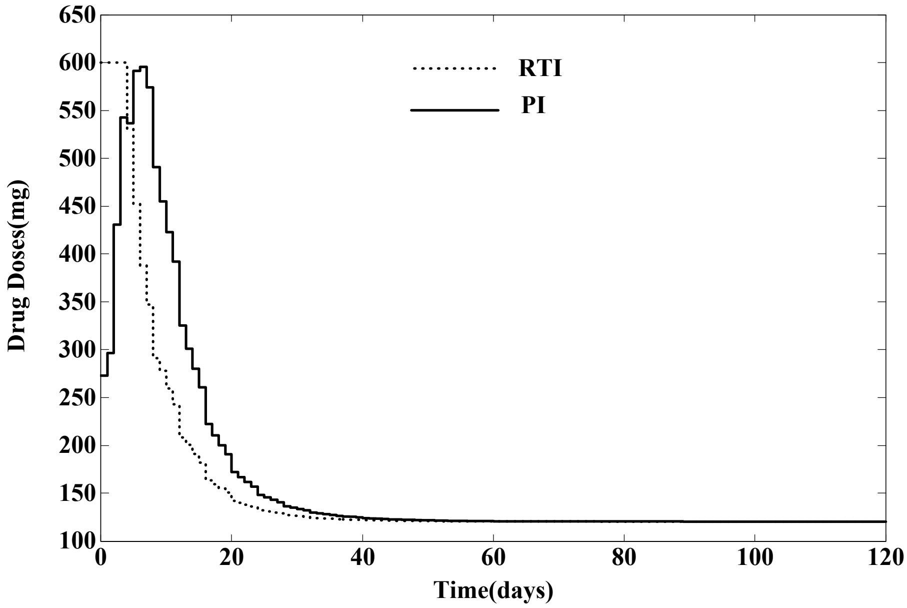

Fig. 29. Reverse transcriptase inhibitors and Protease inhibitors under RHC.

The Appendix

1- Example. Control of a Harmonic Oscillator

Consider the following system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + u(t), \qquad |u(t)| \leq 1. \end{aligned}$$

We want to determine the time-optimal controller which steers the initial state x_0, y_0 to the origin in a minimal time. The complete solution is given in [61] where the optimal control $u^*(t)$ is a relay control with values $+1$ and -1. We consider the solution-curve families in the phase plane of the extremal differential systems:

$$\left\{ \begin{array}{c} \dot{x} = y \\ \dot{y} = -x + 1 \end{array} \right. , \tag{S_+}$$

and

$$\left\{ \begin{array}{c} \dot{x} = y \\ \dot{y} = -x - 1 \end{array} \right. . \tag{S_-}$$

The corresponding extremal response traces out the solutions curves of S_+ and S_- through the origin that are

$$\begin{aligned} \Gamma_+ &: \ (x-1)^2 + y^2 = 1, \ y < 0 \\ \Gamma_- &: \ (x+1)^2 + y^2 = 1, \ y > 0. \end{aligned}$$

The exact discrete-time model is:

$$\left\{ \begin{array}{c} x(k+1) = x(k)\cos(T) + y(k)\sin(T) - \cos(T) + 1 \\ y(k+1) = -x(k)\sin(T) + y(k)\cos(T) + \sin(T) \end{array} \right. , \tag{L_+}$$

and

$$\left\{ \begin{array}{c} x(k+1) = x(k)\cos(T) + y(k)\sin(T) + \cos(T) - 1 \\ y(k+1) = -x(k)\sin(T) + y(k)\cos(T) - \sin(T) \end{array} \right. . \tag{L_-}$$

The solutions of continuous and discrete-time systems are shown in Figures 30 and 31-33, respectively.

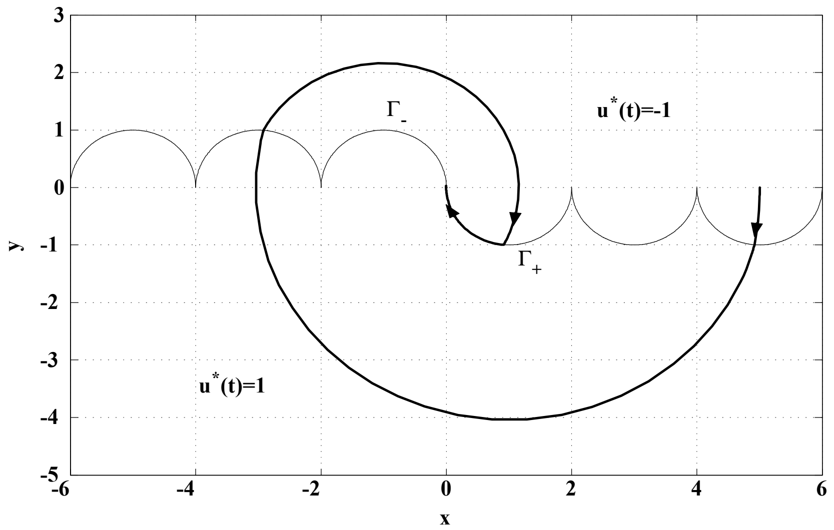

Fig. 30. Trajectory of the continuous-time system.

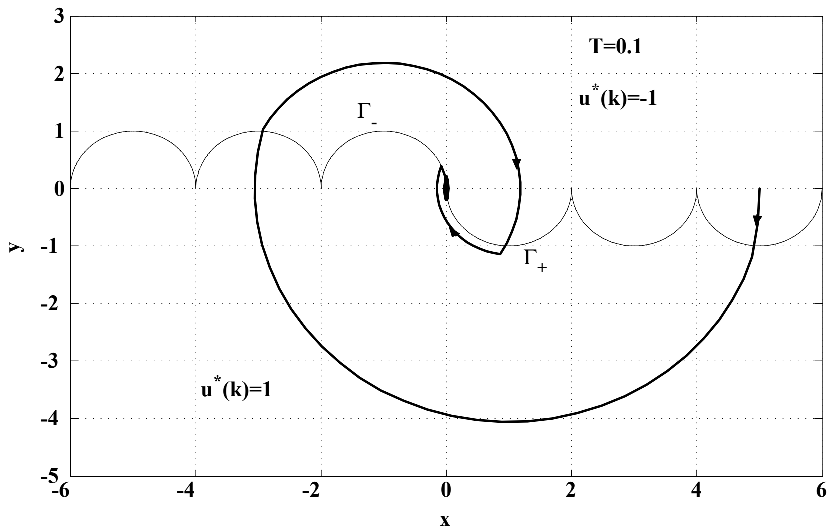

Fig. 31. Trajectory of the exact discrete-time system when $T = 0.1$.

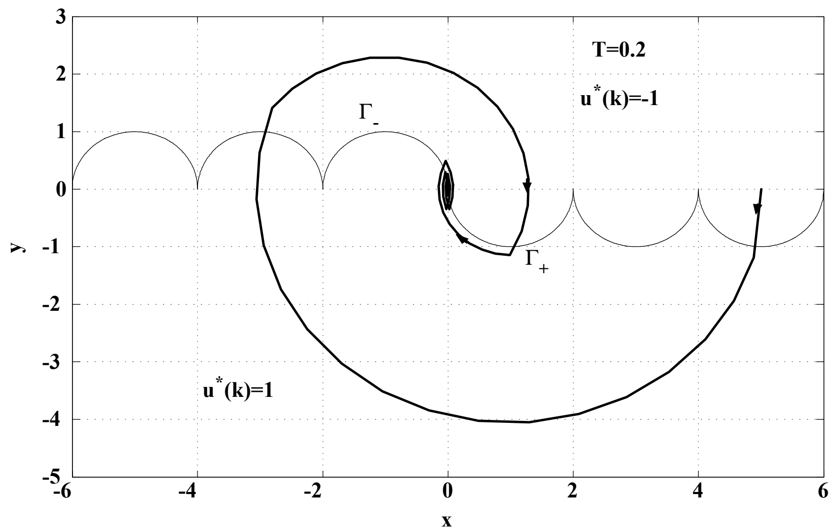

Fig. 32. Trajectory of the exact discrete-time system when $T = 0.2$.

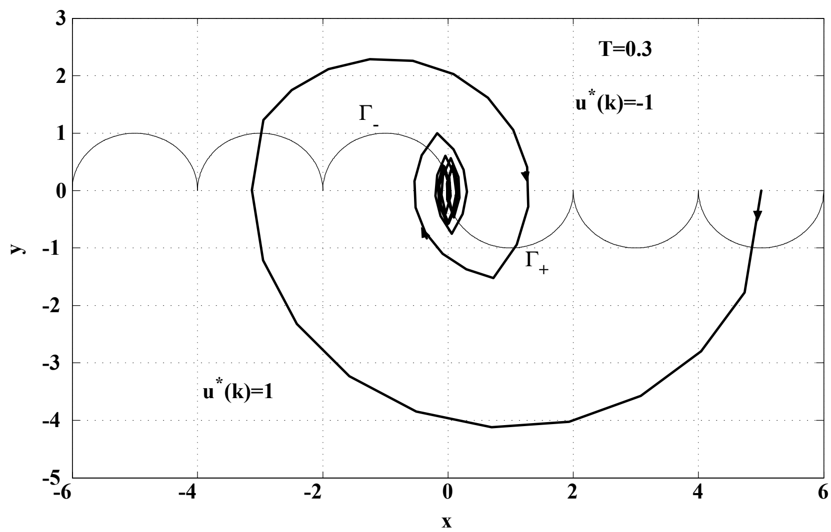

Fig. 33. Trajectory of the exact discrete-time system when $T = 0.3$.

This example shows that if we have a stabilizing controller for continuous-time system and discretize this system, we observe that the trajectory of the discrete-time system tends to a ball around the origin and remains there (i.e., the exact discrete-time system is PAS about the origin). The radius of this ball increases with increasing the sampling period T.

2- The existence of a class-KL function

We shall use the notation $V_N(k)$ in place of $V_N(\phi_k^E(x, \mathbf{u}_v(x)))$, where $\mathbf{u}_v$ is the control sequence produced by v_T^A and F_T^E. Using the pair $(d, V_{\max}^A)$ in the claim, it follows that there exists $T^* > 0$ such that, for all $T \in (0, T^*]$ and all $x \in \Gamma_{\max}$, $\max\{V_N(k+1), V_N(k)\} \geq d$ implies

$$\begin{aligned} V_N(k+1)) - V_N(k) &\leq -\frac{T}{2}\varphi_1(\psi_2^{-1}(V_N(k))) \\ &= -T\alpha(V_N(k)). \end{aligned}$$

We introduce a variable $t \in \mathbb{R}$ and define

$$y_T(t) := V_N(k) + (t - kT)\frac{V_N(k+1) - V_N(k)}{T} \qquad t \in [kT, (k+1)T), \quad k \geq 0.$$

Then $y_T(t)$ is a continuous function of "time" t. Moreover, it is absolutely continuous in t (in fact, piecewise linear) and we can write for almost all t:

$$\dot{y}_T(t) = \frac{V_N(k+1) - V_N(k)}{T}, \ t \in [kT, (k+1)T).$$

From the definition of $y_T(t)$ we have that if

$$\begin{aligned} \{y_T(t) &\geq d, \quad t \in [kT, (k+1)T)\} \\ &\Longrightarrow \max\{V_N(k+1), V_N(k)\} \geq d \\ &\Longrightarrow V_N(k+1)) - V_N(k) \leq -T\alpha(V_N(k)) \\ &\Longrightarrow y_T(t) \leq V_N(k) \end{aligned}$$

and so we can write for almost all t that: $y_T(t) \geq d$, implies

$$\begin{aligned} \dot{y}_T(t) &\leq -\alpha(V_N(k)), \quad k : t \in [kT, (k+1)T). \\ &\leq -\alpha(y_T(t)). \end{aligned}$$

It follows from the arguments in ([95], Section VI) that there exists $\beta_1 \in \mathcal{KL}$ that is determined by $\alpha = \frac{1}{2}\varphi_1 \circ \psi_2^{-1}$ such that

$$y_T(t) \leq \beta_1(y_T(0), t), \quad y_T(0) \in \Gamma_{\max}.$$

This implies, using $y_T(k) = V_N(k)$, with $t = kT$, $y_T(0) = V_N(0)$, that

$$V_N(k) \leq \beta_1(V_N(0), kT), \ \ k \geq 0$$

for all $x \in \Gamma_{\max}$.

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